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# Outline of Synthetic Differential Geometry

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[Initial results in Categorical Dynamics were proved in 1967 and presented in a series of three lectures at Chicago. Since that time a flourishing branch of it called Synthetic Differential Geometry has given rise to four excellent textbooks by Kock, Lavendhomme, Moerdijk & Reyes, and Bell. To help make this subject more widely known and to further encourage its application, I gave some talks in February 1998 in the Buffalo Geometry Seminar. The following outline with 7 appendices was distributed as seminar notes. I have made a few corrections now (Nov. 1998), such as switching some A's and B's. But I have added Appendix 8, concerning the interesting work circulated in October 1998 by Kock & Reyes, in which they verify my main claim and also make several further observations.]

A cartesian closed category is one having exponential or internal hom right adjoint to cartesian product:

There is a natural bijection

$$\frac{X \to Y^A}{A \times X \to Y}$$

between the indicated sets of maps, for all X. It follows that this bijection is mediated by natural transformations

$$X \xrightarrow{\ \lambda_X\ } (A \times X)^A$$

$$A \times Y^A \xrightarrow{\ \varepsilon_Y\ } Y$$

for each A. It also follows that the exponential is a contravariant functor of the given A; i.e. given a map $A \xrightarrow{\alpha} B$, there is an induced natural map in the opposite direction

$$Y^B \xrightarrow{\ Y^{\alpha}\ } Y^A$$

in the same category. Calling 'points of X' the figures of shape 1 in X, i.e. the maps $1 \to X$ where 1 is the terminal object, it follows in particular that the points of an exponential $Y^A$ indeed do parameterize the maps $A \to Y$, and that $\varepsilon_Y$ acts as 'evaluation', etc. (Recall the computer philosophy, according to which a stored program is merely a special sort of data.)

The induced maps may be seen as a special case of the fact that for any three objects there is in the category a 'composition' map

$$Y^X \times Z^Y \longrightarrow Z^X$$

However, cohesiveness and variability of the objects in general comes from the figures of more general shape than punctural. The category $\mathcal{X}/S$, where S is a given object of $\mathcal{X}$, has as objects the objects of X further structured by a given map to S, and as morphisms has commutative triangles over S in $\mathcal{X}$. This is the usual geometric way of dealing with families of spaces parameterized by S, namely the spaces in the family are the fibers of the structural map; the forgetful functor $\mathcal{X}/S \to \mathcal{X}$ takes the *total* space of the family. Note that the identity map $1_S$ is the terminal object of $\mathcal{X}/S$ so that a 'point' in the latter category is 'the locus of a moving point' in the view of the original $\mathcal{X}$. There is the functor

$$\mathcal{X} \xrightarrow{(\ )_S} \mathcal{X}/S,$$

assigning to each X the 'constant family' $X \times S \xrightarrow{proj} S$ (all of whose fibers are X); this functor is not full, since even between two constant families of spaces there are usually many non-constant S-parameterized families of maps. Indeed, it is easily seen that there is a bijection

$$\mathcal{X}(S, Y^A) = (\mathcal{X}/S)(A_S, Y_S)$$

between hom sets, i.e. that the S-figures in $Y^A$ are just the maps 'from A to Y', but in $\mathcal{X}/S$.

A locally cartesian closed category $\mathcal{X}$ is one in which each of the categories $\mathcal{X}/A$ is cartesian closed; since products in $\mathcal{X}/A$ are just the fiber products over A in $\mathcal{X}$, that is easily seen to be equivalent to the requirement that for each map $A \xrightarrow{\alpha} B$, the pullback functor $\alpha : \mathcal{X}/B \to \mathcal{X}/A$ has a right adjoint, often denoted by $\prod_{A/B}$ when $\alpha$ is understood.

Spaces of intensive (contravariant) or extensive (covariant) variable quantities on X are seen to partake of the same kind of cohesion/variation as the domain spaces X by using the cartesian closed structure. Thus for a given rig R in $\mathcal{X}$, $R^X = \mathcal{F}(X)$ is another rig *in* $\mathcal{X}$ (i.e. continuous, smooth, bornological, etc.). Using the action of the multiplicative monoid R on $R^X$. one can carve out from $R^{R^X}$ (using an equalizer) the subobject $\mathrm{Hom}_R(R^X,R) = \mathcal{M}(X)$ which plays the role of 'distributions of compact support' on X. (In the smooth $\mathcal{X}$, the homogeneous maps are usually automatically linear.)

The above method of obtaining the infinite-dimensional spaces of analysis from the finite-dimensional ones of geometry, is in principle well known. Implicit for 300 years and explicit for at least 30 is also (see my 1968 paper on 'diagonal arguments' or Michor and Kriegl's recent AMS book, but also see Fox, Brown, Kan, Steenrod much earlier for the sequentially continuous and simplicial cases) the fact that even these infinite-dimensional spaces W are determined by their categories $\mathcal{X}_{\mathrm{fin}}/W$ of finite-dimensional figures. For example, if S is a 1-dimensional segment, then a figure of shape S in W could also be called a *path*, and in fact many spaces are essentially path-determined. The maps in the category of small figures in W are commutative triangles in $\mathcal{X}$

$$\begin{array}{ccc} S' & \longrightarrow & S \\ & \searrow \quad \swarrow & \\ & W & \end{array}$$

and determine all significant 'incidence relations' among figures. The adequacy of small figures in W means that a map W --------> R is determined by the functor it induces from the category of small figures in W to the category of small figures in R, for any space R. The mathematically reasonable condition on any proposed determination of 'small spaces vs. general spaces' requires this adequacy for all W.

But exponentiation also provides, in another manner, the way to determine the small (finite dimensional) spaces from the infinitesimal ones, the essential goal of differential geometry and continuum physics ! This is done by postulating that among the finite dimensional spaces there are some 'amazingly tiny' ones which in a suitable sense determine everything. (Although papers in algebraic geometry refer to these as 'zero-dimensional', I prefer to reserve that term for the much more restricted class of nearly discrete objects, and consider that the a.t.o.m.s (=amazingly tiny object models) have an ‘infinitesimal dimension’ in a sense which can be made precise.) All the cited books do this via a postulated rig object $R$ (for coordinatizing a line) which is assumed to have enough nilpotent elements: Defining $D_n \rightarrow R \underset{0}{\overset{(\ )^{n+1}}{\rightrightarrows}} R$ to be subobject of $R$ such that a figure $x$ in $R$ belongs to $D_n$ iff $x^{n+1} = 0$ (and $D = D_1$ for short), the key axiom is that the map $R^{n+1} \longrightarrow R^{D_n}$ (which parameterizes functions representable by n-th degree polynomials) is an isomorphism. Identifying $D_1$-figures as tangent vectors and hence $X^{D_1} \longrightarrow X$ as the tangent bundle of any space $X$, we thus have in particular that the tangent bundle of the line is 2-dimensional, relative to R. Since

$$(Y^A)^{D_1} = (Y^{D_1})^A$$

(we bypass Banach manifolds, etc. etc.) to arrive at the tangent bundle of any function space, e.g. those where $Y = R$, $A = 5$ or $A = R^5$.

Since the tangent bundle is thus a *representable* functor, a vector field $X \rightarrow X^{D_1}$ on $X$ can equivalently be viewed as an infinitesimal action $D_1 \times X \rightarrow X$ (by the basic transformation of cartesian closure); such always automatically induces a derivation (in the Leibniz sense) of the function algebra $R^X$. Note that there is automatically a *space* Vect(X) of all vector fields on X, with its own intrinsic cohesion and variation, but that there is also the (cartesian closed) category Vect(*X*) in which an object is a space equipped with a given vector field and in which a morphism is an infinitesimally equivariant map. Restriction along the inclusion $D_1 \rightarrow R$ induces a functor

$$\text{Flows}\,(X) \longrightarrow \text{Vect}\,(X)$$

whose adjoints help to organize thinking about actually solving the ODE implicit in a given vector field. (Here a flow means an action of the additive monoid R ).

Actually, $D_1$ is a *coordinatized* bit of time, $D_1 \approx T_1$, where the only structure $T_1$ has is a unique point $1 \overset{o}{\rightarrow} T_1$. The axiom above implies that ${T_1}^{T_1}$ contains $R$ as a canonical retract and that the multiplication in $R$ comes from the canonical monoid structure on ${T_1}^{T_1}$ that any space of endomorphisms has. Each $D_n$ thus consists of those retardations whose n+1 iterate is the point in the instant. Thus one can actually reconstruct R and its algebraic structure from the purely geometrical data of a cartesian closed category with a given pointed object $T_1$.

For any pointed object T, $T^T$ is canonically bipointed for there are the names of both the identity map and the constant map. These points are both in the object R, now defined to be the part of $T^T$ which consists of 0-preserving endomaps. An important axiom is that the bipointed object R be a connected object. Now we see that the finite dimensional spaces $R^n$ are all retracts of function spaces of 'infinitesimal' objects.

But in what sense are these a.t.o.m.s T really tiny? There are several answers, some based on the intrinsic Heyting logic of subobjects, etc. A strong condition (and amazingly, all this is actually concretely realizable) is that the tangent bundle functor have a further right adjoint:

$$\frac{X \longrightarrow Y^{1/T}}{X^T \longrightarrow Y}$$

This permits representation of differential forms as merely 'functions' $X \longrightarrow R^{1/D}$ valued in a bigger fixed rig. It also permits to show that the category of *2nd order* ODE's is also locally cartesian closed (even a topos), as is briefly seen as follows:

Given a map $A \xrightarrow{\alpha} B$, one can consider the type of structure which for any X would consist of a given map

$$X^A \longrightarrow X^B$$

which is a section of $\alpha_X$. This may be thought of as a "B-tuple of A-ary operations' on X subject to one defining identical equation. In case A = 1, such an $X \longrightarrow X^B$ is equivalent to a pointed action $B \times X \longrightarrow X$ and it is well-known that such actions constitute a new topos over which the original one is 'essential'. In case $B = D_1$, this topos consists of all first order ODE's. Even with A = 1 the property that B is an a.t.o.m. was important in Kock's proof of a theorem of Sophus Lie concerning the flow lines of an ODE.

If A is an a.t.o.m., such a section of $\alpha_X$ is equivalent to a map $X \longrightarrow X^{B/A}$ satisfying the one equation, which implies that the forgetful functor $\mathcal{X}^\alpha \xrightarrow{f^*} \mathcal{X}$ (from the category of prolongation structures being considered, back to the topos of spaces) thus has not only the 'free' left adjoint $f_!$ expected algebraically, but also 'cofree' right adjoint $f_*$ which in turn implies in particular that $\mathcal{X}^{\alpha}$ is a topos.

The higher-order 'monoids' implicit in the above are not familar because in the category of abstract sets there are no a.t.o.m.s, except $A = 1$. The case of particular interest for physical dynamics is that where $\alpha$ is the inclusion$T_1 \longrightarrow T_2$ into a second-order instant (coordinatized by the inclusion $D_1 \longrightarrow D_2$ into the space of $r$ in $R$ for which $r^3 = 0$). A structure in $\mathcal{X}^{\alpha}$ is exactly a smooth prolongation of tangent vectors in a space $X$ to 2nd order jets in X, i.e. a second order ODE or dynamical law on $X$. With the obvious definition of morphism, these therefore form a topos, by the above argument. Of course if $X$ itself is a function space $E^B$, then among the ingredients for a prolongation operator are differential operators considered as maps; in other words, these ODE's include PDE's as a special case.

Actual motions which follow a given $\alpha$-dynamical law in $X$ are to be considered as morphisms $T \longrightarrow X$ in $\mathcal{X}^{\alpha}$ where $T$ is equipped also with an $\alpha$-prolongation law modeling time; for example, if $T$ is coordinatized by $R$ (usually not itself an a.t.o.m. if there is non-trivial Grothendieck topology in the picture) there is the obvious prolongation structure given by $R^D \xrightarrow{\tau} R^{D_2}$ for which

$$\tau(x,v)(t) = x + vt \qquad \text{for all } t^3 = 0$$

i.e. 'zero acceleration'.

APPENDIX 1

## Figures as structure: Graphs

A simple example of the role of figures in deriving the internal structure of objects is provided by the ubiquity of (reflexive directed multi) graphs. If $1 \rightrightarrows I$ is any choice of two distinct points in any chosen object of *any* category $\mathcal{X}$, there is an induced functor, from $\mathcal{X}$, to the category of such graphs: each object $X$ of $\mathcal{X}$ has not only the set $\mathcal{X}(1,X)$ of points, but the set $\mathcal{X}(I, X)$ of 'directed edges'; composition with the chosen points gives the needed 'source and target' structure. Any map $X \longrightarrow Y$ in the category $\mathcal{X}$ induces a map of these graphs which preserves this source and target structure, merely because of the associativity of composition. In case $\mathcal{X}$ is a topos and I is adequate, we can conclude that $\mathcal{X}$ *is* the category of graphs if moreover I has no endomorphisms other than the obvious three.

APPENDIX 2

## Nilpotent Calculus

As engineers have implicitly known for centuries, the use of nilpotent quantities is very effective in reducing differential calculus to high school algebra. For example, we can prove that the length C(r) of the boundary of a disk is $2\pi r$ if we define $\pi$ to be the area of a unit disk, as follows: By homogeneity the area of a disk of radius r is $A(r) = \pi r^2$ and the area of a perturbed disk of radius r + h is $A(r+h) = \pi(r + h)^2$. Thus the area of a thin strip around the boundary circle is

$$A(r + h) - A(r) = \pi(2rh + h^2) = 2\pi rh$$

where in the last step we interpret 'thin' to mean $h^2 = 0$. But the same area is also equal to C(r) h, for all such h, and since there are enough of these nilpotents to permit cancelling them from universally-quantified equations, the result follows.

Similarly, one can prove for example that the electrical attraction of an infinitesimal dipole is inversely proportional to the *cube* of the distance from it.

APPENDIX 3

## Higher (and lower) connectivities

Basic topological intuition can be applied synthetically in terms of the contrast between a category $\mathcal{X}$ of spaces with some cohesion and variability and a category of sets $\mathcal{S}$ with none (or qualitatively less). This applies not only to smooth spaces, but also to continuous or combinatorial ones. Namely, the points functor $\mathcal{X}(1, -)$ usually has a right adjoint (the inclusion of codiscrete spaces) and two successive left adjoints (the inclusion of discrete spaces and the functor 'set of components' which represents the attempts to map a given space to discrete spaces). An object is 'connected' iff its set of components is 1, i.e. iff its only maps to discrete spaces are constant. The study of the set of components of the space $X^S$ of S-figures implies the homotopy and homology of X. (S ranging over reference spaces such as spheres and balls).

Indeed, the needed set-theory $S$ is best *derived from* the geometry $X$ by defining discrete spaces to be those C for which $C \to C^S$ is an isomorphism for a few selected figure forms S which are considered to be connected and which at least have no nontrivial coproduct decompositions.

APPENDIX 4

**Lie and the Discovery of a.t.o.m.s**

Indeed, my 'fractional-exponent' definition of the a.t.o.m. property above was based on a key discovery of my student Anders Kock which I was able to correlate with work of my student Marta Bunge who had characterized presheaf toposes. Kock's discovery was about what was needed to prove a theorem of Sophus Lie concerning the space of flow lines of an ODE

$$D_1 \times X \rightrightarrows X \longrightarrow X/D_1$$

In computing such coequalizers, the fact that $D_1$ is internally projective is crucial; coupling that with the fact that $D_1$ is internally connected (i.e. that $(\ )^{D_1}$ preserves sums), one sees that $(\ )^{D_1}$ preserves colimits; then the special adjoint functor theorem (of one of my teachers, Peter Freyd) implies the existence of the fractional exponents.

APPENDIX 5

**Time Speed-ups and Coordinates**

The sense in which the non-commutative monoid $T^T$ of time speed-up (for the case $T = T_1$ of a first-order microspace with one point) is 'just slightly bigger' than its commutative submonoid R of 0-preserving elements, can be seen quite clearly in terms of a choice $T \approx D_1$ of unit of time, where $D_1$ is the subspace of the coordinatized line R whose figures are those figures t of R for which $t^2 = 0$ with respect to the multiplication of R. For then the elements of $T^T$ , in terms of the endomappings which they parameterize, are easily seen to be pairs <$a_0$, $a_1$> in R for which

$$(a_0)^2 = 0, \text{ and } a_0 a_1 = 0$$

Then, as usual, the elements $a_i$ can be figures of any shape S, but we are using the multiplication of R ; the (S-family of) endomorphisms of T which the pairs parameterize are described by the formula

$$t \;¾\!\rightarrow\; a_0 + a_1 t$$

and the composition rule is the restriction to D of the substitution of one affine-linear transformation into another, which could be called 'the high-school monoid'. The inclusion R ¾→ $T^T$ is the part where $a_0 = 0$, and $a_1$ arbitrary, but D itself is a subspace (not a submonoid) in a perpendicular direction where $a_1 = 0$. It is then easy to see that all the invertible elements of the monoid $T^T$ (for $T = T_1$) belong to the commutative group of invertible elements of R. Disjoint from that, one can calculate, for example, the parts of $T^T$ whose elements f satisfy $f \circ f = 0$, or $f \circ f \circ f = 0$.

Any tangent bundle $X^T$ obviously has a natural right action of $T^T$ , not only of the latter's submonoid R.

APPENDIX 6

**Strong Adjoints versus General Exponents**

Two endofunctors F and U of a cartesian closed category are strongly adjoint iff there is a natural isomorphism

$$Y^{FX} = (UY)^X$$

of function spaces, for all spaces X,Y in the category. Taking points of both sides of the above isomorphism we obtain a natural bijection

$$\frac{FX \to Y}{X \to UY}$$

of map-sets, the usual notion of adjointness. But the converse does not hold; there may not exist any way of making some given adjoint pair strong. For example, $A \times (\ )$ is strongly adjoint to $(\ )^A$ for any fixed space A, which is one of the laws of exponential algebra which is true in any cartesian closed category. But if T is an a.t.o.m., the adjointness of $(\ )^T$ to $(\ )^{1/T}$ is usually not strong, unless $T = 1$. We can measure 'how strong is it?' by considering the subcategory $\mathcal{S}_T$ (certainly including 1) of those test-figures C which perceive it so; that turns out to mean just that the canonical 'inclusion of constant figures (or zero tangent vectors)'

$$C \xrightarrow{\sim} C^T$$

is an isomorphism. This is a special case of the idea of defining a subcategory of 'discrete' spaces in terms of a whole category of spaces (as mentioned in Appendix 3), which might be called the 'microdiscrete' case; it turns out that the inclusion $\mathcal{S}_T \longrightarrow \mathcal{X}$ has a right adjoint, which may be considered as one version of 'the space of points' of an arbitrary space, for indeed the composite of these two adjoints is a kind of '0-skeleton' endofunctor of $\mathcal{X}$, with $sk_0(X) \longrightarrow X$. Given such an adjoint pair, we can define

$$\mathcal{X}(A,Y) = sk_0(Y^A)$$

and define a intermediate notion of $\mathcal{S}$-strong adjointness.

Actually, a strongly adjoint pair of endofunctors is determined by a single object F1, since for all Y

$$UY = Y^{(F1)}$$

as is seen by substituting $X = 1$ in the definition. But for a given $\mathcal{S} \rightleftarrows \mathcal{X}$, the $\mathcal{S}$-strong adjoint endofunctors in general form a larger category of 'exponents'. We consider these exponents as a category by considering as morphisms the natural transformations betwen the left-adjoints, but the exponents act as 'exponents' (also notationally) via the right adjoints; this acting is thus a right action.

$$E' \longrightarrow E \text{ implies } Y^E \longrightarrow Y^{E'}$$

$$(Y^{E1})^{E2} = Y^{E1 \cdot E2}$$

for exponents E where the multiplication of exponents means their composition considered as functors. Thus for a cartesian closed category $\mathcal{X}$,

$X$ itself is canonically embedded in its category of exponents, and indeed

$$A_1 \cdot A_2 = A_1 \times A_2$$

for those special exponents which come from $X$. For any given a.t.o.m. T, the fractional exponent $B/T$ is well-defined, for any object B; but for $T \neq 1$ the multiplication of these is no longer commutative. Still more general examples can be obtained by taking *colimits* of known exponents (which of course actually involves taking the opposite *limits* of the corresponding right-adjoints in the pairs). For example, the sum of two exponents can be defined via their right actions as

$$Y^{E1+E2} = Y^{E1} \times Y^{E2}$$

for all Y, which obviously will agree with the coproduct of spaces $A_1 + A_2$ for those 'integral' exponents coming from $X$.

Call an exponent E=<E,F>*strict* if F itself has a further left adjoint $E_!$; then F1=1 so that no nontrivial integral exponent is strict.

APPENDIX 7

## Galilean Monoids

With respect to a given bifunctor 'multiplication' in a category, a *monad* is defined as in Eilenberg-Moore to be an object E together with a given internal unit and a given 'internal multiplication' map

$$1 \longrightarrow E, \qquad E \cdot E \longrightarrow E$$

(satisfying the associative and 2-sided unit laws), where 1 is the object acting as unit for the functorial multiplication. In case the multiplication functor is merely the cartesian product in a category $X$ which has such, monads are usually called *monoids*. The category of actions in $X$ of a given monoid in $X$ is typically another cartesian closed category. In case the category in which we consider the monads is the category of exponents of a given category $X$, it seems reasonable to call such monads 'Galilean monoids'. They are a real generalization of the monoids in $X$, yet far more special than the usual monads in the category of all endofunctors. In particular, the category $\mathcal{X}^{E}$ of actions on 'configuration spaces' for a Galilean monoid E has *both* left and right adjoints (free and cofree) to the same forgetful functor $\mathcal{X} \longleftarrow \mathcal{X}^{E}$, as was proved by Eilenberg and Moore in 1965. A *strictly -generated* Galilean monoid in $\mathcal{X}$ is one generated by a pointed strict exponent $1 \longrightarrow E$. If the generating process in that case involves a filtered colimit in a topos $\mathcal{X}$ then the category $\mathcal{X}^{E}$ of actions of E is not only a topos (in particular cartesian-closed), but the functor "underlying configuration space" will be a "local geometric morphism" from $\mathcal{X}^{E}$ to $\mathcal{X}$, because the left adjoint aspect will be lex. A distinct system of adjoints is "equilibrium configurations", $\mathcal{X}^{E}$ to $\mathcal{X}$, with trivial action and cotrivial action as left and right adjoints.

While 1st order ODE's in $\mathcal{X}$ and their equivariant maps constitute a topos because they are just the actions of an 'ordinary' monoid infinitesimally generated by $T_1 \approx D_1$ together with the constraint that the point of the instant act as the identity, the 2nd order ODE's in $\mathcal{X}$ form a topos because they are the actions of a strictly–generated Galilean 'monoid' infinitesimally generated by the inclusion $T_1 \longrightarrow T_2$ (coordinatizable by $D_1$ $D_2$) together with the section constraint.

APPENDIX 8

**(Additional comments added Nov. 4, 1998)**

Recently Kock and Reyes verified my main claim in this outline, namely that second-order ODE's (and many similar prolongation structures) in a given topos constitute another topos receiving an essential morphism from the first, provided certain fractional exponents exist. Rather than using filtered colimits of fractional exponents as in the outline, their proof uses general properties of Top/$\mathcal{S}$ such as the existence of fibered products. They also emphasize that the particular "α-prolongation law modeling time" given as a simple example at the end of the outline, namely a one-dimensional interval with zero acceleration, does not represent all lawful motions. Indeed, second order time, the object T representing as an abstract general the concrete generality of all lawful motions in all objects of the topos $\mathcal{X}^{\alpha}$, is a richer "Algebra of Time" than just one-dimensional, as I pointed out in my lectures of that title at the Hamilton Sesquicentennial (Dublin 1993) and at La Sapienza (Rome 1995). Even richer is Galilean time, the object serving the analogous role in the category of dynamical systems, which are structures involving a pair of second-order ODE's on the same configuration space, one of the pair being homogeneous to serve as the 'inertial' zero of specific force. Precise descriptions of those representing objects are needed.

A further observation is suggested by my June 1998 talk 'Why functionals need analyzing' to the Canadian Mathematical Society, where I pointed out that solution-operators for boundary-value problems are also $\alpha$-prolongation structures. Of course, the domain of $\alpha$ (i.e. the 'boundary' of the codomain) is in such cases not usually an a.t.o.m.; however, any topos is a subtopos of another one in which any given object becomes an a.t.o.m., so that the instrument whose development has been taken up by Kock & Reyes may ultimately also shed light on those problems.