Recitation (V): NP-Completeness.

Ex: "Show the following problem is in NP"

A (decision) problem $X$ is in NP if:

1. $\forall$ "Yes"-inst $s_x$ of $X$
2. $\exists$ poly-time certifier $A$ that accepts $s_x$ and another object $c$, verifies that $s_x$ is indeed a "Yes"-inst
3. The object $c$ is called a "certificate"
Exmp (I) Knapsack

Input: n items with weight $w_1, \ldots, w_n$, value $v_1, \ldots, v_n$
weight bound $W$, value threshold $V$

Output: Decide if $\exists S \subseteq [n]$ s.t. $\sum_{i \in S} w_i \leq W$ and $\sum_{i \in S} v_i > V$

- Certificate: optimal solution $S^*$
- Certificate: verify

1. $\sum_{i \in S^*} w_i \leq W$
2. $\sum_{i \in S^*} v_i > V$
Example (II): 3-Coloring

Input: graph $G = (V, E)$, color set $\{R, G, B\}$

Output: Decide if $\exists \pi : V \rightarrow \{R, G, B\}$ s.t.

\[ \forall (u, v) \in E, \pi(u) \neq \pi(v) \]

Certificate: $\pi^*$

Certifier: check for every $(u, v) \in E$

whether $\pi^*(u) \neq \pi^*(v)$.
Example (III): LIS

**Input:** array of \( n \) numbers \( A = [a_1, \ldots, a_n] \), integer \( k \)

**Output:** Decide if there is an increasing subsequence of \( A \) with length \( \geq k \)

- **Certificate:** the LIS \( S \) of \( A \)
- **Certificate:** checks
  1. \( S \) is indeed increasing
  2. \( |S| \geq k \)
Reduction: $Y \leq_p X$

Given an instance $s_Y$ of problem $Y$, show how to construct (in polynomial time) an instance $s_X$ of problem $X$ such that:

- $s_Y$ is a Yes-instance of $Y$ $\Rightarrow$ $s_X$ is a Yes-inst. of $X$
- $s_X$ is a Yes-inst. of $X$ $\Rightarrow$ $s_Y$ is a Yes-inst. of $Y$. 
Example (I): $3$-Coloring $\leq_p 4$-Coloring

3-Coloring inst

$G$

$\{R, G, B\}$

4-Coloring inst

$G$

$\{R, G, B, O\}$

Reduction
Exmp (ii) Hamiltonian Path $\leq_p$ Degree-3 Spanning Tree

Def (Degree-3 Spanning Tree)

Input: graph $G = (V, E)$, $|V| = n$

Output: Decide if $\exists$ spanning tree $T$ s.t. $\forall v \in V$, $\deg_T(v) \leq 3$
0 “Yes”-inst of HP $\Rightarrow$ “Yes”-inst of 3-ST:

Combining the HP in $G$ with all the extra edges we added gives a 3-ST.
2. "Yes"-inst of 3-ST $\Rightarrow$ "Yes"-inst of HP

Given a 3-ST $T$ in $G'$, we remove all the extra edges, then the remained part of $T$ (let's say $T'$) is a $(s, t)$-HP in $G$:

Let $d_v := \deg_{T'}(v)$, then

1. $d_s = d_t = 1$
2. $\forall v \neq s, t, d_v \leq 2$

But by the hand-shake lemma

$$d_s + d_t + \sum_{v \neq s, t} d_v = 2(n-1) \Rightarrow \sum_{v \neq s, t} d_v = 2n - 4$$

$\Rightarrow d_v = 2, \forall v \neq s, t$.

Thus $T'$ is a HP.