Recitation (II)

Prob 1: Weighted Completion Time on Single Machine

Given a set of \( n \) jobs \( \{1, 2, 3, \ldots, n\} \), each job \( j \) with a processing time \( t_j > 0 \) and a weight \( w_j > 0 \), we need to schedule the \( n \) jobs on a machine in some order. Let \( C_j \) be the completion time of \( j \) on in the schedule. Then the goal of the problem is to find a schedule to minimize the weighted sum of the completion times, i.e., \( \sum_{j=1}^{n} w_j C_j \).

\[ J = \{ \begin{array}{c}
\text{ } \\
\frac{w_1 = 5}{t_1 = 10} \\
\frac{w_2 = 8}{t_2 = 5} \\
\end{array} \} \]

Schedule:

\( S_1 \):

1 \[ \overline{\text{10}} \] 2  \quad  w_1 t_1 + w_2 (t_1 + t_2) = 170

\( S_2 \):

2 \[ \overline{\text{5}} \] 1  \quad  w_2 t_2 + w_1 (t_1 + t_2) = 115
First try:

Schedule the job with maximum weight first.

$\text{Jobs} = \{ w=100, t=100 \} \quad w=99 \quad 99 \quad 99$

$100 \times 100 + 99 \left(1 + 2 + 10 + \cdots + 10\right) + 99 (1 + 2 + 3 + \cdots + 10) + 100 \times 110$

Second try:

Pick the job \( j \) with largest \( w_j/t_j \) to execute first.
Proof of the safety property:

- $\pi^*$ is an optimal schedule: $\pi^*: \{\text{jobs}\} \rightarrow \{1, 2, 3, \ldots, n\}$

  $\pi^*(j) = 2 \Rightarrow \text{job } j \text{ is the 2nd job to be executed.}$

- Let $j_0 := \arg\max_{\text{job } i} \frac{W_i}{t_i}$, suppose $\pi^*(j_0) \neq 1$. $\pi^*(j_0) = k$

\[\pi^*: \begin{array}{cccc} j & \cdots & j_0 & \cdots \end{array} \quad \text{WCT}(\pi^*) = \sum_{i=1}^{k-1} W_{j_i} C_{j_i} + W_{j_0} (T + t_{j_0})\]

\[\pi: \begin{array}{cccc} j & \cdots & j_0 & \cdots \end{array} \quad \text{WCT}(\pi) = W_{j_0} t_{j_0} + \sum_{i=1}^{k-1} W_{j_i} (C_{j_i} + t_{j_i})\]

$\Rightarrow \text{WCT}(\pi^*) - \text{WCT}(\pi) = W_{j_0} T - \sum_{i=1}^{k-1} W_{j_i} t_{j_i}$

$= \frac{W_{j_0}}{t_{j_0}} \cdot t_{j_0} \cdot T - \sum_{i=1}^{k-1} \frac{W_{j_i}}{t_{j_i}} \cdot t_{j_i} \cdot t_{j_0}$
\[ = t_j \cdot \left( \frac{\sum_{i=1}^{k} w_{j_i}}{t_{j_0}} \cdot t_{j_i} \right) \]

\[ = t_j \cdot \sum_{i=1}^{k} \left( \frac{w_{j_i}}{t_{j_0}} - \frac{w_{j_0}}{t_{j_i}} \right) \cdot t_{j_i} \geq 0 \]

\[ \Rightarrow WCT(\pi^*) = WCT(\pi) \Rightarrow \pi \text{ is also optimal.} \]
You wish to drive from point $A$ to point $B$ along a highway minimizing the time that you are stopped for gas. You are told beforehand the capacity number $L$ of miles you can drive when the tank is full, the locations $x_1, \ldots, x_n$ of the gas stations along the highway, where $x_i$ indicates the distance from the $i$-th gas station from $A$. Design a greedy algorithm to compute the minimum number of times you need to fill the gas tank.

**Greedy Strategy**: drive as long as possible and fill gas at the last gas station encountered.
Proof of safety property:

Let $Y_1$ to be the first stop chosen by the greedy strategy.

1. Let $\pi^*$ be an optimal strategy, and $\pi^*$ chooses $Y_1^*$ to be the first stop, and $Y_2^*$ to be the 2nd stop.

2. Let $\pi$ to stop at $Y_1$, then follows $\pi^*$ in the rest of the time.

Only need to show that the car can reach $Y_2^*$ after filling gas at $Y_1$.

Since $Y_1^* \leq Y_1$, 

$$ \Rightarrow Y_2^* - Y_1 \leq L $$

Given a tree $T = (V, E)$, find the maximum independent set of the tree. For example, maximum independent set of the tree of following tree has size 7.

```
Example:  

→ Observation: It's better to pick children nodes.
```
Greedy Strategy:
Add all leaves to the independent set, then remove all the neighboring nodes (i.e., their parents), and continue.

Proof of Safety:

$S^*$: optimal IS, doesn't contain some leaf $v$

1. $\text{parent}(v) \notin S^*$
   $\Rightarrow$ add $v$ to $S^*$
   $\Rightarrow$ cannot happen

2. $\text{parent}(v) \in S^*$
   $\Rightarrow$ switch $v$ with $\text{parent}(v)$ $\Rightarrow$ get another IS of same size. \qed
D Shor test Path. and MST

- Simulate Dijkstra Alg.: starting from a
Simulate Prim's Alg

<table>
<thead>
<tr>
<th></th>
<th>b</th>
<th>c</th>
<th>d</th>
<th>e</th>
<th>f</th>
<th>g</th>
</tr>
</thead>
<tbody>
<tr>
<td>a</td>
<td>13</td>
<td>a</td>
<td>8</td>
<td>a</td>
<td>40</td>
<td>a</td>
</tr>
<tr>
<td>c</td>
<td>13</td>
<td>a</td>
<td></td>
<td>c</td>
<td>17</td>
<td>a</td>
</tr>
<tr>
<td>d</td>
<td>13</td>
<td>a</td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>g</td>
<td>13</td>
<td>a</td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>b</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>e</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

Graph:
- Nodes: a, b, c, d, e, f, g
- Edges and weights: 8 (a to b), 40 (a to c), 25 (c to d), 18 (d to e), 5 (e to f), 18 (d to g), 16 (g to f)
Dijkstra’s Alg’s Failure on Graphs with negative weights.

Q: Shortest path from s to d?

Ans: -\infty
DST paper:
Move to appendix:

\[ \text{Lmm 16} \quad \text{Claim 15} \quad \text{Section 5.4 Lmm 23} \]

Fix reference to: Thm 6, Def 15, P1, P2, ..., P6
OT-L paper: reduce 8 pg of main body.

10 pg of full paper

To appendix

Lmm 17: 0.5 pg. Lmm 18: 0.5 pg

Lmm 21 + Lmm 22 + Lmm 23 = 1 pg.

Sec 2.1: 0.5 pg. Sec 2.4: 0.5 pg

Lmm 11: 0.6 pg. Thm 7, 8: 0.5