Recitation (V): NP-Completeness.

Ex: "Show the following problem is in NP"

A (decision) problem X is in NP if:

$\forall$ "Yes"-inst s_X of X

- $\exists$ poly-time certifier A that accepts s_X and another object c, verifies that s_X is indeed a "Yes"-inst
- The object c is called a "certificate"

Exmp (I) Knapsack

Input: n items with weight $w_1, \cdots, w_n$, value $v_1, \cdots, v_n$

weight bound W, value threshold V

Output: Decide if $\exists S \subseteq [n]$ s.t. $\sum_{i \in S} w_i \leq W$ and $\sum_{i \in S} v_i > V$

- certificate: optimal solution S^*

- Certifier: verify

① $\sum_{i \in S^*} w_i \leq W$ ② $\sum_{i \in S^*} v_i > V$

Exmp (II) : 3-Coloring

Input : graph $G = (V, E)$, color set $\{R, G, B\}$

Output: Decide if $\exists \pi : V \mapsto \{R, G, B\}$ s.t.

$\forall (u, v) \in E$, $\pi(u) \neq \pi(v)$

- certificate: π^*

- Certifier: check for every $(u, v) \in E$ whether $\pi^*(u) \neq \pi^*(v)$.

Exmp (III) : LIS

Input : array of n numbers $A = [a_1, \cdots, a_n]$, integer K

Output : Decide if $\exists$ increasing subseq of A with length $\geq K$

- certificate : the LIS S of A
- Certifier : checks

① S is indeed increasing

② $|S| \geq K$

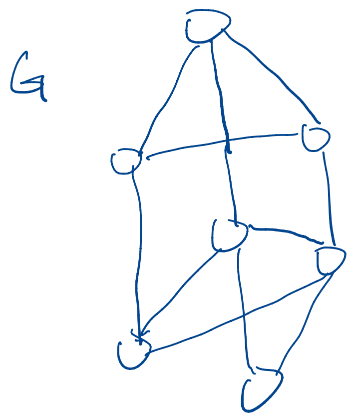
▷ Reduction : $Y \leq_p X$

Given an instance s_Y of problem Y, show how to construct (in polynomial time) an instance s_X of problem X such that:

- s_Y is a Yes-instance of $Y \Rightarrow s_X$ is a Yes-inst. of X
- s_X is a Yes-inst of $X \Rightarrow s_Y$ is a Yes-inst of Y.

Exmp (I) : 3-Coloring $\le_p$ 4-Coloring

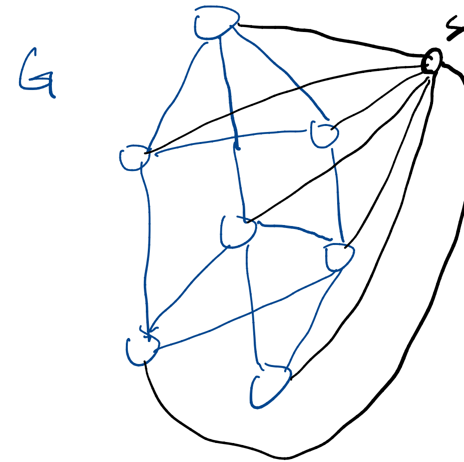
3-Coloring inst



$\{R, G, B\}$

reduction $\Longrightarrow$

4-coloring inst



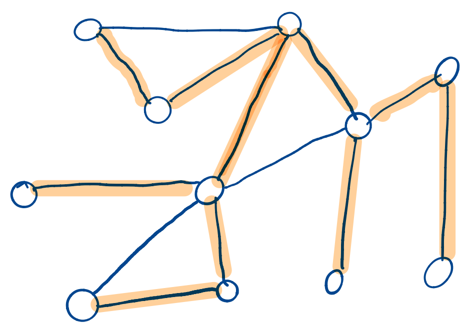
$\{R, G, B, O\}$

Exmp (II) Hamiltonian Path $\leq_p$ Degree-3 Spanning Tree

Def (Degree-3 Spanning Tree)

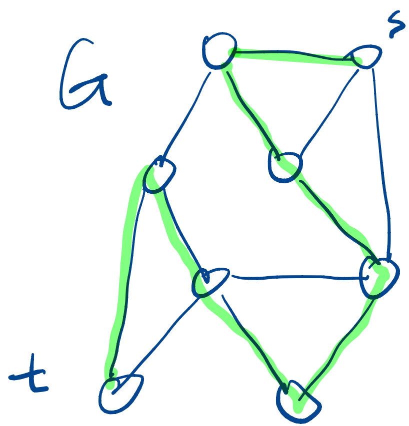
Input: graph $G = (V, E)$, $|V| = n$

Output: Decide if $\exists$ spanning tree T s.t. $\forall v \in V$, $\deg_T(v) \leq 3$

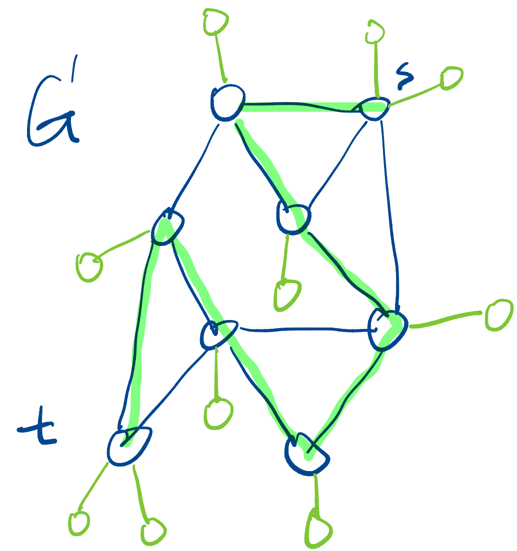


HP inst

3-ST inst



$\Longrightarrow$



① "Yes"-inst of HP $\Rightarrow$ "Yes"-inst of 3-ST:

Combining the HP in G with all the extra edges we added gives a 3-ST.

② "Yes"-inst of 3-ST $\Rightarrow$ "Yes"-inst of HP

Given a 3-ST T in G', we remove all the extra edges, then the remained part of T (let's say T') is a (s,t)-HP in G:

Let $d_v := \deg_{T'}(v)$. then

① $d_s = d_t = 1$ ② $\forall v \neq s,t,\ d_v \leq 2$

But by the hand-shake lemma

$$d_s + d_t + \sum_{v \neq s,t} d_v = 2(n-1) \Rightarrow \sum_{v \neq s,t}^{n-2} d_v = 2n-4$$

$\Rightarrow d_v = 2.\ \forall v \neq s,t.$

Thus T' is a HP.