Practice Problems for Exam 1

You can use class notes and/or textbook during the exam, but no electronic devices please. Print out the pages if you have an electronic version of the textbook.

1. Solve PDE with auxiliary condition:

$$u_x + yu_y = -u$$
$$u(0, y) = \cos y$$

2. A rod occupying interval $-L \le x \le L$ and initially held a zero temperature $u(x, 0) = 0$is subject to a periodic heat source $f(x) = \cos\left(\frac{\pi x}{2L}\right) + H$.

(a) With homogeneous Neumann boundary conditions determine for what value of H does the equilibrium state of the rod exist?

Hint: Look at thermal energy of the rod.

(b) Find the equilibrium state from part (a).

Hint: One of the constants can be determined from thermal energy conservation argument.

3. Solve initial value problem:

$$u_{tt} - 3u_{tx} + 2u_{xx} = 0$$
$$u(x, 0) = \cos x$$
$$u_t(x, 0) = -\sin x$$

4. Use D'Alembert's formula to solve and sketch solution at time $t = \frac{1}{2c}$, $t = \frac{1}{c}$ and $t = \frac{3}{2c}$ as a function of x.

$$u_{tt} - c^2 u_{xx} = 0$$
$$u_t(x, 0) = 0$$
$$u(x, 0) = \begin{cases} 1 - |x|, & |\mathrm{x}| < 1 \\ 0, & |\mathrm{x}| \ge 1 \end{cases}$$

5. Given wave equation in a 2D domain D:

$$u_{tt} - c^2 \nabla^2 u = 0$$

(a) Write the energy of the string:

$$E = \frac{1}{2} \int_D \left[u_t^2 + c^2 (\nabla u)^2 \right] dS$$

and show that it is a conserved quantity with homogeneous Dirichlet conditions.

(b) Consider Robin boundary conditions $\dfrac{\partial u}{\partial n} = -\dfrac{1}{c^2} u$ and modify the potential energy with extra term to obtain a conserved quantity.

6. Given a first order PDE with auxiliary condition:

$$y u_x - x u_y = 0$$

$$u(0, y) = f(y)$$

(a) Solve the problem by method of characteristics.

(b) Draw characteristics in the xy-plane. What conditions $f(y)$ must satisfy for this to be a wellposed problem.

7. A flat disc is filled with a bacteria that multiply at a fixed rate α. Bacteria cannot escape from the boundary of the disc. Assume that the flux of bacteria is proportional to the gradient of concentration. Determine the mathematical model describing bacteria concentration in the disc.

8. Solve the diffusion equation $u_t = k u_{xx}$ with the initial condition $u(x, 0) = x^2$ by the following method:

(a) Show that $u_{xxx}(x, t)$ satisfies the heat equation with zero initial conditions. What is $u_{xxx}(x, t)$?

(b) Integrate u_{xxx} three times in x. Are the constants time-dependent?

(c) Determine the integration constants by substitution of your ansatz into the heat equation and initial condition.

9. Solve 1D heat equation with convection on the line:

$$u_t - ku_{xx} + Vu_x = 0$$
$$u(x, 0 = \varphi(x)$$

Hint: Go to a moving frame of reference by introducing new variable $y = x - Vt$.