Partial Differential Equations and Waves

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Prolegomenon

These are the lecture notes for Amath 353: Partial Differential Equations and Waves. This is the first year these notes are typed up, thus it is guaranteed that these notes are full of mistakes of all kinds, both innocent and unforgivable. Please point out these mistakes to me so they may be corrected for the benefit of your successors. If you think that a different phrasing of something would result in better understanding, please let me know.

These lecture notes are not meant to supplant the textbook used with this course. The main textbook is Roger Knobel's "An introduction to the mathematical theory of waves", American Mathematical Society 1999, Student Mathematical Library Vol 3.

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Chapter 1

Introduction

1.1 An introduction to waves

In this course we will learn different techniques for solving partial differential equations (PDEs), specifically with an emphasis on wave phenomena. We begin by defining, in a very loose sense, what we mean by a wave.

Let's start with some examples. We have students majoring in very many different fields enrolled in this class. Waves are relevant for all of you. I will not always pick the most obvious examples below, as you'll have to come up with some examples on the first homework set.

- **ACMS and Mathematics:** The study of wave phenomena has produced some of the biggest mathematical breakthroughs of the last decades. Waves have helped our understanding of geometry, algebraic geometry, analysis, and just about any other area of mathematics. And, obviously, as an ACMS major, all applications listed here are relevant to you.

- **Aeronautical and Aerospace Engineering:** Radar is an important example of the use of sound waves: waves are sent out and their reflection is observed. This is the same process used by bats to determine where they are in the middle of the night or in a dark cave. Sonic booms are another example we will talk about.

- **Atmospheric Science:** Patterns in clouds are an obvious example of waves. Specific examples are the Morning Glory phenomenon off the coast of Australia (`https://en.wikipedia.org/wiki/Morning_Glory_cloud`) and the Kevin-Helmholtz instability `http://en.es-static.us/upl/2014/05/kelvin-helmhotz-clouds1.jpg`.

- **Biology:** Dispersal of seeds by wind waves is one of the most important means of plant regeneration.

- **Computer Science and Computer Engineering:** Electromagnetic waves are responsible for how we see, how we communicate (using our smart phones, for instance, or even through old-fashioned wired communications). These signals are put in on one end, they are transmitted, and finally they are received on the other end. If we're dealing with digital communication, the waves we're talking about are sequences of zeros and ones.

- **Bioresource Engineering:** the movement of plant habitats as a function of a changing climate is described by a wave that moves a LOT slower than most of the ones we describe here: a noticable change can take decades to be observable.

- **(Bio) Chemistry and BioEngineering:** The Belousov-Zhabotinsky reactions displays both temporal and spatial oscillations that are easily observed with the naked eyes, see `https://www.youtube.com/watch?v=IBa4kgXI4Cg`. This is not your typical chemistry 101 reaction, where the reactants react, and the final product appears.

- **Mechanical Engineering:** The understanding of tidal and other water waves is important for harbor and ship design, see `https://www.dropbox.com/s/mlittwqaxk6j3we/hexagons-bw.jpg?dl=0`.

- **Oceanography:** Do I need to say tsunami? Or rogue wave? In fact, a lot of the terminology we will introduce originates from the study of water waves. That is most likely because water waves are so obvious to observe: at the very least we can see the patterns we're talking about.

- **Physics:** Waves in Quantum Mechanics: the electron microscope allows us to "see" matter at an atomic scale by sending in an electron wave (a beam of electrons) which interacts with the surface of the material we are "looking at". By analyzing the reflected and the transmitted wave, we can determine the nature of the surface.

- **Political Science and Economics, International Studies:** The propagation of political or economical ideas can be described by a wave. As we are leading up to an election, different candidates generate different waves of enthusiasm based on a variety of external forces (media, advertizing, *etc*). These waves are also affected by the medium in which they propagate: Iowa first, New Hampshire second, and so on.

- **Psychology:** A delta wave is a brain wave that occurs during what is known as deep sleep. It is known that delta wave activity is vastly reduced in people with schizophrenia.

What do all the phenomena mentioned above have in common? It turns out that it is not so easy to give a precise definition of a wave that captures everything we want. We can agree on the following.

1. A wave is the result of a disturbance propagating through a medium, with finite velocity, and

2. Associated with waves are signals: the result of any kind of measurement of a wave. The outcome could be called **Amplitude**, **Frequency**, *etc.*

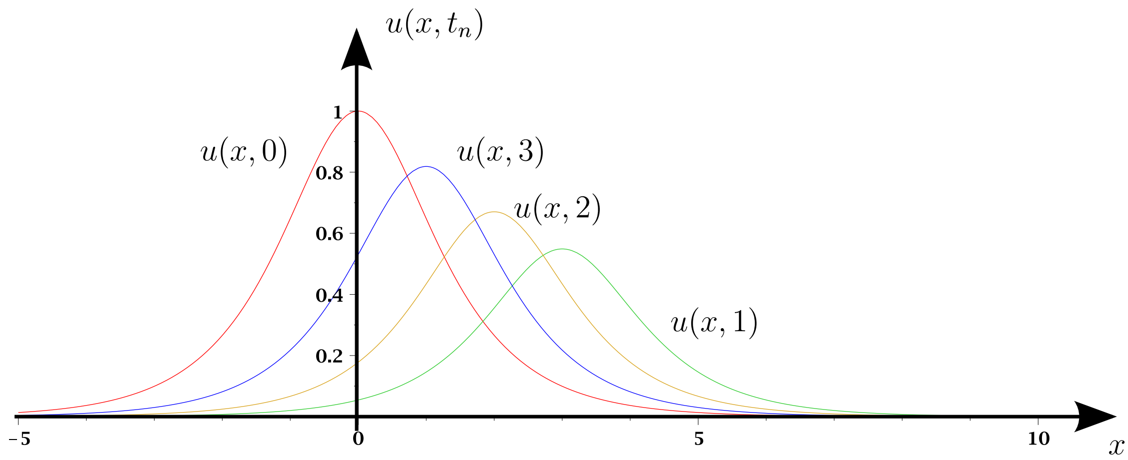

Figure 1.1: A wave of decreasing height (*i.e.*, amplitude) traveling to the right.

Example. Ripples in a pond. Waves travel horizontally across the surface of the pond. A signal might be the vertical displacement of the crests.

Example. Waves in traffic disturbances can be caused by an accident, a police car, a traffic light, a car merging, *etc.* We will discuss this example near the end of the course. And I apologize: this knowledge will not let you race through Seattle traffic. But it will tell you why you are stopped. That should make you happier, in a zen-like way, no?

Example. The wave in a sports stadium.

1.2 A mathematical representation of waves

Let's start with one-dimensional waves. These are waves that propagate in one direction only, say the x direction. For the sake of argument, think of a signal along a string (like a guitar or violin string) or a channel, or a queue of people.

Clearly, the wave signal will depend on where we measure it, and when we measure it. In other words, the signal is a function of both x (space) and t (time). Thus, we are looking for a function of two variables, $u(x, t)$.

At a fixed time t_0, we can take a snapshot of the wave signal $u(x, t_0)$. Similarly, we can put a probe in a specific location x_0 to get the time signal $u(x_0, t)$. To visualize wave signals, we often take a series of snapshots at different fixed times t_1, t_2, t_3, ... This is illustrated in Fig. 1.1 and Fig. 1.2. Of course, we can always make a three-dimensional plot too (with x and t as the horizontal axes, $u(x, t)$ the vertical), or even a movie. Both those last two options don't work as well on a two-dimensional sheet of paper.

Example. The function

$$u(x, t) = f(x - vt), \quad v > 0,$$

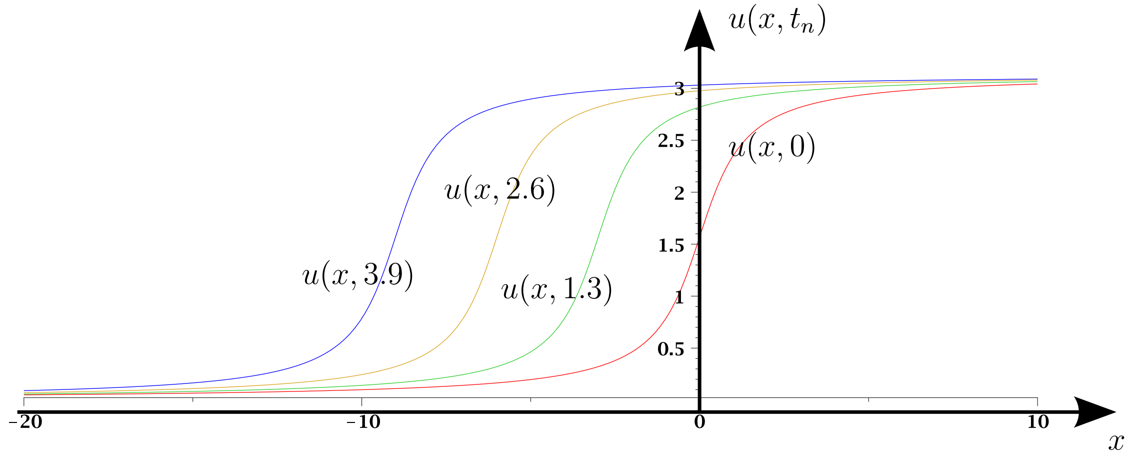

Figure 1.2: A shock wave traveling to the left.

gives a profile $f(x)$ that is moving to the right (because v is positive) with speed v. If $v < 0$, the profile would be moving to the left, with speed $|v|$. Indeed, at $t = 0$, we have

$$u(x, 0) = f(x),$$

so this is our initial profile. At a later time $t > 0$, we get the same profile, but it has moved. Suppose we wanted to track the value $f(0)$. At any time t, this value can be found at the position given by

$$x - vt = 0 \quad \Rightarrow \quad x = vt,$$

which is positive, and increasing linearly with time t. Specifically, the speed at which the value $f(0)$ moves to the right is given by

$$\frac{dx}{dt} = v,$$

which verifies our claim. The same reasoning works for tracking any value of $f(x)$[1].

Example. As an example of our example (Really? Yup, really.), we consider

$$u(x, t) = \operatorname{sech}^2(x - 5t).$$

I know you all love hyperbolic functions, so I won't waste much time recalling what they are. We have that

$$\operatorname{sech}(x) = \frac{1}{\cosh(x)},$$

and

$$\cosh(x) = \frac{e^x + e^{-x}}{2}.$$

[1]When I make such gratuitous statements, you should check them. I do not make them to sound smart. If I did, I'd make them in Latin.

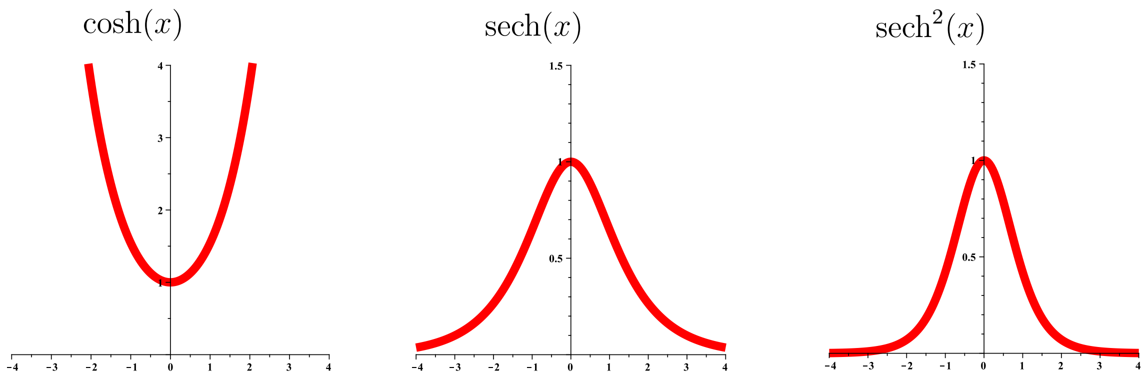

Figure 1.3: The graphs for $\cosh(x)$ (left), $\operatorname{sech}(x)$ (middle), and $\operatorname{sech}^2(x)$ (right).

Figure 1.3 shows what the plot of this function looks like at time $t = 0$. Once we know that, we know that $u(x, t)$ has the same plot at all time, but translated to the right by an amound $5t$.

Example. The wave $u(x, t) = \sin(x + t)$ represents a sine function at $t = 0$, which moves to the left with speed 1.

Example. Consider

$$u(x, t) = H(x - 7t),$$

where $H(x)$ is the Heaviside step function:

$$H(x) = \begin{cases} 1 & \text{if} \quad x \geq 0, \\ 0 & \text{if} \quad x < 0. \end{cases}$$

Its graph is drawn in Fig. 1.4. Thus $u(x, t)$ is a step profile, moving to the right with velocity 7.

Visualizing functions of two variables

We have different options for visualizing functions $u(x, t)$ of two continues variables x and t. Which one we prefer often depends on how we are communicating our results. Check out the visualization.mw file in the software folder.

- **Animation.** We show a movie of $u(x, t)$ as t changes from an initial to a final value. Not so great on paper (unless you have a flip book, which people sometimes do!), but works really well in a presentation.

- **Slice plots.** We create a bunch of slices and we plot them together, either in 2-D or in 3-D. Great for paper communication.

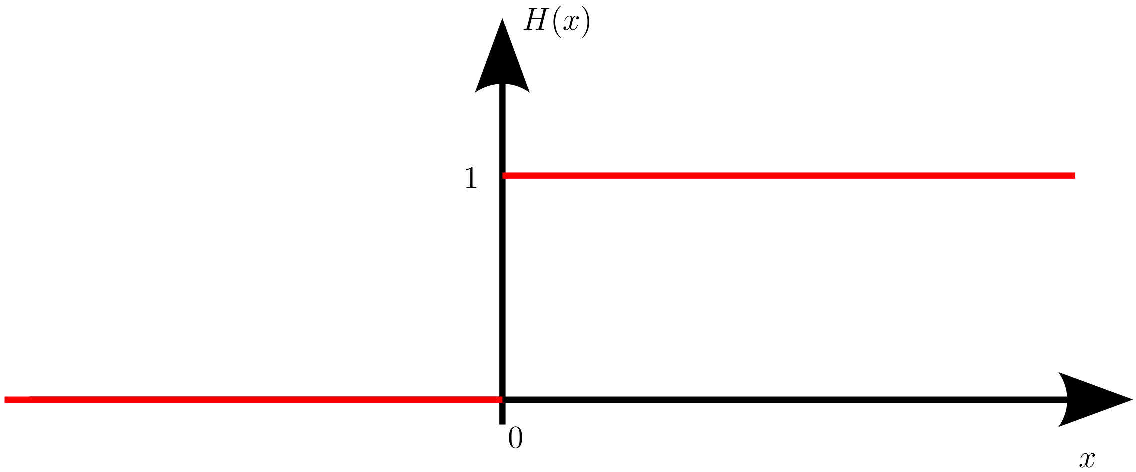

Figure 1.4: The graphs for $\cosh(x)$ (left), $\operatorname{sech}(x)$ (middle), and $\operatorname{sech}^2(x)$ (right).

- **Surface plots.** We plot $u(x,t)$ as a surface depending on the two variables x and t. Also good for paper communication, provided that the three dimensions come out well. This depends heavily on $u(x,t)$.
- (x,t)**-plots, contour plots.** We plot $u(x,t)$ as a function in the (x,t) plane, but looked at from above. We use different colors or grey scale to indicate the height $u(x,t)$. Great on paper.

1.3 Partial differential equations

A partial differential equation (PDE) for a function $u(x,t)$ is a differential equation that relates different derivatives of $u(x,t)$ to each other and to $u(x,t)$. That's the same definition as for ordinary differential equations, but now, because $u(x,t)$ depends on more than one variable, some of the derivatives will be with respect to x, others will be with respect to t.

We often use shorthand:

$$u_t := \frac{\partial u}{\partial u}, \quad u_x := \frac{\partial u}{\partial x}, \quad u_{xt} := \frac{\partial^2 u}{\partial x \partial t},$$

and so on.

Example. The advection equation is given by

$$u_t + c u_x = 0,$$

where c is a parameter. This equation is **linear**, since it contains no products of u with itself or any of its derivatives, and first order in both x and t, since no derivatives of higher than first order appear. The equation is also **homogeneous**, because $u = 0$ is a solution, albeit[2] a not very interesting one.

Example. **The diffusion equation** is given by

$$u_t = \sigma u_{xx},$$

where $\sigma > 0$ is a parameter. This equation is also linear, but is is of second order in x, first order in t. The equation is homogeneous. Since it arises in the study of heat transport, it is also known as the **heat equation**.

Example. The three-dimensional diffusion or heat equation is

$$u_t = \sigma(u_{xx} + u_{yy} + u_{zz}), \quad \sigma > 0.$$

As before, the equation is linear and homogeneous. It is first order in t, and second order in the spatial variables x, y and z.

Example. The Burgers equation is given by

$$u_t + uu_x = 0.$$

It is first order in x and t. It is our first example of a nonlinear PDE, because of the second term. It is still homogeneous, but for nonlinear equations that won't buy us much.

Example. The equation

$$u_t + uu_x + u_{xxx} = g(t)$$

is nonlinear (evil second term), first order in t, third order in x, and nonhomogeneous if $g(t) \not\equiv 0$.

Example. The sine-Gordon equation[3] is given by

$$u_{xt} = \sin u.$$

It is nonlinear (why?), first order in x and t. It is homogeneous.

Example. In contrast, the sadly unnamed equation

$$u_{xt} = \cos u,$$

is not homogeneous. It is nonlinear (again, why?), first order in x and t.

Let's bring some intuition in. The quantity $u_t(x_0, t)$ (where x_0 is fixed) denotes the rate of change of u as t changes, at a fixed location. In other words, we put a probe at a certain spot, and we look at how fast the time signal changes. Similarly, $u_x(x, t_0)$ (where t_0 is fixed)

[2] That's a funny word!

[3] Yes, that'a a pun, which you will recognize if you are a quantum physicist. If you are a quantum physicist, this class may be a bit too basic for you.

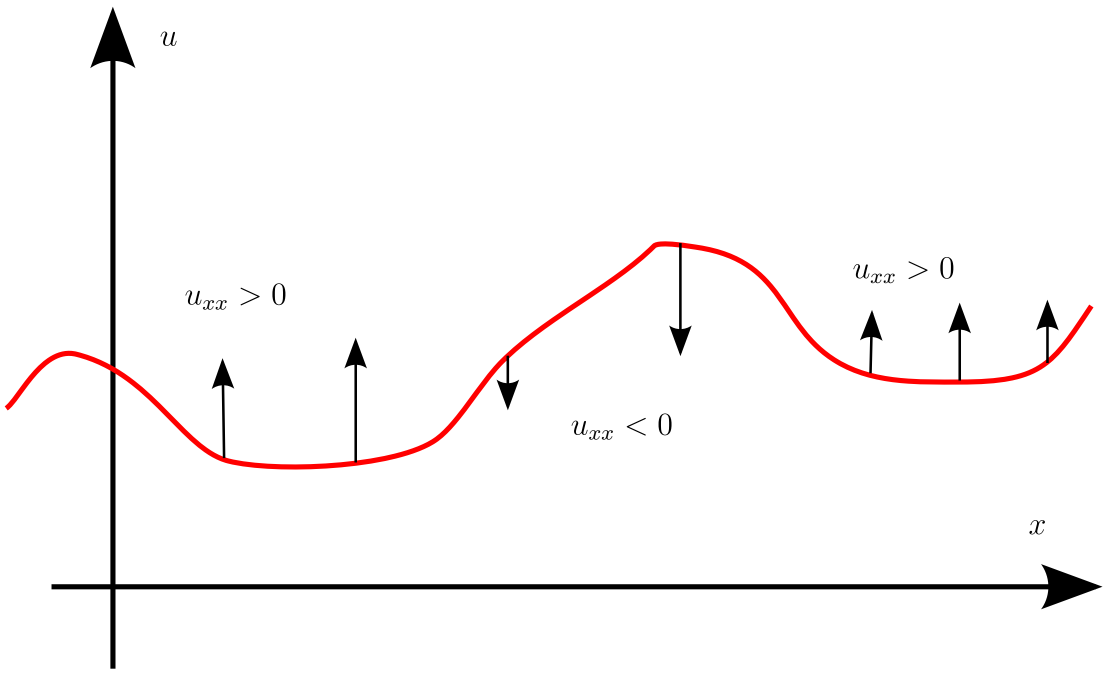

Figure 1.5: An initial profile for the heat equation and how it will evolve.

is the rate of change of u as x changes, at a fixed time. This corresponds to taking a snapshot of u, and looking at the slope of the curve in the picture. It follows that both u_x and u_t are interpreted as velocities. Similarly, both u_{xx} and u_{tt} can be seen as accelerations.

Example. Let u represent the temperature in a metal rod at position x, at time t. Then u satisfies the so-called heat equation

$$u_t = Du_{xx},$$

where $D > 0$ is the heat conductivity of the rod, assumed to be constant here. Suppose we start with an initial temperature profile, as in Fig. 1.5. In those regions where $u_{xx} > 0$ (in other words, the function is concave up), the heat equation states that u_t will be positive, so u will increase in time. For those regions where the profile is concave down, the heat equation gives that u_t will be negative, *i.e.*, u will decrease in time, as indicated in the figure. It follows that the heat equation likes to smear out profiles to a constant value.

Example. Next, we consider u to be a solution of the *transport equation*

$$u_t = u_x.$$

We choose an initial profile $u(x, 0)$ as in Fig. 1.6. Where $u(x, 0)$ is increasing as a function of x, $u_x > 0$ and therefore $u_t = u_x > 0$, thus u is increasing in time. Similarly, if $u(x, 0)$ decreases as a function of x, $u_x < 0$ and it follows that u decreases in time, as indicated by the arrows in Fig. 1.6. It follows that the overall shape of the profile will move to the left.

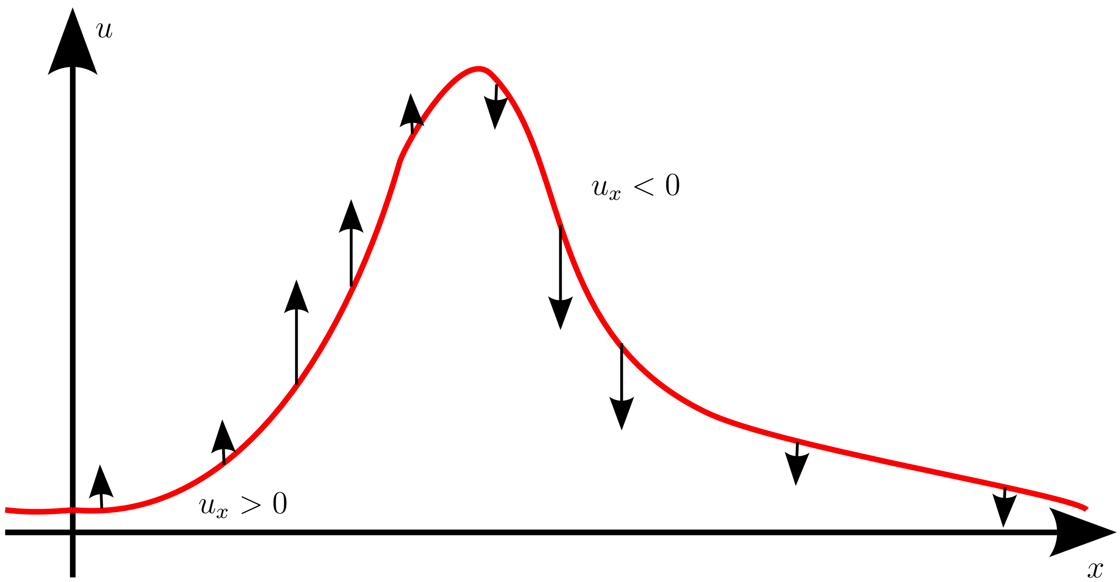

Figure 1.6: An initial profile for the transport equation and how it will evolve.

Chapter 2

Traveling and standing waves

2.1 Traveling wave solutions of PDEs

We have already seen that functions of the form

$$u(x,t) = f(x - vt)$$

represent profiles $f(x)$ that move to the right with velocity v is $v > 0$, and to the left if $v < 0$.

In general, solutions of PDEs are functions of both x and t independently. Sometimes they have solutions where x and t always show up in the special combination $x - vt$. We call such solutions **traveling waves**. We are especially interested in the case when $f(x)$ is not constant, or in the case where $f(x)$ is bounded for all values of x. Unbounded signals (*i.e.*, $u \to \pm\infty$) are usually[1] not relevant for applications. Then $f(x - vt)$ represents a disturbance moving through a medium with velocity v.

Example. Consider

$$u(x,t) = \sin(3x - t).$$

This is a traveling wave moving to the right with velocity $v = 1/3$. Indeed,

$$\begin{aligned} u(x,t) &= \sin(3x - t) \\ &= \sin 3(x - t/3) \\ &= f(x - vt), \end{aligned}$$

so that $v = 1/3$ and

$$f(z) = \sin 3z.$$

Example. Let's find traveling wave solutions of the wave equation

$$u_{tt} = a^2 u_{xx},$$

[1]In the "always" sense.

where we may assume that the constant $a > 0$. We let

$$u = f(x - vt).$$

Our task is to find the function f and the constant v. Let $z = x - vt$. We get

$$\begin{aligned} & & u(x,t) &= f(z) \\ &\Rightarrow & u_x &= f', \\ &\Rightarrow & u_{xx} &= f'', \\ &\Rightarrow & u_t &= -vf', \\ &\Rightarrow & u_{tt} &= (-v)^2 f'' = v^2 f'', \end{aligned}$$

where we have used the chain rule and the fact that

$$\frac{\partial z}{\partial x} = 1, \quad \frac{\partial z}{\partial t} = -v.$$

Thus, traveling wave solutions of the PDE

$$u_{tt} = a^2 u_{xx}$$

satisfy the ODE

$$\begin{aligned} & & v^2 f'' &= a^2 f'' \\ &\Rightarrow & (v^2 - a^2) f'' &= 0. \end{aligned}$$

Either $f'' = 0$ or $v^2 = a^2$. Using the first possibility, we get

$$f'' = 0 \quad \Rightarrow \quad f(z) = Az + B,$$

independent of what v is. Here A and B are constants. The second possibility results in

$$v^2 = a^2 \quad \Rightarrow \quad v = \pm a,$$

independent of what $f(z)$ is. In summary, we have obtained the following solutions:

- $A(x - vt) + B$, for any value of A, B, v. These solutions are not very interesting: in order for them to be bounded, we need A to be zero. But that leaves us with a constant solution, which is unexciting.
- $f_1(x - at)$, for any function $f_1(z)$, and
- $f_2(x + at)$, for any function $f_2(z)$.

Since the equation is linear, we can superimpose the solutions to get a more general solution:

$$u(x,t) = A(x - vt) + B + f_1(x - at) + f_2(x + at).$$

Some remarks are in order.

- Usually, we would include multiplicative constants c_1, c_2 and c_3 to get a solution of the form
$$u(x,t) = c_1(A(x - vt) + B) + c_2 f_1(x - at) + c_3 f_2(x + at).$$
Since these constants can be absorbed in the forms of f_1, f_2, and the values of A and B, we may omit them.

- Note that the superposition of different traveling waves is not necessarily a traveling wave! Our superposition consists of three different parts. Two of these parts ($f_1(x-at)$ and $f_2(x + at)$) even move in opposite directions.

2.2 The sine-Gordon equation

We consider a more complicated example. The sine-Gordon equation is

$$u_{tt} = u_{xx} - \sin u.$$

First we substitute $u = f(z)$, $z = x - vt$ into the equation. We want to find an ODE for $f(z)$, which we want to use to determine $f(z)$ and perhaps also v. In this case, we get

$$\begin{aligned} & v^2 f'' = f'' - \sin f \\ \Rightarrow \quad & (1 - c^2) f'' = \sin f. \end{aligned}$$

This is a messy second-order ODE. We can reduce it to a first order ODE by multiplying by f', which results in an equation we may integrate once:

$$\begin{aligned} & (1 - c^2) f' f'' = f' \sin f \\ \Rightarrow \quad & \frac{d}{dz}\left[(1 - c^2)\frac{f'^2}{2}\right] = \frac{d}{dz}\left[-\cos f\right] \\ \Rightarrow \quad & (1 - c^2) f'^2 = A - 2\cos f, \end{aligned}$$

where we have used that $f' f''$ is the derivative of $f'^2/2$ and that $f' \sin f$ is the derivative of $-\cos f$. Here A is an arbitrary constant. Our new equation is a first-order ODE for $f(z)$. This is progress! It's still a messy ODE, but we can actually solve this ODE using separation of variables. Here I just give one class of solutions. You should check that these are, in fact, solutions[2].

$$f(z) = 4\arctan\left(e^{z/\sqrt{1-c^2}}\right),$$

[2] You should never, ever, trust me.

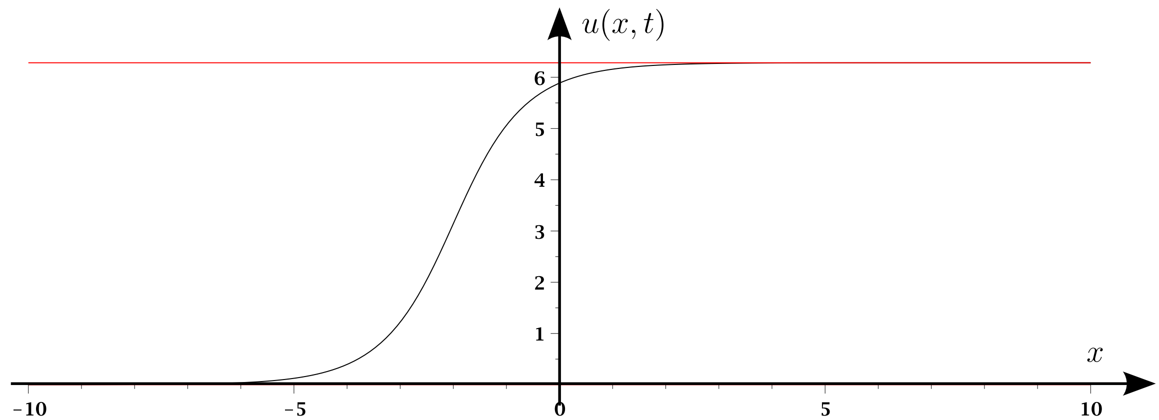

Figure 2.1: The profile of a traveling wave solution of the sine-Gordon equation. For this specific profile, $c = 1/2$ and the whole graph moves to the right with velocity $1/2$.

which corresponds to $A = 2$, and is valid for $c \in (0, 1)$.

It follows that

$$u(x,t) = 4 \arctan\left(e^{(x-ct)/\sqrt{1-c^2}}\right)$$

is a traveling wave solution of $u_{tt} = u_{xx} - \sin u$. Since this equation is nonlinear, we can't simply superimpose a bunch of these solutions to get new solutions. A plot of a traveling wave solution is shown in Fig. 2.1. Note the horizontal asymptotes at 0 and 2π.

2.3 The Korteweg-de Vries equation

In 1834, John Scott Russell, a Scottish engineer[3] was on horseback, following a ship as it was pulled by horses along one of the canals in Scotland. At some point, the ship hit something in the water. What happened next is best told in his own words, and preferably with a Scottish accent. More information can be found at `http://www.macs.hw.ac.uk/~chris/scott_russell.html`.

> "I was observing the motion of a boat which was rapidly drawn along a narrow channel by a pair of horses, when the boat suddenly stopped - not so the mass of water in the channel which it had put in motion; it accumulated round the prow of the vessel in a state of violent agitation, then suddenly leaving it behind, rolled forward with great velocity, assuming the form of a large solitary elevation, a rounded, smooth and well-defined heap of water, which continued its course along the channel apparently without change of form or diminution of speed. I followed it on horseback, and overtook it still rolling on at a rate of some eight or nine miles an hour, preserving its original figure some thirty feet long and a foot to a foot and a half in height. Its height gradually

[3] I am not making this up!

> diminished, and after a chase of one or two miles I lost it in the windings of the channel. Such, in the month of August 1834, was my first chance interview with that singular and beautiful phenomenon which I have called the Wave of Translation"

It took 10 years for J. S. Russell to publish his results. It took another 50 for them to become appreciated (no Twitter yet). Two Dutchmen Korteweg (PhD advisor) and de Vries (his student) derived the equation that now bears their name, although we usually abbreviate it the KdV equation:

$$u_t + uu_x + u_{xxx} = 0.$$

They derived the equation to describe long waves in shallow water, like tsunamis. But it describes much more than this: in general it describes the propagation of long waves in a dispersive medium[4]. This covers water waves, but also waves in plasmas, like the northern lights, or like the light flickering we sometimes see in (large) LED lights.

Let's look for traveling wave solutions of the KdV equation. Thus we let

$$u(x,t) = f(z), \quad z = x - vt,$$

where v is the velocity of the wave. For simplicity we will look for waves traveling to the right, thus $v > 0$. Further, we will limit our investigations to waves like the ones that Russell saw, namely waves that decay to zero as $x \to \infty$ or $x \to -\infty$. As before, we have

$$u_x = f', \;\; u_{xxx} = f''', \;\; u_t = -vf'.$$

The KdV equation becomes

$$\begin{aligned} & & -vf' + ff' + f''' = 0 \\ &\Rightarrow & \frac{d}{dz}\left(-vf + \frac{1}{2}f^2 + f''\right) = 0 \\ &\Rightarrow & -vf + \frac{1}{2}f^2 + f'' = A, \end{aligned}$$

where A is an integration constant. Since A is constant, we may use any value of x to evaluate the left-hand side. The most convenient values of x are $\pm\infty$, because we are looking for solutions for which f, f', *etc*, all decay to zero there. Evaluating our expression as $\pm\infty$ gives $A = 0$, so that we are left with

$$-vf + \frac{1}{2}f^2 + f'' = 0.$$

Multiplying this by f', we have

$$\begin{aligned} & & -vff' + \frac{1}{2}f^2 f' + f'f'' = 0 \\ &\Rightarrow & \frac{d}{dz}\left(-\frac{v}{2}f^2 + \frac{1}{6}f^3 + \frac{1}{2}f'^2\right) = 0 \\ &\Rightarrow & -\frac{v}{2}f^2 + \frac{1}{6}f^3 + \frac{1}{2}f'^2 = B, \end{aligned}$$

[4] We'll learn what "dispersive" means soon.

where B is a second constant of integration. evaluating the left-hand side once again at $\pm\infty$, we find that $B = 0$. Thus, we find the first-order ordinary differential equation

$$-\frac{v}{2}f^2 + \frac{1}{6}f^3 + \frac{1}{2}f'^2 = 0.$$

It follows that

$$3f'^2 = (3v - f)f^2.$$

Since $v > 0$ (by choice), we find that $f \leq 3v$. That's good: we are looking for a bounded solution, and it looks like we'll find one. It remains to solve the first-order equation using separation of variables. Ready to have a good time with integration? We have

$$\begin{aligned} & \frac{\sqrt{3}f'}{f\sqrt{3v - f}} = 1 \\ \Rightarrow \quad & \sqrt{3}\int \frac{df}{f\sqrt{3v - f}} = z + \alpha, \end{aligned}$$

where α is a constant of integration. We use substitution to simplify the integral. The hardest part is the square root in the denominator. Let

$$g = \sqrt{3v - f} \quad \Rightarrow \quad f = 3v - g^2.$$

It follows that $df = -2gdg$. We get

$$\begin{aligned} & \sqrt{3}\int \frac{-2gdg}{(3v - g^2)g} = z + \alpha \\ \Rightarrow \quad & -2\sqrt{3}\int \frac{dg}{3c - g^2} = z + \alpha. \end{aligned}$$

This integral can be done using partial fractions. This results in

$$\begin{aligned} & \ln\left(\frac{\sqrt{3v} + g}{\sqrt{3v} - g}\right) = -\sqrt{v}(z + \alpha) \\ \Rightarrow \quad & g = \sqrt{3v}\frac{e^{-\sqrt{v}(z+\alpha)} - 1}{e^{-\sqrt{v}(z+\alpha)} + 1} \\ & \quad = -\sqrt{3v}\tanh\frac{\sqrt{v}}{2}(z + \alpha). \end{aligned}$$

Returning to $f = 3v - g^2$, we get

$$f(z) = 3v\text{sech}^2\frac{\sqrt{v}}{2}(z + \alpha).$$

It follows that

$$u(x,t) = 3v\,\text{sech}^2\frac{\sqrt{v}}{2}(x - vt + \alpha)$$

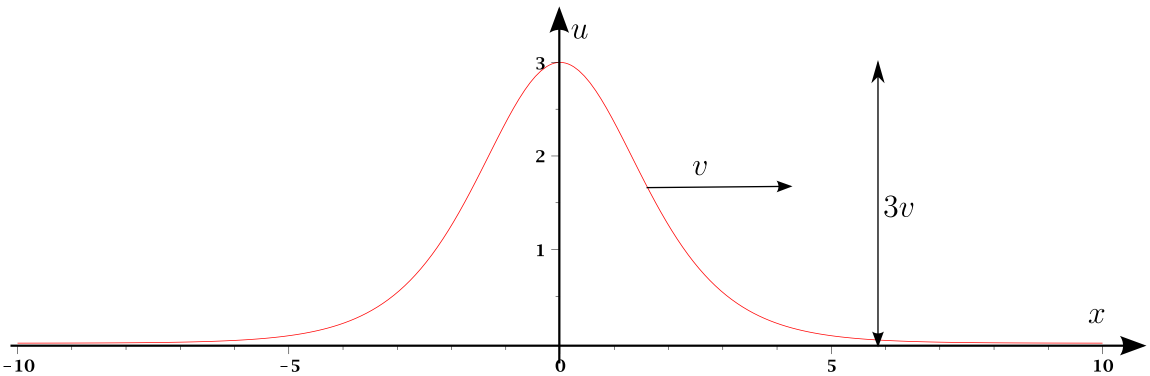

Figure 2.2: The profile of a traveling wave solution of the KdV equation. The profile moves to the right with velocity v and has height $3v$.

is a traveling wave solution of the KdV equation, for any $v > 0$. It is illustrated in Fig. 2.2. Its amplitude (*i.e.*, height) is $3v$, while its velocity is v. Thus higher waves of the KdV equation travel faster, as we observe at the beach. Further, the width of the profile is proportional to $1/\sqrt{v}$, thus taller, faster waves are more narrow. Lastly, the maximum of the wave profile occurs when $x - vt + \alpha = 0$, or at $x = vt - \alpha$.

2.4 Wave fronts and pulses

A traveling wave

$$u(x,t) = f(x - ct)$$

is called a **front** if for a fixed t, we have

$$\lim_{x \to -\infty} u = k_1, \qquad \lim_{x \to \infty} u = k_2,$$

and

$$k_1 \neq k_2.$$

In other words, our profile approaches different limit values at $-\infty$ and $+\infty$. The traveling wave solution of the sine-Gordon equation is a good example. Note, however, that the transition between the two limiting values does not have to be monotone.

If on the other hand,

$$k_1 = k_2,$$

then we call the solution a **pulse**. The profile drawn in Fig. 1.6 is an example. Thus, for pulses, the beginning and end state are the same: the medium returns back to its original state after a pulse passes through. On the other hand, the passing of a front forever alters the state of the medium.

Examples of fronts.

- Weather fronts: different meteorological signals are altered by the passing of a weather front, such as the air pressure.
- Sonic booms: the density in the air is altered by the passing of the sonic boom. Of course, dissipation effects eventually return it to its original value, but for quite a while, the air density is changed. That is different from a subsonic plane passing through, which creates a localized disturbance in the density.
- Bob Dylan, Beethoven: music was forever altered by both of them.

Examples of pulses.

- Optical flashes
- Bits
- One-hit wonders: they have no lasting effect on the musical landscape.

2.5 Wave trains and dispersion

Different types of traveling waves, other than fronts or pulses, exist.

Example. Consider

$$u(x,t) = 3\cos(7x - 5t) = 3\cos 7(x - 5t/7).$$

This is a traveling wave train of amplitude 3 and velocity 5/7. It has wave number 7 and frequency 5.

In general, (linear) traveling wave trains are expressions of the form

$$u = A\cos(kx - \omega t + \phi),$$

where instead of cos we may have sin or $\exp(i \ldots)$. Using one of these functional forms, the following quantities are defined.

- A is the **amplitude**. This is the largest value the wave train attains. Similarly, $-A$ is its smallest value.
- The **wave number** k denotes how many oscillations occur during an interval of length 2π. Indeed, the period of the wave train is $2\pi/k$. Since one oscillation occurs per period, k oscillations occur over an interval of length 2π.
- The **frequency** ω denotes how many oscillations happen during a time interval of length 2π. The same argument as above shows this, but we consider the temporal dependence instead of the spatial dependence. It follows that signals with high wave number or frequency are very oscillatory. That means that if we want to plot them accurately, we'll need to use many points at which to sample the function.

- The **phase shift** ϕ simply shift the origin of space or time. That sounds very philosophical[5], but it really isn't. We can rewrite the wave train signal as (using the cosine form, for instance)

$$u(x,t) = A\cos(k(x-x_0)-\omega t) \quad \text{or} \quad u(x,t) = A\cos(kx-\omega(t-t_0)),$$

  where $-kx_0 = \phi$ and $\omega t_0 = \phi$. Thus ϕ simply changes when we start measuring where or when we start.

- The **velocity** of the traveling wave is $v = \omega/k$, since we can rewrite the signal as

$$u(x,t) = A\cos(k(x-vt)+\phi),$$

  where $v = \omega/k$.

Often, the frequency ω and the wave number k are related. This relation is called the **dispersion relation**. Since sin and cos can be written as linear combinations of exponentials, it suffices to consider expressions of the form

$$u = Ae^{ikx-i\omega t}.$$

The idea is to substitute this into the PDE and find the relationship between ω and k, if there is one. This is easy: note that whenever we take an x derivative, we get

$$u_x = ikAe^{ikx-i\omega t} = iku,$$

and so on for higher-order derivatives. Thus taking an x derivative simply multiplies our solution by a factor ik. Similarly, taking a time derivative,

$$u_t = -i\omega Ae^{ikx-i\omega t} = -i\omega u,$$

and taking a time derivative results in multiplying u by $-i\omega$. Taking more time derivatives results in more powers of $-i\omega$.

Example. Consider the wave equation

$$u_{tt} = a^2 u_{xx}, \qquad a > 0.$$

We already know that this equation has traveling wave solutions that travel to the left with velocity $-a$ and to the right with velocity a. Plugging in

$$u = Ae^{ikx-i\omega t},$$

we get

$$\begin{aligned} & (-i\omega)^2 u = a^2(ik)^2 u \\ \Rightarrow \quad & -\omega^2 = -k^2 \\ \Rightarrow \quad & \omega^2 = k^2a^2, \end{aligned}$$

[5] That's OK: I have a doctorate in philosophy. Really.

from which it follows that

$$\omega_1 = ka, \qquad \omega_2 = -ka.$$

Thus there are two possible branches of the dispersion relation. They give rise to the solutions

$$u_1 = A_1 e^{ik(x-at)},$$

and

$$u_2 = A_2 e^{ik(x+at)}.$$

Example. Consider the PDE

$$u_{tt} = a u_{xx} - bu, \quad a, b > 0.$$

Looking for the dispersion relation, we find

$$\begin{aligned} & (-i\omega)^2 u = a(ik)^2 u - bu \\ \Rightarrow \quad & -\omega^2 = -ak^2 - b \\ \Rightarrow \quad & \omega^2 = ak^2 + b \\ \Rightarrow \quad & \omega_{1,2} = \pm\sqrt{ak^2 + b}, \end{aligned}$$

leading to the two classes of solutions

$$u_1 = A_1 e^{ikx - i\omega_1 t},$$

and

$$u_2 = A_2 e^{ikx - i\omega_2 t}.$$

Example. Consider the linear free[6] Schrödinger equation:

$$i\varphi_t = -\varphi_{xx}.$$

We look for solutions of the form

$$\varphi = A e^{ikx - i\omega t},$$

to get

$$\begin{aligned} & i(-i\omega)\varphi = -(ik)^2 \varphi \\ \Rightarrow \quad & \omega = k^2. \end{aligned}$$

[6] because there is no potential

2.6 Dispersion relations for systems of PDEs

We will examine how to do this by example. Consider the PDE system

$$\begin{aligned} u_t &= \alpha u_x + v_{xxx} \\ v_t &= \beta v_x - u_{xxx}. \end{aligned}$$

We wish to look for wave train solutions, whose x and t dependence is of the form

$$e^{ikx-i\omega t}.$$

For the scalar case, the amplitude A never came into play, since for linear equations, we could simply divide it out. That is no longer true here, since there is no reason why u and v should have the same amplitude.

This situation is similar to what we do with systems of ODEs: if we wish to solve

$$ay'' + by' + cy = 0,$$

we guess

$$y = e^{\lambda t}.$$

On the other hand, if we want to solve the system

$$y' = Ay,$$

where A is a matrix and y is a vector, then we guess

$$y = e^{\lambda t} v,$$

where v is a vector.

We do the same for our system of PDEs. We guess

$$\begin{pmatrix} u \\ v \end{pmatrix} = \begin{pmatrix} U \\ V \end{pmatrix} e^{ikx-i\omega t}.$$

Substituting this into the PDE, we get

$$\begin{aligned} -i\omega U &= \alpha ikU + (ik)^3 V \\ -i\omega V &= \beta ikV - (ik)^3 U \\ \Rightarrow \quad \begin{pmatrix} -i\omega - \alpha ik & ik^3 \\ ik^3 & -i\omega - \beta ik \end{pmatrix} \begin{pmatrix} U \\ V \end{pmatrix} &= 0. \end{aligned}$$

Since we are not interested in the zero solution

$$\begin{pmatrix} U \\ V \end{pmatrix} = 0,$$

we need that

$$\begin{pmatrix} -i\omega - \alpha ik & ik^3 \\ ik^3 & -i\omega - \beta ik \end{pmatrix}$$

is a singular matrix. Thus

$$\det \begin{pmatrix} -i\omega - \alpha ik & ik^3 \\ ik^3 & -i\omega - \beta ik \end{pmatrix} = 0.$$

This the dispersion relation, determining the frequency ω as a function of the wave number k. The rest is algebra. We get

$$\begin{aligned}
& (-i\omega - \alpha ik)(-i\omega - \beta ik) - k^6 = 0 \\
\Rightarrow \quad & -\omega^2 + i\omega(\alpha ik + \beta ik) - \alpha\beta k^2 - k^6 = 0 \\
\Rightarrow \quad & -\omega^2 - \omega(\alpha+\beta)k - \alpha\beta k^2 - k^6 = 0 \\
\Rightarrow \quad & \omega^2 + \omega(\alpha+\beta)k + \alpha\beta k^2 + k^6 = 0 \\
\Rightarrow \quad & \omega = \frac{1}{2}\left(-(\alpha+\beta)k \pm \sqrt{(\alpha+\beta)^2k^2 - 4(\alpha\beta k^2 + k^6)}\right).
\end{aligned}$$

Thus, there are two possibilities for ω: ω_1 and ω_2, corresponding to the $+$ and $-$ signs above.

2.7 Information from the dispersion relation

Suppose we have found and solved the dispersion relation, so that we have $\omega(k)$. What do we get out of this? Well, of course we can conclude that the PDE has wave train solutions whose x and t dependence if of the form

$$u = e^{ikx - i\omega t}.$$

Since the equation is linear, we may superimpose such solutions. But what can we say about these solutions? What are their properties?

- **Phase velocity.** We may rewrite the solution as

  $$u = e^{ik(x - c_p(k)t)},$$

  where

  $$c_p(k) = \frac{\omega(k)}{k}$$

  is called the phase velocity. It is the velocity with which a single wave train travels.

- **Group velocity.** Another velocity matters for wave trains:

  $$c_g(k) = \frac{d\omega}{dk},$$

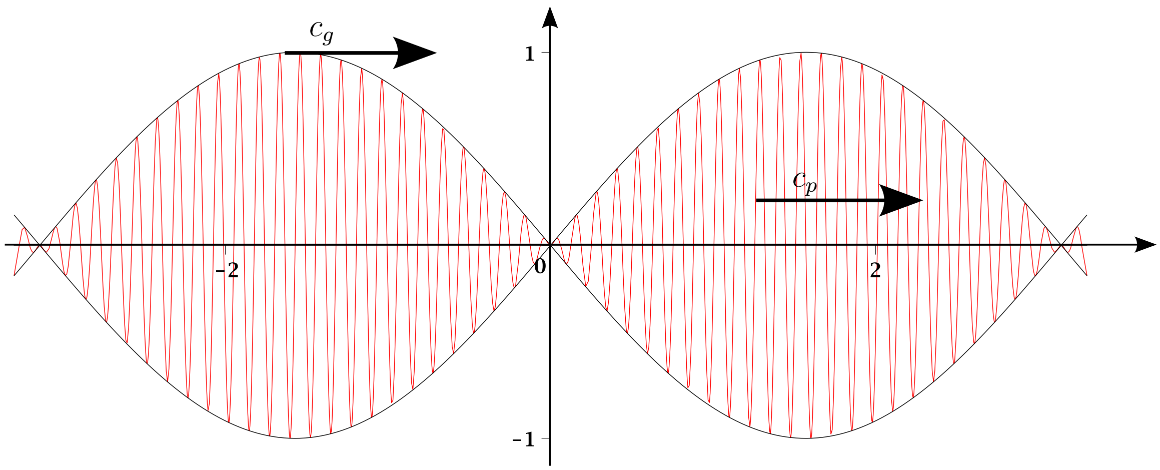

Figure 2.3: A wave packet consisting of two traveling wave trains. The phase speed is the speed of waves inside the packets, whereas the envelope of the packets moves with the group speed.

which we call the group velocity. The full importance of the group velocity is hard to explain (see Amath569), but let's give it a shot.

Suppose that in a given signal, the most important wave number is k_0, with corresponding frequency $\omega_0 = \omega(k_0)$. We rewrite $\exp(ikx - i\omega(k)t)$ as

$$\begin{aligned}
e^{ikx-\omega(k)t} &= e^{ik_0-i\omega_0 t}e^{i(k-k_0)x-i(\omega(k)-\omega_0)t} \\
&= e^{ik_0-i\omega_0 t}e^{i(k-k_0)x-i(\omega(k_0)-(k-k_0)\omega'(k_0)+\mathcal{O}((\delta k)^2)-\omega_0)t} \\
&= e^{ik_0-i\omega_0 t}e^{i(k-k_0)x-i((k-k_0)\omega'(k_0)+\mathcal{O}((\Delta k)^2))t} \\
&= e^{ik_0-i\omega_0 t}e^{i\Delta k(x-\omega'(k_0)t)+\dots}.
\end{aligned}$$

The second exponential factor acts as a slowly varying amplitude to the first one. Indeed, the wave number Δk and frequency $\omega'(k_0)\Delta k$ are both small, assuming that k is close to k_0. The second, slowly-varying factor moves with the group velocity! A wave packet is used to illustrate this in Fig. 2.3.

Thus individual waves move with the phase velocity, while wave packets move with the group velocity. Note that it is perfectly possible for the group and phase velocity to have opposite signs.

Although we will not show this here, the group velocity is also the velocity with which the energy associated with a wave moves. Often we care about this much more than we care about individual waves.

- **Dispersive *vs.* non-dispersive waves.** An equation or a system is called **dispersive** if (a) the phase speed c_p is real for real k, and (b) the phase speed is not constant. This implies that wave trains with different wave numbers will move with different speeds.

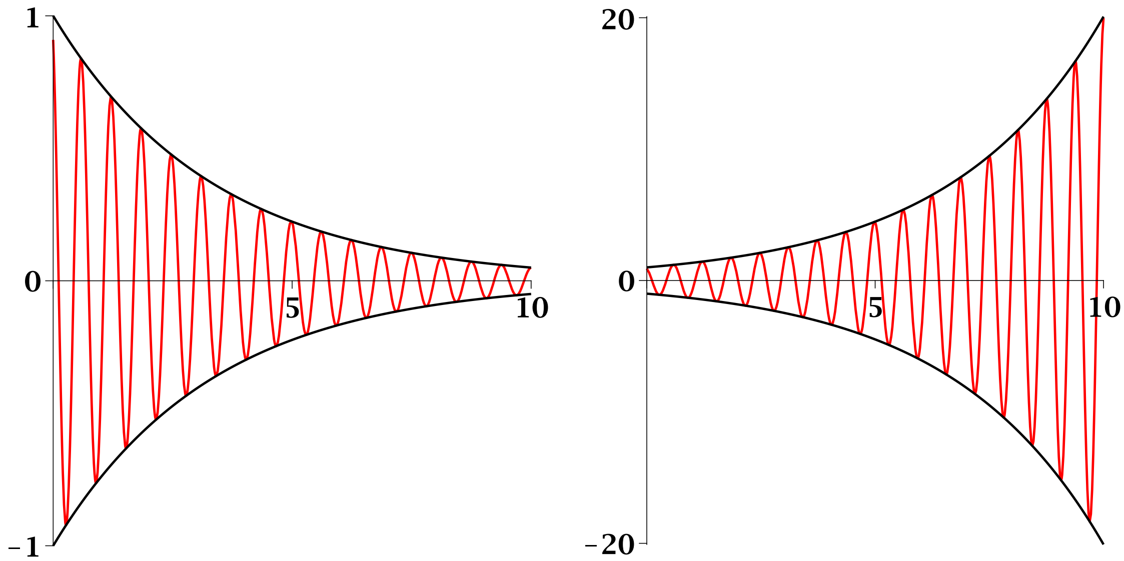

Figure 2.4: A decaying wave train ($\omega_I < 0$, left) and a growing wave train ($\omega_I > 0$, right).

But there is more! In general, the dispersion relation may be complex, even for real k. If that happens, then for those values of k the equation is not dispersive.

Example. Consider the PDE

$$u_t = u_{xx} + cu_x.$$

This equation gives rise to the dispersion relation

$$\begin{aligned} -i\omega &= (ik^2) + cik \\ \Rightarrow \qquad \omega &= -ik^2 - ck. \end{aligned}$$

In general, we can split $\omega(k)$ into its real and imaginary parts: $\omega = \omega_R + i\omega_I$, where ω_R and ω_I are the real and imaginary parts of ω, respectively. It follows that our wave train solutions are of the form

$$\begin{aligned} e^{ikx-i\omega t} &= e^{ikx-it(\omega_R+i\omega_I)} \\ &= e^{ikx-i\omega_R t+\omega_I t} \\ &= e^{\omega_I t} e^{ikx-i\omega_R t}. \end{aligned}$$

Thus, if $\omega_I > 0$, the signal will grow in time. If $\omega_I < 0$, the signal will decay in time. Both situations are illustrated in Fig. 2.4.

If $\omega_I < 0$ and amplitudes decay, we say the PDE is **dissipative**. if $\omega_I > 0$ and amplitudes grow, we call the system **unstable**.

2.8 Pattern formation

Often, linear systems are obtained by ignoring nonlinear term, which are assumed to be less important, provided the solution is small. Such small solutions can be considered disturbances to the zero solution. When we see instabilities in a linear PDE or a system of linear PDES, they result in the growth of disturbances to the zero solution. In actual application settings, this growth cannot go on forever, as there is only a finite amount of energy in the system, for instance.

Recall that the linear system is only valid provided the solution is small. If the disturbances grow exponentially, the linear system will not be valid for long, and nonlinear terms will have to be considered. Often they have the effect of arresting the growth due to linear instabilities. The nonlinear effects that we ignored to get to the linear system may be small initially, but they will start to matter as the solution grows.

Even so, the study of the dispersion relation often allows us to predict many aspects of the solution of the full, nonlinear problem, even if we cannot solve that problem completely. Let's see how we can predict what kinds of patterns form in the so-called **Kuramoto-Sivasinski (KS) equation**

$$u_t + uu_x + u_{xx} + au_{xxxx} = 0, \qquad a > 0.$$

This equation arises a lot in applications. Among other things, it describes the dynamics of flame fronts.

- We see immediately that $u = 0$ is a solution.
- Let's investigate the dynamics of solutions close to this solution. Such solutions are small, so that we may ignore the term uu_x. Indeed, if

  $$u \sim Ae^{ikx-i\omega t},$$

  with A small, then

  $$u_x \sim ikAe^{ikx-i\omega t},$$

  and

  $$uu_x \sim ikA^2e^{2ikx-2i\omega t},$$

  which is a lot smaller than any of the linear terms, since $A^2 \ll A$, for small A. Thus, we are justified in studying the linearized KS equation:

  $$u_t + u_{xx} + au_{xxxx} = 0.$$

- We find its dispersion relation:

$$\begin{aligned} & -i\omega + (ik)^2 + a(ik)^4 = 0 \\ \Rightarrow \quad & -i\omega - k^2 + ak^4 = 0 \\ \Rightarrow \quad & -i\omega = k^2 - ak^4, \end{aligned}$$

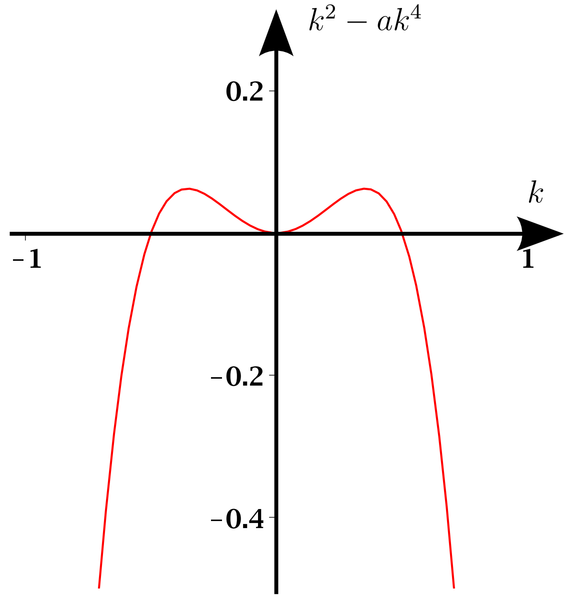

Figure 2.5: The graph of $k^2 - ak^4$ with $a = 4$.

so that

$$e^{ikx - i\omega t} = e^{ikx + (k^2 - ak^4)t}.$$

It follows that the solution grows in t if $k^2 - ak^4 > 0$ and it decays if $k^2 - ak^4 < 0$. We plot $k^2 - ak^4$ in Fig. 2.5.

If $k \in (-1/\sqrt{a}, 1/\sqrt{a})$, the growth rate is positive. Otherwise it is negative. Since the equation is linear, the general solution is a superposition of a bunch of solutions of the form

$$e^{ikx + (k^2 - ak^4)t}.$$

We have just concluded that any part of this superposition that has $k > 1/sqrta$ or $k < -1/\sqrt{a}$ will decay. Thus after some time, these parts will not come into play anymore. Since for all of these parts, $|k|$ is large, the period is small. In other words, these are highly oscillatory signals. The equation appears to get rid of them quickly.

All the contributions from the other wave numbers k grow (except from $k = 0$). Which one grows the most? Let

$$f(k) = k^2 - ak^4 \quad \Rightarrow \quad f'(k) = 2k - 4ak^3 = 2k(1 - 2ak^2).$$

Thus the growth rate is maximal for $k^* = \pm 1/sqrt2a$. All other solutions grow slower than this one. Thus, in comparison, they decay. Indeed, if we have

$$y = c_1 e^{ax} + c_2 e^{bx},$$

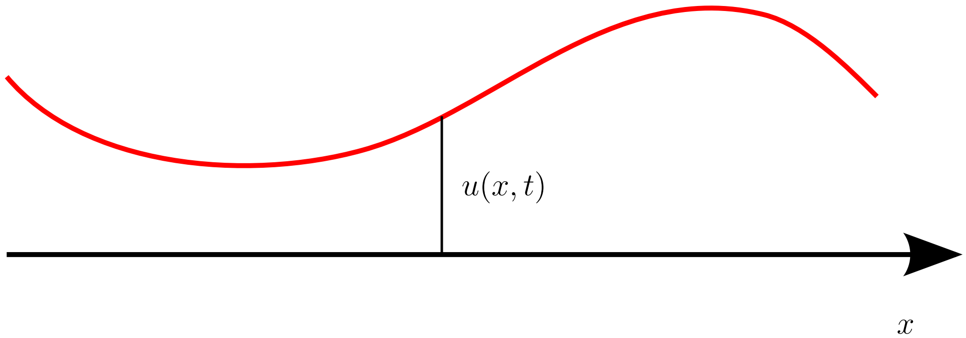

Figure 2.6: The set-up for the derivation of the wave equation modeling a plucked string.

with $a > b > 0$ (so both exponentials grow as $x \to \infty$), then

$$y = e^{ax}\left(c_1 + c_2 e^{(b-a)x}\right),$$

and the exponential in the parentheses decays! Thus, so sufficiently large x, we see $y \sim c_1 e^{ax}$.

The same conclusion holds for our case: for large enough t, we see only

$$e^{ik^* x + (k^{*2}) - ak^{*4} t} \quad \text{and} \quad e^{ik^* x + (k^{*2}) - ak^{*4} t}.$$

Thus, the equation naturally creates periodic patterns with period $2\pi/k^* = 2\pi\sqrt{2a}$, independent of the initial conditions!

- After some t, these solutions become too big and we can no longer ignore the nonlinear terms. But, the stage is set and the linear problem has already selected the period of the solution!

2.9 A derivation of the wave equation

The wave equation

$$u_{tt} = c^2 u_{xx},$$

shows up in many applications. Let's actually derive it in one setting, namely that of a plucked string, as illustrated in Fig. 2.6

We begin with the following assumptions.

- The equilibrium position of the string is at $u(x,t) = 0$.
- The string has constant density ρ.
- The vibration of the string stays in the plane: there is no dependence on the transverse variable y.

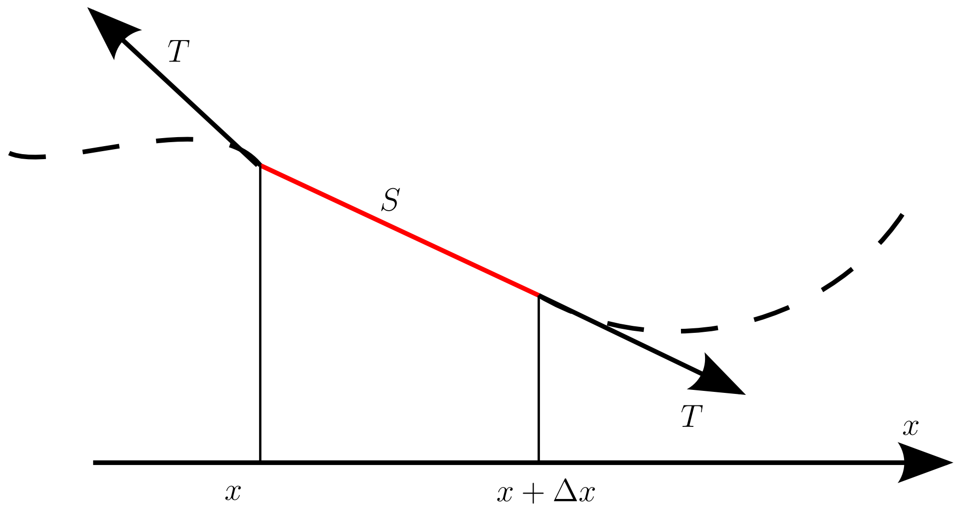

Figure 2.7: We apply Newton's law to a little piece of string.

- Tension is uniform: a string extends a force only in the direction parallel to the string. In other words, the force a piece of string exerts on neighboring pieces , keeping the string together, is tangential to the string.
- We assume that tension is constant anywhere along the string.
- There are no other forces.
- All vibrations are small. This is a physical way of saying that, mathematically, we will be ignoring nonlinear effects.

Next, we apply Newton's law of motion to an itty bitty[7] piece of string, lying between x and $x + \Delta x$, where Δx is very small. We have

$$\text{Mass of } S \times \text{Acceleration of } S = \text{Net force on } S.$$

This is Newton's law, perpendicular to the x axis. We could also write it parallel to the x axis, but that would result in a perfect force balance. This would offer no information about $u(x,t)$, which is the *vertical* displacement.

First, we get an expression for the mass of S.

$$\begin{aligned}\text{Mass of } S &= \rho \times \text{arclength}\\ &= \rho \int_x^{x+\Delta x} \sqrt{1 + u_x^2(s,t)}ds\\ &\approx \rho \int_x^{x+\Delta x} \left(1 + \frac{1}{2}u_x^2(s,t) + \dots\right) ds\\ &\approx \rho \Delta x,\end{aligned}$$

[7]Technical term.

where we have used that the vibrations are small, thus all nonlinear terms are ignored.

Next, the acceleration of S is simply $u_{tt}(x,t)$, by definition. Last, we turn to the net force. The net force is pulling on the left and right ends of S by the string parts to the immediate left and right of S. On the left end, the tension pulls with magnitude T in the direction of the tangent vector. This normalized tangent vector is given by

$$\frac{-(1, u_x(x,t))}{\sqrt{1+u_x^2(x,t)}}.$$

Thus the left force is

$$\begin{aligned} \frac{-T(1, u_x(x,t))}{\sqrt{1+u_x^2(x,t)}} &\approx -T(1, u_x)\left(1 - \frac{1}{2}u_x^2 + \dots\right) \\ &\approx -T(1, u_x). \end{aligned}$$

It follows that the left force in the vertical direction is $-Tu_x(x,t)$. We repeat these considerations on the right end. The tangent vector is

$$\frac{(1, u_x(x+\Delta x,t))}{\sqrt{1+u_x^2(x+\Delta x,t)}},$$

so that the force is

$$\begin{aligned} \frac{T(1, u_x(x+\Delta x,t))}{\sqrt{1+u_x^2(x+\Delta x,t)}} &\approx -T(1, u_x(x+\Delta x))\left(1 - \frac{1}{2}u_x^2(x+\Delta x,t) + \dots\right) \\ &\approx -T(1, u_x(x+\Delta x,t)). \end{aligned}$$

Thus, after ignoring the nonlinear terms, the right force in the vertical direction is given by $Tu_x(x+\Delta x,t)$.

Combining all of this, Newton's law becomes

$$\begin{aligned} && \rho\Delta x u_{tt}(x,t) &= T(u_x(x+\Delta x,t) - u_x(x,t)) \\ \Rightarrow && \rho u_{tt}(x,t) &= T\frac{u_x(x+\Delta x,t) - u_x(x,t)}{\Delta x}. \end{aligned}$$

Taking the limit $\Delta x \to 0$, we obtain

$$\rho u_{tt} = T u_{xx},$$

or

$$u_{tt} = c^2 u_{xx},$$

where

$$c^2 = \frac{T}{\rho},$$

and we have derived the wave equation! We have already seen that $f(x-ct)$ and $f(x+ct)$ are traveling wave solutions of the wave equation. It follows that their velocity is given by

$$\pm c = \pm\sqrt{\frac{T}{\rho}}.$$

2.10 d' Alembert's solution of the wave equation

We want to find the **general** solution of the wave equation

$$u_{tt} = c^2 u_{xx}.$$

We already know that this equation has traveling wave solutions $f(x - ct)$ and $g(x + ct)$, for arbitrary profiles f and g. We show that **any** solution of the wave equation is a linear combination of such traveling waves.

Inspired by the form of the traveling waves, we use a coordinate transformation

$$\begin{aligned} \xi &= x - ct, \\ \eta &= x + ct. \end{aligned}$$

Let us find out how the derivatives transform. First, we consider u_x.

$$\begin{aligned} \frac{\partial u}{\partial x} &= \frac{\partial u}{\partial \xi}\frac{\partial \xi}{\partial x} + \frac{\partial u}{\partial \eta}\frac{\partial \eta}{\partial x} \\ &= u_\xi \cdot 1 + u_\eta \cdot 1 \\ &= u_\xi + u_\eta. \end{aligned}$$

Next up, the second derivative u_{xx}:

$$\begin{aligned} \frac{\partial^2 u}{\partial x^2} &= \frac{\partial}{\partial x}(u_\xi + u_\eta) \\ &= \frac{\partial}{\partial \xi}(u_\xi + u_\eta)\frac{\partial \xi}{\partial x} + \frac{\partial}{\partial \eta}(u_\xi + u_\eta)\frac{\partial \eta}{\partial x} \\ &= \frac{\partial}{\partial \xi}(u_\xi + u_\eta) \cdot 1 + \frac{\partial}{\partial \eta}(u_\xi + u_\eta) \cdot 1 \\ &= \frac{\partial}{\partial \xi}(u_\xi + u_\eta) + \frac{\partial}{\partial \eta}(u_\xi + u_\eta) \\ &= (u_{\xi\xi} + u_{\eta\xi}) + (u_{\xi\eta} + u_{\eta\eta}) \\ &= u_{\xi\xi} + 2u_{\xi\eta} + u_{\eta\eta}, \end{aligned}$$

where we have assumed that u is smooth, so that $u_{\xi\eta} = u_{\eta\xi}$. Next, we do the same for the t derivatives.

$$\begin{aligned} \frac{\partial u}{\partial t} &= \frac{\partial u}{\partial \xi}\frac{\partial \xi}{\partial t} + \frac{\partial u}{\partial \eta}\frac{\partial \eta}{\partial t} \\ &= u_\xi \cdot (-c) + u_\eta \cdot c \\ &= -cu_\xi + cu_\eta. \end{aligned}$$

Next up, the second derivative u_{tt}:

$$\begin{aligned}
\frac{\partial^2 u}{\partial x^2} &= \frac{\partial}{\partial x}(-cu_\xi + cu_\eta) \\
&= \frac{\partial}{\partial \xi}(-cu_\xi + cu_\eta)\frac{\partial \xi}{\partial t} + \frac{\partial}{\partial \eta}(-cu_\xi + cu_\eta)\frac{\partial \eta}{\partial t} \\
&= \frac{\partial}{\partial \xi}(-cu_\xi + cu_\eta) \cdot (-c) + \frac{\partial}{\partial \eta}(-cu_\xi + cu_\eta) \cdot c \\
&= -c\frac{\partial}{\partial \xi}(-cu_\xi + cu_\eta) + c\frac{\partial}{\partial \eta}(-cu_\xi + cu_\eta) \\
&= -c(-cu_{\xi\xi} + cu_{\eta\xi}) + c(-cu_{\xi\eta} + cu_{\eta\eta}) \\
&= c^2 u_{\xi\xi} - 2c^2 u_{\xi\eta} + c^2 u_{\eta\eta}.
\end{aligned}$$

Substituting the expressions for u_{xx} and u_{tt} into the wave equation we get

$$\begin{aligned}
& & u_{tt} &= c^2 u_{xx} \\
\Rightarrow & & c^2 u_{\xi\xi} - 2c^2 u_{\xi\eta} + c^2 u_{\eta\eta} &= c^2(u_{\xi\xi} + 2u_{\xi\eta} + u_{\eta\eta}) \\
\Rightarrow & & u_{\xi\eta} &= 0.
\end{aligned}$$

This is fantastic progress! Now we proceed to solve this equation. From $u_{\xi\eta} = 0$, it follows that $(u_\xi)\eta = 0$, thus u_ξ is independent of η: it is a function of ξ only. Thus

$$u_\xi = F(\xi),$$

where F is any function. We integrate once more to find

$$u = \int^\xi F(\xi) d\xi + g(\eta),$$

where g is any function of η. The first term is the anti-derivative of any function of ξ, so it's another arbitrary function of ξ. Let's denote it by f. Thus we have that any solution of the wave equation can be written in the form

$$u = f(\xi) + g(\eta),$$

or, returning to the original variables:

$$u = f(x - ct) + g(x + ct),$$

which proves our claim.

Example. You can easily show that $u = \cos t \sin x$ solves the wave equation with $c = 1$. Strange.... It doesn't appear to be of the form given above. But it is, as we show now. Fun with trig identities! From the trig addition formulas, we have

$$\begin{aligned}
\cos t \sin x + \sin t \cos x &= \sin(x + t) \\
\cos t \sin x - \sin t \cos x &= \sin(x - t),
\end{aligned}$$

from which it follows that

$$u(x,t) = \cos t \sin x = \frac{1}{2}(\sin(x+t) + \sin(x-t)),$$

which shows that u can indeed be written as the sum of a function which depends on $x+t$ only, and another one which depends on $x-t$ only.

Next, we will solve our first **initial-value problem** in this course. Consider the problem

$$\begin{aligned} u_{tt} &= c^2 u_{xx} \\ u(x,0) &= f(x) \\ u_t(x,0) &= g(x), \end{aligned}$$

where f and g are given functions. Here $f(x)$ represents the initial position of the string and $g(x)$ represents its initial velocity.

We know that $u(x,t)$ can be written as

$$u(x,t) = F(x-ct) + G(x+ct).$$

Our task, should we choose to accept it[8], is to find F and G in terms of f and g. At $t=0$, we have $u(x,t) = f(x)$, thus

$$f(x) = F(x) + G(x).$$

Taking a time derivative of our solution formula, we have

$$u_t(x,t) = -cF'(x-ct) + cG(x+ct).$$

Evaluating this at $t=0$, we get

$$g(x) = -cF'(x) + cG'(x).$$

Thus we need to solve the equations

$$\left\{ \begin{array}{rcl} F(x) + G(x) &=& f(x) \\ -cF'(x) + cG'(x) &=& g(x) \end{array} \right. ,$$

for the unknown functions F and G.

From the second equation,

$$\begin{aligned} -F'(x) + G'(x) &= \frac{1}{c} g(x) \\ \Rightarrow \qquad -F(x) + G(x) &= \frac{1}{c}\int_0^x g(s)ds + \alpha, \end{aligned}$$

[8] We will. We're awesome. We're fearless.

where α is an integration constant. Our system becomes

$$\begin{cases} F(x) + G(x) &= f(x) \\ -F(x) + G(x) &= \frac{1}{c}\int_0^x g(s)ds + \alpha. \end{cases},$$

Adding these two equations, we find that

$$G(x) = \frac{1}{2}f(x) + \frac{1}{2c}\int_0^x g(s)ds + \frac{\alpha}{2}.$$

Subtracting the two equations gives F:

$$F(x) = \frac{1}{2}f(x) - \frac{1}{2c}\int_0^x g(s)ds - \frac{\alpha}{2}.$$

Putting all of this together, we find that

$$\begin{aligned} u(x,t) &= \frac{1}{2}f(x-ct) - \frac{1}{2c}\int_0^{x-ct} g(s)ds - \frac{\alpha}{2} + \frac{1}{2}f(x+ct) + \frac{1}{2c}\int_0^{x+ct} g(s)ds + \frac{\alpha}{2} \\ &= \frac{1}{2}f(x-ct) + \frac{1}{2c}\int_{x-ct}^{0} g(s)ds + \frac{1}{2}f(x+ct) + \frac{1}{2c}\int_0^{x+ct} g(s)ds \\ &= \frac{1}{2}(f(x-ct) + f(x+ct)) + \frac{1}{2c}\int_{x-ct}^{x+ct} g(s)ds. \end{aligned}$$

This is the **d' Alembert** solution for the initial-value problem for the wave equation.

Don't take this for granted. The wave equation is one of very few PDEs for which we can write down the solution of the initial-value problem so explicitly!

Example. Consider the initial-value problem

$$\begin{aligned} u_{tt} &= c^2 u_{xx} \\ u(x,0) &= e^{-x^2} \\ u_t(x,0) &= 0, \end{aligned}$$

then d'Alembert gives

$$u(x,t) = \frac{1}{2}\left(e^{-(x-ct)^2} + e^{-(x+ct)^2}\right).$$

Thus the solution splits the initial condition in two parts: one goes to the left, one goes to the right. Both propagate with speed c.

2.11 Characteristics for the wave equation

Consider the transport equation

$$u_t + cu_x = 0.$$

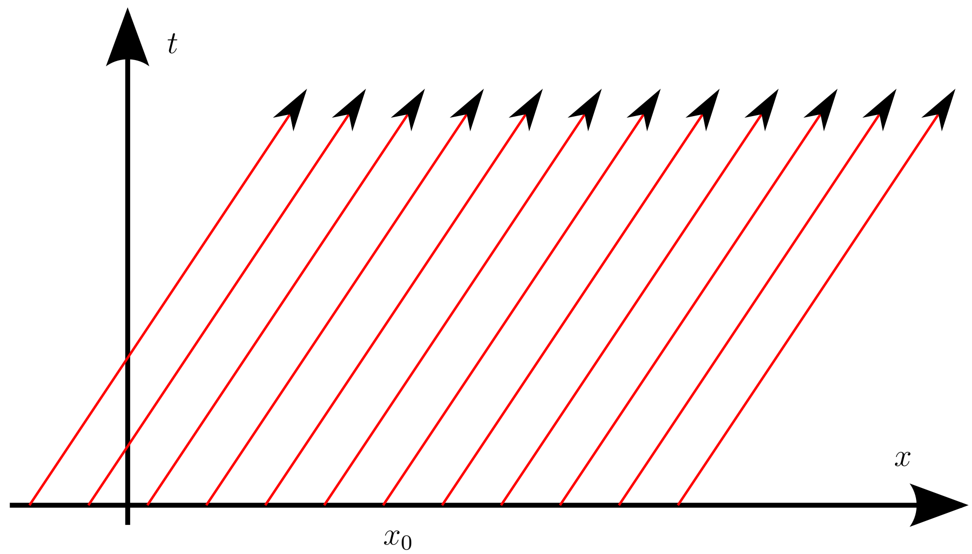

Figure 2.8: The initial condition is transported along the characteristics.

Using the transformation

$$\xi = x - ct,$$
$$\tau = t,$$

we see that the general solution of this problem is

$$u = f(x - ct),$$

where $f(x) = u(x, 0)$, the initial profile. Thus, the transport equation simply moves the initial profile to the right with speed c, where we have assumed that $c > 0$.

This is illustrated in Fig. 2.8. The straight lines $x - ct = x_0$, originating at $(x_0, 0)$ are called the **characteristics**. It follows that the value of $u(x, t)$ along these characteristics is given by

$$u(x, t) = f(x - ct) = f(x_0).$$

Thus, u is constant along the characteristics: if we know u anywhere along a characteristic, when we know it anywhere on that characteristic.

We aim to get a similar understanding for the dynamics of the wave equation

$$u_{tt} = c^2 u_{xx}.$$

Recall that the d'Alembert solution gives

$$u(x, t) = \frac{1}{2}(u(x - ct, 0) + u(x + ct, 0)) + \frac{1}{2c}\int_{x-ct}^{x+ct} u_t(s, 0) ds.$$

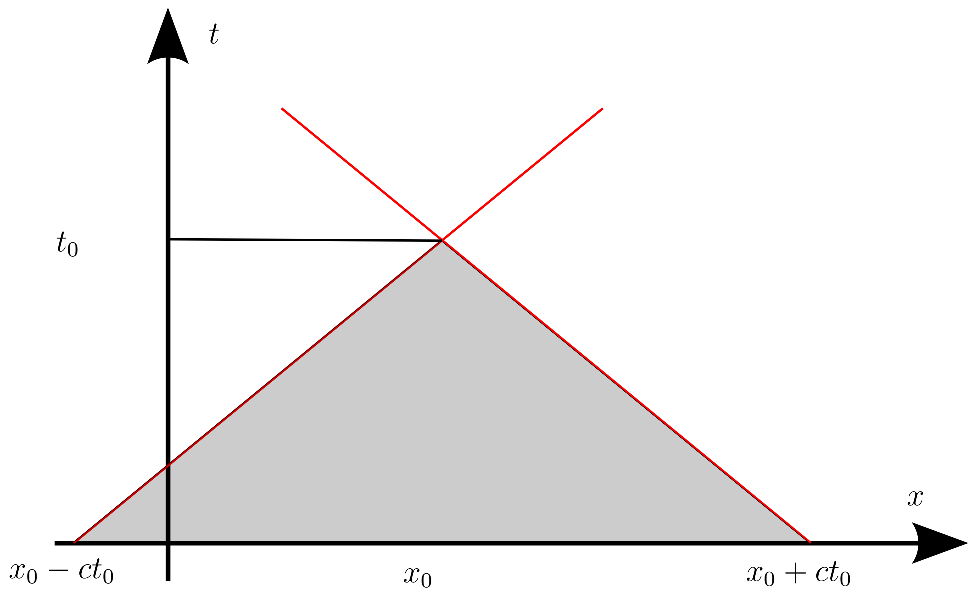

Figure 2.9: The domain of dependence (grey) for the solution of the wave equation.

It is clear from this formula that the value of u at $(x = x_0, t = t_0)$ is determined only by the initial values for $x \in [x_0 - ct_0, x_0 + ct_0]$. This is illustrated in Fig. 2.9.

The cone given by the grey region in Fig. 2.9 is called the **domain of dependence**. It indicates that the solution at place x_0 and time t_0 depends on all values in that cone, but on no values outside of it.

If we consider the special situation where

$$u_t(0, t) = 0,$$

in other words, the string starts with an initial profile, but is released without any initial velocity, then d'Alembert giveth

$$u(x, t) = \frac{1}{2}(u(x - ct, 0) + u(x + ct, 0)),$$

which shows that u depends only on the initial condition on the boundary of the domain of dependence. These two boundary lines are called the **characteristics for the wave equation**. Thus, the characteristics determine the domains of dependence.

By turning all of the above around, we can answer the question of which (x, t) points have a given point (x_0, t_0) in their domain of dependence. This is known as the **domain of influence**. It is illustrated in Fig. 2.10. The domain of influence is also bordered by the characteristics, but now on the lower side.

The domain of influence shows that information in the wave equation travels at a finite speed c: it takes a definite time for the effect from x_0 to be felt at any other x.

Example. Sometimes the characteristics may be used to examine the solution of an

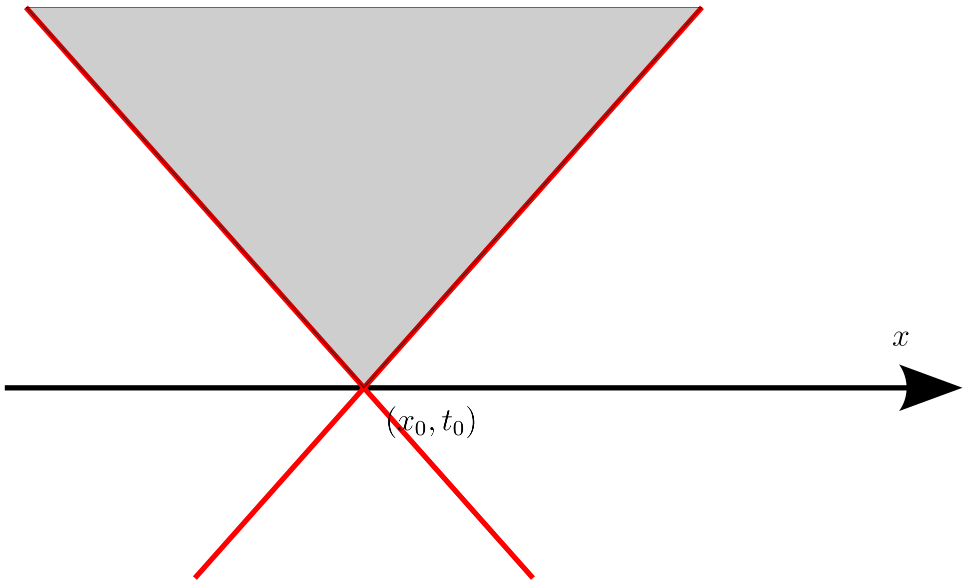

Figure 2.10: The domain of influence (grey) of the point (x_0, t_0) for the wave equation.

initial-value problem. Consider the problem

$$\begin{aligned} u_{tt} &= 4u_{xx} \\ u(x,0) &= \begin{cases} 1 & \text{if } x \in [0,1] \\ 0 & \text{if } x \notin [0,1] \end{cases} \\ u_t(x,0) &= 0. \end{aligned}$$

Here the speed of propagation is 2:

$$c^2 = 4 \;\Rightarrow\; c = 2.$$

We have the set-up illustrated in Fig. 2.11.

From this characteristic plot we can immediately read off the values of $u(x,t)$, for any x and t. For instance, we can plot the solution $u(x,t)$ at different instances of t, by taking horizontal slices of the characteristic plot above. This is illustrated in Fig. 2.12.

2.12 The wave equation on the semi-infinite domain

Fixed end

Let's consider a string of semi-infinite extent, fixed at the left end, as drawn in Fig. 2.13.

The condition $u(0,t) = 0$ for all t is our first example of a **boundary condition**: a

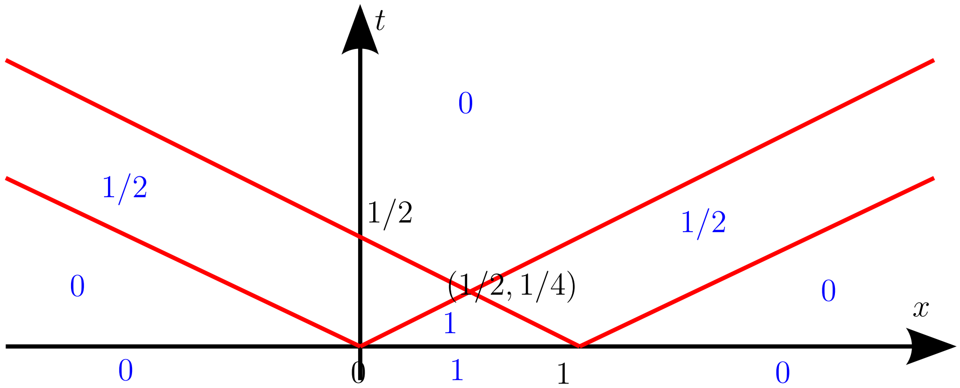

Figure 2.11: The different solution regions for the example problem. The values in blue are values of $u(x,t)$ in that region. The characteristics drawn have slope 1/2.

condition on the function we are looking for, given at the end of our physical domain.

Thus, we want to solve the boundary-value problem

$$\left\{\begin{array}{rcl} u_{tt} &=& c^2 u_{xx}, \quad x\in(0,\infty) \\ u(0,t) &=& 0, \quad \text{for all } t>0 \\ u(x,0) &=& f(x), \quad x\in(0,\infty) \\ u_t(x,0) &=& g(x), \quad x\in(0,\infty) \end{array}\right. ,$$

where $f(x)$ and $g(x)$ are given functions. We will solve this using d'Alembert's solution and the insight we gained from using characteristics. From d'Alembert,

$$u(x,y)=\frac{1}{2}(f(x-ct)+f(x+ct))+\frac{1}{2c}\int_{x-ct}^{x+ct} g(s)ds.$$

This formula can only be valid if $x-ct>0$. Otherwise, we'd be asking to evaluate $f(x)$ at arguments that are negative, but we have been given $f(x)$ only for positive arguments. A conundrum! Thus the solution ceases to be valid when

$$x-ct<0 \quad \Rightarrow \quad t>x/c.$$

This region is illustrated in Fig. 2.14.

This is not surprising: the shaded region in Fig. 2.14 would be a part of the domain of influence of negative x values, but we have no information there. So, what happens? Boundary condition to the rescue: we have not used the condition $u(0,t)=0$ for all t. We will do so now.

In the shaded region, we still have that

$$u(x,t)=F_1(x-ct)+G(x+ct),$$

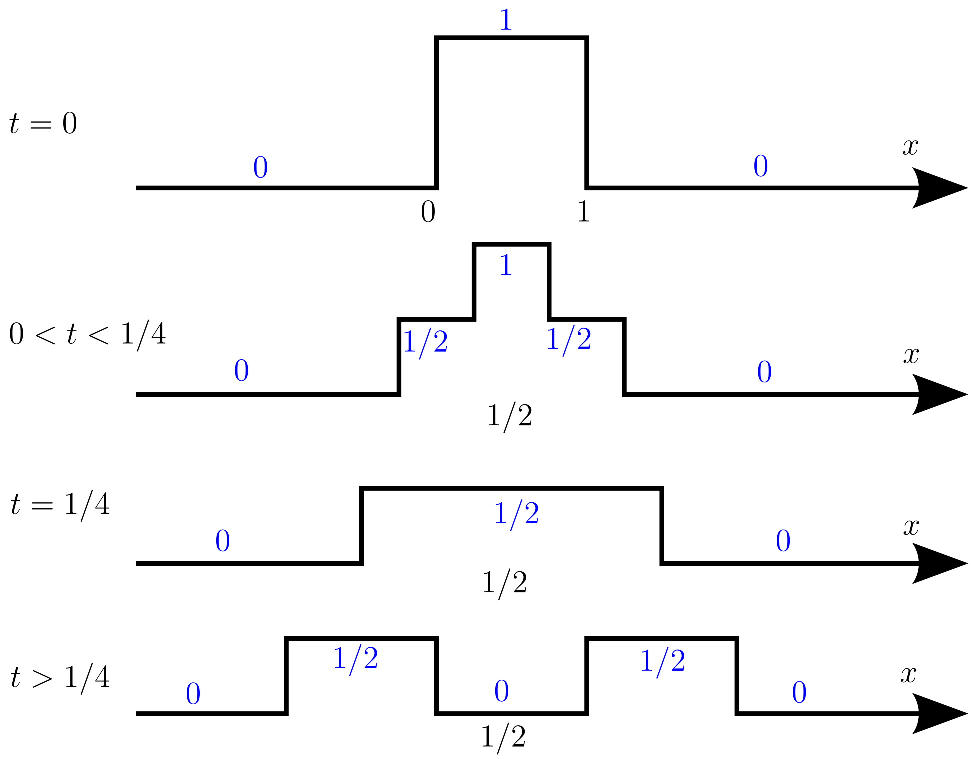

Figure 2.12: The different stages of the solution. As before, values of u are in blue.

since all solutions of the wave equation are of this form. Using the boundary condition, we have

$$\begin{aligned} & 0 = F_1(-ct) + G(ct) \\ \Rightarrow \quad & F_1(z) = -G(-z), \end{aligned}$$

so that

$$u(x,t) = -G(-x+ct) + G(x+ct).$$

Using the initial conditions, we find that

$$G(z) = \frac{1}{2}f(z) + \frac{1}{2c}\int_0^z g(s)ds,$$

as before. Note that to determine $G(x+ct)$, it is fine to use the initial conditions, since the information in $G(x+ct)$ comes only from positive values of x. Alternatively, we can impose the continuity of our solution at $x = ct$, finding the same result for $G(x)$. Either way, it

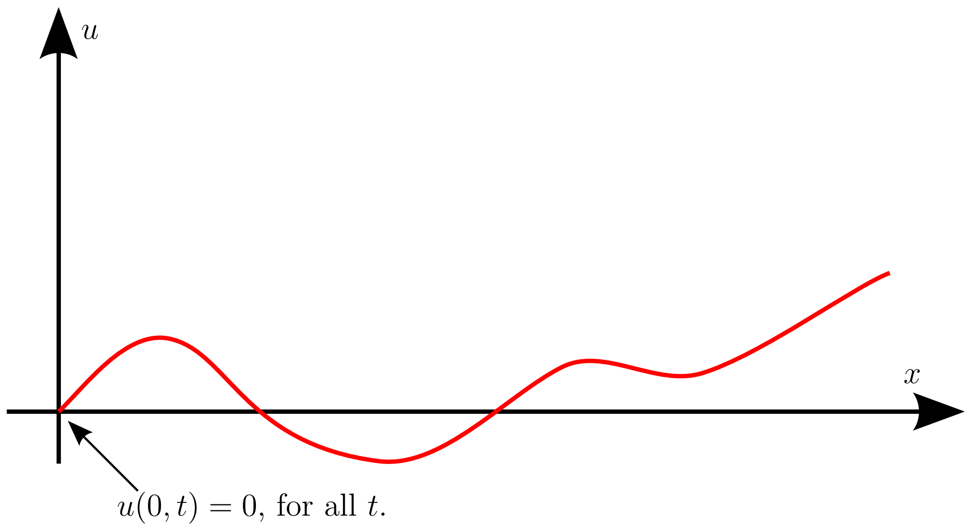

Figure 2.13: The wave equation string, on the half line $x > 0$ with fixed end $u(0, t) = 0$, for all t.

follows that

$$\begin{aligned} u(x,t) &= \frac{1}{2}f(x+ct) + \frac{1}{2c}\int_0^{x+ct} g(s)ds - \frac{1}{2}f(-x+ct) - \frac{1}{2c}\int_0^{-x+ct} g(s)ds \\ &= \frac{1}{2}(f(x+ct) - f(ct-x)) + \frac{1}{2c}\int_{ct-x}^{ct+x} g(s)ds, \end{aligned}$$

which is valid for $t > x/c$, while we still have that

$$u(x,t) = \frac{1}{2}(f(x-ct) + f(x+ct)) + \frac{1}{2c}\int_{x-ct}^{x+ct} g(s)ds,$$

for $t \le x/c$.

Let's examine the characteristics to make sense out of this, see Fig. 2.15. At $x = 0$, the region of dependence would usually come partially from $x < 0$, but that is not allowed now. Rather, all the information comes from $x > 0$, and from the boundary itself. We can **pretend** there is an $x < 0$ region, as long as we always satisfy $u(0, t) = 0$, for all t. This is easily done: let us extend u for $x < 0$ to be the opposite of u for the corresponding positive value $-x$. In other words, the overall $u(x, t)$ is odd, as a function of x: $u(-x, t) = -u(x, t)$:

$$\hat{u}(x,t) = \begin{cases} u(x,t), & x \ge 0 \\ -u(-x,t) & x \le 0 \end{cases},$$

then clearly $\hat{u}(0, t) = 0$. Also, the function behaves as if it switches sign every time it hits the boundary, as the derivative will be even[9]. Indeed, when the value from point (a) in Fig. 2.15 hits the boundary, it meets the value from (b), which has the opposite value. This opposite value continues on for $x > 0$. Thus, in effect, the value flips at the boundary.

[9] Recall that the derivative of an even function is odd and the derivative of an odd function is even. Whoa!

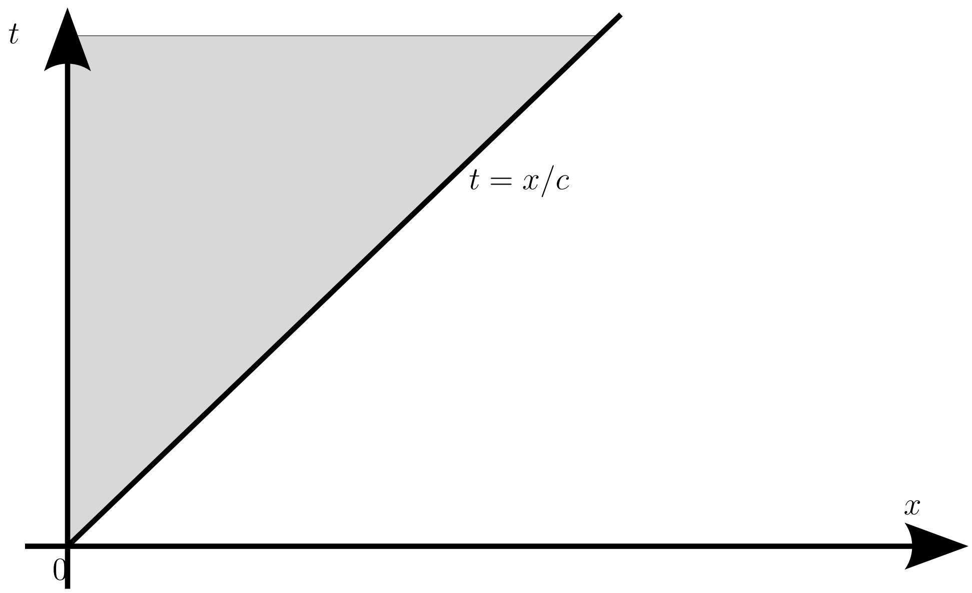

Figure 2.14: Our original d'Alembert solution is not valid in the shaded region, which corresponds to $t > x/c$.

Free end

We consider the boundary-value problem

$$\left\{ \begin{array}{rcl} u_{tt} & = & c^2 u_{xx}, \quad x \in (0,\infty) \\ u_x(0,t) & = & 0, \quad \text{for all } t > 0 \\ u(x,0) & = & f(x), \quad x \in (0,\infty) \\ u_t(x,0) & = & g(x), \quad x \in (0,\infty) \end{array} \right. ,$$

Thus, according to the boundary condition, the string is horizontal at the boundary, for all time. We proceed as before. By d'Alembert,

$$u(x,t) = \frac{1}{2}(f(x-ct) + f(x+ct)) + \frac{1}{2c}\int_{x-ct}^{x+ct} g(s)ds.$$

By the same reasoning as for the fixed-end case, this is only valid when $x - ct > 0$, and another solution form has to be found in the shaded region of Fig. 2.14. Again this is expected: in the shaded region, part of the solution would come from $x < 0$, but we have no information there.

As always, we have that in the shaded region[10]

$$u(x,t) = F_2(x-ct) + G(x+ct),$$

[10]Because we're solving the wave equation.

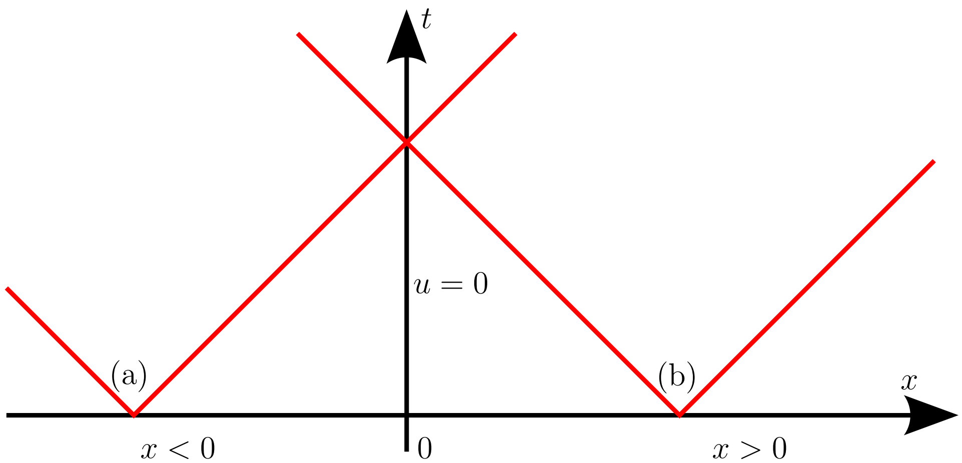

Figure 2.15: The characteristics for the wave equation on the half line, including the "phantom" characteristics for $x < 0$.

where $G(z)$ is determined as before (once again: the information in $G(z)$ comes only from positive x values):

$$G(z) = \frac{1}{2}f(z) + \frac{1}{2c}\int_0^z g(s)ds.$$

Thus

$$u(x,t) = F_2(x-ct) + \frac{1}{2}f(x+ct) + \frac{1}{2c}\int_0^{x+ct} g(s)ds.$$

In order to impose the boundary condition, we calculate

$$u_x(x,t) = F_2'(x-ct) + \frac{1}{2}f'(x+ct) + \frac{1}{2c}g(x+ct).$$

Evaluating this at $x = 0$, we get

$$\begin{aligned} & 0 = F_2'(-ct) + \frac{1}{2}f'(ct) + \frac{1}{2c}g(ct) \\ \Rightarrow \qquad & F_2'(-ct) = -\frac{1}{2}f'(ct) - \frac{1}{2c}g(ct). \end{aligned}$$

Equating $-ct = z$, we get

$$\begin{aligned} & F_2'(z) = -\frac{1}{2}f'(-z) - \frac{1}{2c}g(-z) \\ \Rightarrow \qquad & F_2(z) = \frac{1}{2}f(-z) + \frac{1}{2c}\int_0^{-z} g(s)ds, \end{aligned}$$

by the fundamental theorem of calculus. Thus

$$F_2(x-ct) = \frac{1}{2}f(ct-x) - \frac{1}{2c}\int_{ct-x}^0 g(s)ds,$$

so that for $t > x/c$,

$$u(x,t) = \frac{1}{2}(f(x+ct) + f(ct-x)) + \frac{1}{2c}\int_0^{ct-x} g(s)ds + \frac{1}{2c}\int_0^{ct+x} g(s)ds.$$

If we let $g(x) \equiv 0$, we can understand this result, using the characteristics, see Fig. 2.15. In order to satisfy the boundary condition, we extend $u(x,0)$ to negative values to be an even function. Then its derivative will be odd, and its value at 0 will be 0. We see that at (x,t), there are two contributions:

- One from $(ct+x,0)$ and one from $(x-ct,0)$.
- The first one is $f(x+ct)/2$, the second is given by $f(ct-x)/2$, by our result above.

2.13 Standing wave solutions of the wave equation

Standing waves are different from traveling waves in that they stay where they are, but their profile may change over time.

Example. The wave profile

$$u(x,t) = 2\cos t \sin x$$

represents a standing wave. It looks like $2\sin x$ at all time t, but multiplied by a time-dependent amplitude $\cos t$. It is illustrated in Fig. 2.16. It is clear[11] that standing waves can often be written as linear combinations of traveling waves. Here we have

$$u(x,t) = \sin(x-t) + \sin(x+t),$$

but this is not always the case.

To look for traveling wave solutions of the wave equation

$$u_{tt} = c^2 u_{xx},$$

we let

$$u(x,t) = T(t)X(x).$$

Thus $X(x)$ represents the spatial profile, while $T(t)$ gives the temporal part of the profile, its time-dependent amplitude. We get

$$\begin{aligned} & T''(t)X(x) = c^2 T(t)X''(x) \\ \Rightarrow \quad & \frac{T''}{T} = c^2\frac{X''}{X}. \end{aligned}$$

[11] As clear as some trig identities.

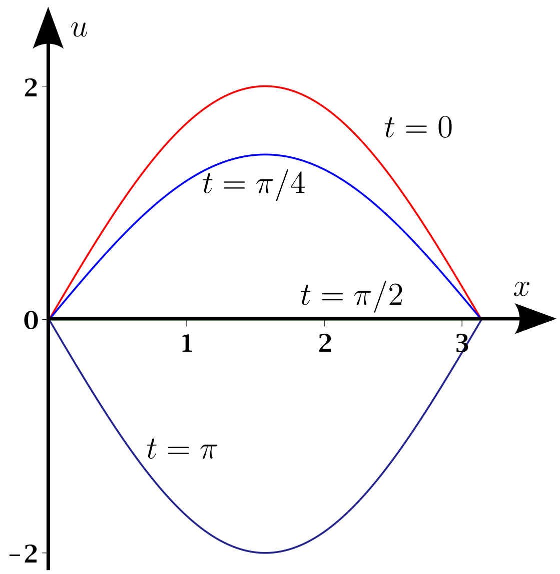

Figure 2.16: The standing wave profile $u = 2\cos t \sin x$, for different values of t.

Note that the left-hand side depends on t only, while the right-hand side is a function of only x. It follows that both sides have to be constant. Indeed, if we take an x derivative of the equation, we get

$$0 = c^2 \left(\frac{X''}{X}\right)',$$

since the left-hand side does not depend on x. Thus $c^2 X''/X$ is constant. The same argument, but by taking a derivative with respect to t, gives that T''/T is constant too. Thus we have

$$\frac{T''}{T} = c^2 \frac{X''}{X} = \lambda,$$

where λ is a constant. We get two ordinary differential equations:

$$\begin{cases} T'' & = & \lambda T, \\ X'' & = & \frac{\lambda}{c^2} X. \end{cases}$$

Here the first equation is an ODE in t, while the second one depends on x.

As you know, we get different kinds of solutions, depending on the sign of λ. Let us investigate all possibilities.

- **$\lambda = 0$.** we get $T'' = 0$ and $X'' = 0$. It follows that $T = A + Bt$ and $X = C + Dx$. Thus

$$u = (C + Dx)(A + Bt).$$

  In particular, we want $D = 0$ and $B = 0$, if $x \in \mathbb{R}$, since we want bounded solutions. On the other hand, if x is restricted to a smaller domain, the solution with $D \neq 0 \neq B$ may be perfectly acceptable.

- $\lambda > 0$. We write $\lambda = r^2$, where $r > 0$. We have to solve

$$\begin{cases} T'' &= r^2 T, \\ X'' &= \frac{r^2}{c^2} X, \end{cases}$$

from which it follows that[12]

$$T = Ae^{rt} + Be^{-rt},$$

and

$$X = Ce^{rx/c} + De^{-rx/c},$$

resulting in

$$u = (Ae^{rt} + Be^{-rt})(Ce^{rx/c} + Be^{-rx/c}).$$

- $\lambda < 0$. For the last case, we write $\lambda = -r^2$, with $r > 0$. We have to solve

$$\begin{cases} T'' &= -r^2 T, \\ X'' &= -\frac{r^2}{c^2} X, \end{cases}$$

from which it follows that

$$T = A\cos(rt + B\sin(rt),$$

and

$$X = C\cos(rx/c) + D\sin(rx/c),$$

resulting in

$$u = (A\cos(rt + B\sin(rt))(C\cos(rx/c) + D\sin(rx/c)).$$

All of these result in standing wave solutions of the wave equation. Next, we impose some boundary conditions.

Standing waves on a finite string

All the profiles above solve the wave equation. If we impose boundary conditions, many of them are rules out and only some remain. As an important example, let's consider the wave equation on a finite interval with fixed ends, like a guitar string of length L. We have, for $x \in (0, L)$, $t > 0$,

$$\begin{aligned} u_{tt} &= c^2 u_{xx}, \\ u(0,t) &= 0, \\ u(L,t) &= 0. \end{aligned}$$

[12] Or, you could use hyperbolic functions. Always fun!

We have that

$$u(x,t) = X(x)T(t),$$

for standing waves. We want $u(0,t) = 0$, which implies that

$$X(0)T(t) = 0.$$

Since we don't want that $T(t) = 0$[13], we need that $X(0) = 0$. Similarly, we need that $X(L) = 0$. We already know the allowed forms for $X(x)$:

$$\begin{aligned} & X(x) = C + Dx, \\ \text{or} \qquad & X(x) = Ce^{rx/c} + De^{-rx/c}, \\ \text{or} \qquad & X(x) = C\cos(rx/c) + D\sin(rx/c). \end{aligned}$$

We now impose the conditions

$$X(0) = 0, \quad X(L = 0),$$

on these possibilities.

- With $X = C + Dx$, we get

  $$\begin{aligned} X(0) &= 0 = C, \\ X(L) &= 0 = C + DL, \end{aligned}$$

  from which it follows that both C and D are zero, so that $X = 0$. Not interesting.

- Next, we consider $X = Ce^{rx/c} + De^{-rx/c}$. Imposing $X(0) = 0$, we get

  $$C + D = 0 \quad \Rightarrow \quad D = -C.$$

  This allows us to rewrite $X(x)$ as

  $$X(x) = C(e^{rx/c} - e^{-rx/c}) = 2C\sinh(rx/c).$$

  Next, we impose $X(L) = 0$. We get

  $$0 = 2C\sinh(rL/c).$$

  Clearly[14], we don't want $C = 0$, since this would result in $u(x,t) = 0$. That is not exciting. If we want excitement, we have to impose

  $$\sinh(rL/c) = 0.$$

  Unfortunately, the sinh function is zero only when it's argument is zero, which would imply $r = 0$, since $L \neq 0$ (otherwise we'd have no string. But $r = 0$ is not allowed for this case, since $r = 0$ implies $\lambda = 0$, which was the previous case. Bummer. All that work and no solutions. What is this? A course on how not to find solutions to PDEs? Patience, my young apprentices...

[13] Resulting in only the extremely boring $u = 0$ solution...

[14] The most hated word in a math text, with the possible exception of "obviously".

- Inevitably, we end up with all our money on the last case[15]. Imposing $X(0) = 0$, with

  $$X(x) = C\cos(rx/c) + D\sin(rx/c),$$

  we get

  $$C = 0.$$

  Not a good start! We've just thrown out half of our last remaining non-zero solutions. This leaves us with

  $$X(x) = D\sin(rx/c).$$

  Next we impose $X(L) = 0$. We obtain

  $$D\sin(rL/c) = 0.$$

  Since we don't want $D = 0$, we need $\sin(rL/c) = 0$, which is satisfied if

  $$\frac{rL}{c} = n\pi,$$

  where n is any integer. Since r is not allowed to be zero, we have to exclude $n = 0$. Further, since we may assume that $r > 0$, we need only consider $n \in \mathbb{Z}_0^+$, the set of strictly positive integers: $n \in \{1, 2, 3, 4, \ldots\}$. Thus we have found

  $$X_n(x) = \sin\frac{n\pi x}{L}, \quad n = 1, 2, \ldots$$

  We can ignore the constant multiplication factor D, since it can be absorbed into the multiplying function $T(t)$, which we still have to determine. We have endowed $X(x)$ with an index n, to distinguish the different solutions we have found.

In summary, we find that the only standing wave solutions of the wave equation that satisfies the boundary conditions for a fixed finite-length string are given by

$$u_n(x,t) = (A\cos\left(\frac{n\pi ct}{L}\right) + B\sin\left(\frac{n\pi ct}{L}\right))\sin\left(\frac{n\pi x}{L}\right),$$

for all $n = 1, 2, \ldots$.

Modes of vibration

We call u_n the n-th mode of vibration of the string. The wave number of the n-th mode is $n\pi/L$. Over the domain $x \in (0, L)$, the solution has exactly $n-1$ zeros[16] Note that all of the X_n are sin functions, with increasing wave number as n increased. In other words, for

[15] This statement is in no way an endorsement of any gambling activity. The University of Washington and its employees do not encourage gambling in any way or form. Except for solving PDEs.

[16] You should convince yourself of this statement. If you have extra time, convince your neighbor too.

larger n, X_n has a smaller period. For increasing n we are simply cramming more periods in the interval $[0, L]$.

We briefly revisit the x-problem we solved:

$$\begin{aligned} c^2 X'' &= \lambda X, \\ X(0) &= 0, \\ X(L) &= 0. \end{aligned}$$

This is called a **Sturm-Liouville** problem for $X(x)$. The constant λ is called the **eigenvalue**, $X(x)$ the eigenfunction corresponding to λ.

The method we have used to solve for the modes of the wave equation is called **Separation of Variables**, because we looked for that solutions that are multiples of functions that depend on x and t separately.

In the next chapter, we look at how we can construct very general solutions from linear superpositions of these standing wave solutions.

Chapter 3

Fourier series and solutions of partial differential equations on a finite interval

3.1 Superposition of standing waves

Since the wave equation is linear and homogeneous, we should be able to superimpose different solutions, in order to get new solutions.

Consider the problem

$$\begin{aligned} u_{tt} &= c^2 u_{xx}, && x \in (0, L), t > 0, \\ u(0,t) &= 0, && t > 0, \\ u(L,t) &= 0, && t > 0. \end{aligned}$$

We show explicitly that

$$v = \alpha_1 u_1(u, t) + \alpha_2 u_2(x, t)$$

solves this problem, provided that u_1 and u_2 do. Here α_1 and α_2 are constants.

First, we check that v satisfies the PDE:

$$\begin{aligned} (\alpha_1 u_1 + \alpha_2 u_2)_{tt} &= \alpha_1 u_{1tt} + \alpha_2 u_{2tt} \\ &= \alpha_1 c^2 u_{1xx} + \alpha_2 c^2 u_{2xx} \\ &= c^2(\alpha_1 u_{1xx} + \alpha_2 u_{2xx}) \\ &= c^2(\alpha_1 u_1 + \alpha_2 u_2)_{xx}. \end{aligned}$$

It follows that $v_{tt} = c^2 v_{xx}$.

Next we verify that v satisfies the boundary conditions. We have

$$v(0,t) = \alpha_1 u_1(0,t) + \alpha_2 u_2(0,t) = \alpha_1 \cdot 0 + \alpha_2 \cdot 0 = 0,$$

and

$$v(L,t) = \alpha_1 u_1(L,t) + \alpha_2 u_2(L,t) = \alpha_1 \cdot 0 + \alpha_2 \cdot 0 = 0.$$

This proves what we had to show.

Example. But, be careful: this does not work for nonhomogeneous problems, even if the nonhomogeneous part is in the boundary conditions. For instance, consider

$$\begin{aligned} u_{tt} &= c^2 u_{xx}, && x \in (0, L), t > 0, \\ u(0,t) &= 0, && t > 0, \\ u(L,t) &= 1, && t > 0. \end{aligned}$$

As you repeat the calculation above for this problem, you encounter a problem with the second boundary condition. Bummer!

The above implies we can add our standing wave solutionsto get new, more general solutions of the wave equation. Let

$$\begin{aligned} u(x,t) &= \sum_{n=1}^{N} u_n(x,t) \\ &= \sum_{n=1}^{N} \left[A_n \cos\left(\frac{n\pi ct}{L}\right) + B_n \sin\left(\frac{n\pi ct}{L}\right)\right] \sin\left(\frac{n\pi x}{L}\right). \end{aligned}$$

This is a superposition of a finite-number of standing waves. Any constant choice of A_n and B_n gives a new solution. Thus $u(x,t)$ depends on $2N$ parameters. The idea would be to pick these parameters A_n and B_n so we can satisfy the initial conditions, if any are given.

Example. Consider the initial-value problem

$$\begin{aligned} u_{tt} &= u_{xx}, && 0 < x < 1,\ t > 0, \\ u(0,t) &= 0, && t > 0, \\ u(1,t) &= 0 && t > 0, \\ u(x,0) &= 0, && 0 < x < 1, \\ u_t(x,0) &= 2\sin(\pi x) - 3\sin(4\pi x), && 0 < x < 1. \end{aligned}$$

Let

$$u(x,t) = \sum_{n=1}^{N} \left[A_n \cos(n\pi t) + B_n \sin(n\pi t)\right] \sin(n\pi x).$$

At this point, we have satisfied the PDE and the boundary conditions. From the initial conditions, we get

$$u(x,0) = 0 = \sum_{n=1}^{N} A_n \sin(n\pi x),$$

and

$$\begin{aligned}
u_t(x,0) &= \sum_{n=1}^{N} \left[(-A_n)n\pi \sin(n\pi t) + B_n n\pi \cos(n\pi t)\sin(n\pi x)\right]\big|_{t=0} \\
&= \sum_{n=1}^{N} B_n n\pi \sin(n\pi x) \\
&= 2\sin \pi x - 3\sin 4\pi x.
\end{aligned}$$

The first equation is easily satisfied by choosing

$$A_n = 0,$$

for all n. Next, we can choose $N = 4$, so that the second equation becomes

$$B_1 \pi \sin \pi x + B_2 2\pi \sin 2\pi x + B_3 3\pi \sin 3\pi x + B_4 4\pi \sin 4\pi x = 2\sin \pi x - 3\sin 4\pi x,$$

which is solved by choosing

$$B_1 = \frac{2}{\pi}, \quad B_2 = 0, \quad B_3 = 0, \quad B_4 = \frac{-3}{4\pi}.$$

This solves the given initial-value problem.

This whole thing feels like a cheat, right? Those were very special initial conditions! Can we do something in general? For starters, can we show that the above is the *unique* solution to the problem? The answers to these questions will be "Yes!" and "Yes!"[1].

Let's kick this up a notch©. How about if we include **all** standing wave solutions we know? Can we let $N \to \infty$? We would have

$$u(x,t) = \sum_{n=1}^{\infty} \left[A_n \cos\left(\frac{n\pi ct}{L}\right) + B_n \sin\left(\frac{n\pi ct}{L}\right)\right] \sin\left(\frac{n\pi x}{L}\right).$$

Does this make sense? Perhaps. If $A_n, B_n \to 0$ sufficiently fast, we might get a convergent series. Such a convergent series is called a **Fourier series**. The big trick is to see whether we can find A_n and B_n such that we can satisfy general initial conditions, not just the really special ones we used above. Perhaps this is possible.

Example. Suppose the initial condition is given as

$$\begin{aligned}
u(x,0) &= \sin \pi x - \frac{1}{9}\sin 3\pi x + \frac{1}{25}\sin 5x - \ldots \\
&= \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{(2n-1)^2} \sin(2n-1)\pi x,
\end{aligned}$$

and $u_t(x,0) = 0$. Clearly, we would need

$$A_1 = 1,\ A_2 = 0,\ A_3 = -\frac{1}{9},\ A_4 = 0,\ A_5 = \frac{1}{25},\ \ldots$$

Can we do this for more general initial conditions?

[1] Yes, with exclamation points.

3.2 Fourier series

Here is the concrete question we will answer: given $f(x)$, $c \in [-L, L]$, we wish to write $f(x)$ as a linear combination of sines and cosines, where all of these have period $2L$. Thus, we'd want the terms in our linear combination to contain

$$\sin\left(\frac{n\pi x}{L}\right) \quad \text{and} \quad \cos\left(\frac{n\pi x}{L}\right),$$

for $n = 0,\ 1,\ 2,\ \ldots$[2]. Before we proceed, here are some identities we need that will be very helpful:

$$\begin{aligned}
\int_{-L}^{L} \sin\left(\frac{n\pi x}{L}\right) \sin\left(\frac{m\pi x}{L}\right) dx &= L\delta_{nm}, \\
\int_{-L}^{L} \cos\left(\frac{n\pi x}{L}\right) \cos\left(\frac{m\pi x}{L}\right) dx &= L\delta_{nm}, \\
\int_{-L}^{L} \sin\left(\frac{n\pi x}{L}\right) \cos\left(\frac{m\pi x}{L}\right) dx &= 0,
\end{aligned}$$

where δ_{nm} is the **Kronecker delta**:

$$\delta_{nm} = \begin{cases} 1, & n = m, \\ 0, & n \neq m. \end{cases}$$

These are easy to prove. Let's look at the first one[3]. Recall the following trig identities:

$$\begin{aligned}
\sin\alpha \sin\beta - \cos\alpha \cos\beta &= -\cos(\alpha + \beta) \\
\sin\alpha \sin\beta + \cos\alpha \cos\beta &= \cos(\alpha - \beta) \\
\Rightarrow \qquad \sin\alpha \sin\beta &= \frac{-cos(\alpha + \beta) + \cos(\alpha - \beta)}{2}.
\end{aligned}$$

This allows for an easy evaluation of the integrals. Assuming that $n \neq m$, we get

$$\begin{aligned}
\int_{-L}^{L} \sin\left(\frac{n\pi x}{L}\right) \sin\left(\frac{m\pi x}{L}\right) dx &= \frac{1}{2} \int_{-L}^{L} \left[-\cos\left(\frac{(n+m)\pi x}{L}\right) + \cos\left(\frac{(n-m)\pi x}{L}\right) \right] dx \\
&= \frac{1}{2} \left[-\frac{\sin\left(\frac{(n+m)\pi x}{L}\right)}{(n+m)\pi/L} \right]_{-L}^{L} + \frac{1}{2} \left[\frac{\sin\left(\frac{(n-m)\pi x}{L}\right)}{(n-m)\pi/L} \right]_{-L}^{L} \\
&= \frac{1}{2} \left(-\frac{\sin(n+m)\pi + \sin(n+m)\pi}{(n+m)\pi/L} \right) + \\
&\qquad \frac{1}{2} \left(\frac{\sin(n-m)\pi + \sin(n-m)\pi}{(n-m)\pi/L} \right) \\
&= 0.
\end{aligned}$$

[2]Fourier was declared crazy by his contemporaries for wanting to answer this question, even though he was successful!

[3]You know the drill: you should look at the other two.

On the other hand, if $n = m$, then

$$\begin{aligned}\int_{-L}^{L} \sin\left(\frac{n\pi x}{L}\right)\sin\left(\frac{m\pi x}{L}\right)dx &= \int_{-L}^{L} \sin^2\left(\frac{n\pi x}{L}\right)dx\\ &= \frac{1}{2}\int_{-L}^{L}\left(1-\cos\left(\frac{2n\pi x}{L}\right)\right)dx\\ &= L,\end{aligned}$$

since the integral of the second term is zero. Combining these two results, we get the desired identity:

$$\int_{-L}^{L} \sin\left(\frac{n\pi x}{L}\right)\sin\left(\frac{m\pi x}{L}\right)dx = L\delta_{nm}.$$

The others are proven in a similar way.

Back to our main business: we wish to find a_n and b_n such that

$$f(x) = A + \sum_{n=1}^{\infty}\left(a_n\cos\left(\frac{n\pi x}{L}\right) + b_n\sin\left(\frac{n\pi x}{L}\right)\right).$$

Let's start by taking the average of this expression:

$$\frac{1}{2L}\int_{-L}^{L} f(x)dx = A + \frac{1}{2L}\int_{-L}^{L} dx\sum_{n=1}^{\infty}\left(a_n\cos\left(\frac{n\pi x}{L}\right) + b_n\sin\left(\frac{n\pi x}{L}\right)\right).$$

Since the average of all the sines and cosines is zero, we get

$$A = \frac{1}{2L}\int_{-L}^{L} f(x)dx.$$

Next, let $m = 1,\ 2, \ldots$. Then

$$f(x)\sin\left(\frac{m\pi x}{L}\right) = A\sin\left(\frac{n\pi x}{L}\right) + \sum_{n=1}^{\infty} a_n\cos\left(\frac{n\pi x}{L}\right)\sin\left(\frac{m\pi x}{L}\right) + \sum_{n=1}^{\infty} b_n\sin\left(\frac{n\pi x}{L}\right)\sin\left(\frac{m\pi x}{L}\right).$$

Once again, we integrate:

$$\begin{aligned}\int_{-L}^{L} f(x)\sin\left(\frac{m\pi x}{L}\right)dx = A\int_{-L}^{L}\sin\left(\frac{m\pi x}{L}\right)dx + + \sum_{n=1}^{\infty} a_n\int_{-L}^{L}\cos\left(\frac{n\pi x}{L}\right)\sin\left(\frac{m\pi x}{L}\right)dx\\ + \sum_{n=1}^{\infty} b_n\int_{-L}^{L}\sin\left(\frac{n\pi x}{L}\right)\sin\left(\frac{m\pi x}{L}\right)dx\\ = \sum_{n=1}^{\infty} b_n\int_{-L}^{L}\sin\left(\frac{n\pi x}{L}\right)\sin\left(\frac{m\pi x}{L}\right)dx.\end{aligned}$$

The only nonzero terms occur for $n = m$, so that

$$b_m = \frac{1}{L}\int_{-L}^{L} f(x)\sin\left(\frac{m\pi x}{L}\right)dx, \qquad m = 1, 2, \ldots.$$

Similarly, we get

$$a_m = \frac{1}{L}\int_{-L}^{L} f(x)\cos\left(\frac{m\pi x}{L}\right)dx, \qquad m = 1, 2, \ldots.$$

Note that $A = a_0/2$, so that we may write

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty}\left(a_n\cos\left(\frac{n\pi x}{L}\right) + b_n\sin\left(\frac{n\pi x}{L}\right)\right),$$

with

$$a_n = \frac{1}{L}\int_{-L}^{L} f(x)\cos\left(\frac{n\pi x}{L}\right)dx, \quad n = 0, 1, 2, \ldots,$$
$$b_n = \frac{1}{L}\int_{-L}^{L} f(x)\sin\left(\frac{n\pi x}{L}\right)dx, \quad n = 1, 2, \ldots.$$

This series for $f(x)$ using trig functions is called its **Fourier series**. It arises not only in wave and PDE problems, but also in image processing, data analysis, *etc.*

Example. Consider

$$f(x) = x^2, \quad x \in [-1, 1],$$

as plotted in Fig. 3.1. For this example, $L = 1$, and we get

$$\begin{aligned} a_n &= \frac{1}{L}\int_{-L}^{L} f(x)\cos\left(\frac{n\pi x}{L}\right)dx \\ &= \int_{-1}^{1} x^2\cos(n\pi x)dx \\ &= \frac{4n\pi(-1)^n}{n^3\pi^3} \\ &= \frac{4(-1)^n}{n^2\pi^2}, \end{aligned}$$

where we have used integration by parts a few times[4]. Next,

$$b_n = \frac{1}{L}\int_{-L}^{L} f(x)\sin\left(\frac{n\pi x}{L}\right)dx = \int_{-1}^{1} x^2\sin(n\pi x)dx = 0,$$

[4] Per the Constitution, you should check this.

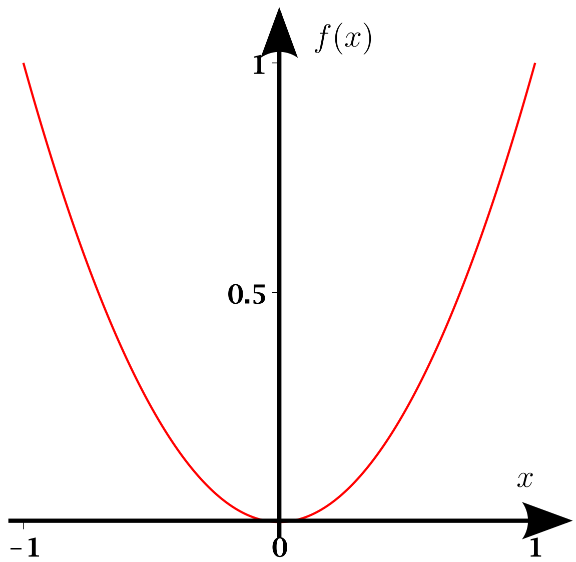

Figure 3.1: The function $f(x) = x^2$ over its domain of definition $x \in [-L, L]$.

since the integrand is an odd function. It follows that for $x \in [-1, 1]$,

$$
\begin{aligned}
x^2 &= \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos n\pi x \\
&= \frac{1}{3} + \frac{4}{\pi^2} \sum_{n=1}^{\infty} \frac{(-1)^n}{n^2} \cos n\pi x.
\end{aligned}
$$

This example immediately leads to some special cases and remarks.

Remarks.

- When we construct the Fourier series for $f(x) = x^2$, for $x \in [-1, 1]$, we are in fact constructing the Fourier series for the periodic extension of $f(x)$, consisting of the periodic repetition of the function between $[-1, 1]$, as plotted in Fig. 3.2.

- If $f(x)$ is even then $b_n = 0$, and

$$
\begin{aligned}
a_n &= \frac{1}{L} \int_{-L}^{L} f(x) \cos\left(\frac{n\pi x}{L}\right) dx \\
&= \frac{2}{L} \int_{0}^{L} f(x) \cos\left(\frac{n\pi x}{L}\right) dx,
\end{aligned}
$$

  and

$$
f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos\left(\frac{n\pi x}{L}\right).
$$

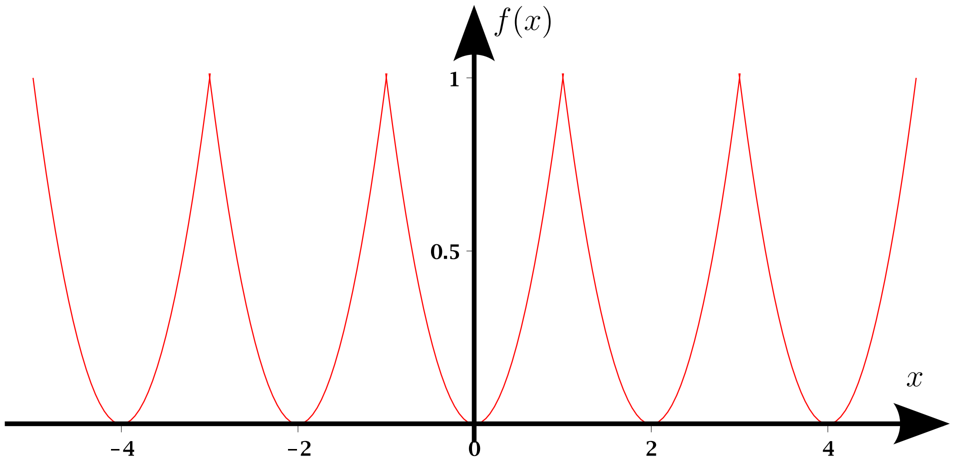

Figure 3.2: The periodic extension function of the function $f(x) = x^2$.

This is called the **Fourier cosine series**.

- If $f(x)$ is odd then $a_n = 0$, and

$$\begin{aligned} b_n &= \frac{1}{L}\int_{-L}^{L} f(x)\sin\left(\frac{n\pi x}{L}\right)dx \\ &= \frac{2}{L}\int_{0}^{L} f(x)\sin\left(\frac{n\pi x}{L}\right)dx, \end{aligned}$$

and

$$f(x) = \sum_{n=1}^{\infty} b_n \sin\left(\frac{n\pi x}{L}\right).$$

This is called the **Fourier sine series**.

Next, we will start to use Fourier series to solve PDEs.

3.3 Fourier series solutions of the wave equation

Let's return to the problem of a finite string with fixed ends:

$$\begin{aligned} u_{tt} &= c^2 u_{xx}, && x \in (0, L),\ t > 0, \\ u(x,0) &= f(x), && x \in (0, L), \\ u_t(x,0) &= g(x), && x \in (0, L), \\ u(0,t) &= 0, && t > 0, \\ u(L,t) &= 0, && t > 0. \end{aligned}$$

We already know that

$$u(x,t) = A + \sum_{n=1}^{\infty} \sin\left(\frac{n\pi x}{L}\right)\left(a_n \cos\left(\frac{n\pi ct}{L}\right) + b_n \sin\left(\frac{n\pi ct}{L}\right)\right),$$

is the most general solution of the wave equation that satisfies the boundary conditions, found using separation of variables. It remains to impose the initial conditions. First, by letting $t = 0$, we get

$$f(x) = A + \sum_{n=1}^{\infty} a_n \sin\left(\frac{n\pi x}{L}\right).$$

Next, we take a derivative of $u(x,t)$ with respect to t, and we let $t = 0$. This results in

$$g(x) = \sum_{n=1}^{\infty} b_n \frac{n\pi c}{L} \sin\left(\frac{n\pi x}{L}\right).$$

Because of the boundary conditions (fixed end), we use an odd extension of $f(x)$. Using the Fourier sine series, we get

$$A = 0,$$

and

$$a_n = \frac{2}{L}\int_0^L f(x)\sin\left(\frac{n\pi x}{L}\right) dx,$$

for $n = 1, 2, \ldots$. Similarly, we use an odd extension for $g(x)$, since the above indicates we wish to use a Fourier sine series. We get

$$\begin{aligned} & b_n \frac{n\pi c}{L} = \frac{2}{L}\int_0^L g(x)\sin\left(\frac{n\pi x}{L}\right) dx, \\ \Rightarrow \quad & b_n = \frac{2}{n\pi c}\int_0^L g(x)\sin\left(\frac{n\pi x}{L}\right) dx. \end{aligned}$$

also for $n = 1, 2, \ldots$.

This completely determines the solution of the wave equation problem with the given initial data.

Let's consider a different problem, namely that with two *free* ends.

$$\begin{aligned} u_{tt} &= c^2 u_{xx}, && x \in (0,L),\ t > 0, \\ u(x,0) &= f(x), && x \in (0,L), \\ u_t(x,0) &= g(x), && x \in (0,L), \\ u_x(0,t) &= 0, && t > 0, \\ u_x(L,t) &= 0, && t > 0. \end{aligned}$$

We begin by using separation of variables to find the standing wave solutions. Let

$$u(x,t) = X(x)T(t).$$

Then

$$\frac{T''}{c^2T} = \frac{X''}{X} = \lambda,$$

so that λ has to be constant. It follows that

$$X'' = \lambda X.$$

Note that the boundary conditions imply that $X'(0) = 0 = X'(L)$. We consider three different cases.

- $\lambda = 0$**.** We have

  $$X'' = 0 \quad \Rightarrow \quad X = D,$$

  where we have already used the boundary conditions. This case results in a constant (as a function of x) standing wave, given by

  $$u_0(x,t) = A_0 + B_0 t,$$

  where we have equated $D = 1$, since we can absorb the constant in the values of A_0 and B_0.

- $\lambda > 0$**.** We set $\lambda = r^2$, where $r > 0$. Then

  $$X(x) = c_1 \cosh(rx) + c_2 \sinh(rx).$$

  To impose the boundary conditions, we need

  $$X'(x) = c_1 r \sinh(rx) + c_2 \cosh(rx).$$

  From $X'(0) = 0$, it follows that

  $$0 = c_2.$$

  Next, from $X'(L) = 0$, we are left with

  $$0 = c_1 r \sinh(rL),$$

  which requires $c_1 = 0$, so that the exponential case does not result in any solutions.

- $\lambda < 0$**.** Now we let $\lambda = -r^2$, again with $r > 0$. We know we get solutions of the form

  $$X(x) = c_1 \cos(rx) + c_2 \sin(rx).$$

  For the boundary conditions, we need

  $$X'(x) = -c_1 r \sin(rx) + c_2 r \cos(rx).$$

From $X'(0) = 0$ we get

$$c_2 r = 0 \quad \Rightarrow \quad c_2 = 0.$$

Next, using $X'(L) = 0$, we get

$$-c_1 r \sin(rL) = 0.$$

Since we wish to avoid $c_1 = 0$, we impose that

$$r = \frac{n\pi}{L},$$

so that

$$X_n(x) = \cos\left(\frac{n\pi x}{L}\right),$$

for $n = 1, 2, 3, \ldots$. We get the standing wave solutions

$$u_n(x,t) = \cos\left(\frac{n\pi x}{L}\right)\left(A_n \cos\left(\frac{n\pi ct}{L}\right) + B_n \sin\left(\frac{n\pi ct}{L}\right)\right)$$

Using superposition, we get that the most general solution satisfying the boundary conditions is given by

$$u(x,t) = A_0 + B_0 t + \sum_{n=1}^{\infty} \cos\left(\frac{n\pi x}{L}\right)\left(A_n \cos\left(\frac{n\pi ct}{L}\right) + B_n \sin\left(\frac{n\pi ct}{L}\right)\right).$$

It remains to impose the initial conditions, so as to determine the constants A_n and B_n. First, we plug in $x = 0$. We get

$$f(x) = A_0 + \sum_{n=1}^{\infty} A_n \cos\left(\frac{n\pi x}{L}\right).$$

Using an even extension of $f(x)$, we get

$$A_n = \frac{2}{L}\int_0^L f(x)\cos\left(\frac{n\pi x}{L}\right)dx, \quad n = 1, 2, \ldots,$$

and

$$A_0 = \frac{1}{L}\int_0^L f(x)dx,$$

using the formulae for the Fourier cosine series. Lastly, to impose the second initial condition, we take a derivative with respect to t, and we let $t = 0$. We get

$$g(x) = B_0 + \sum_{n=1}^{\infty} B_n \frac{n\pi c}{L}\cos\left(\frac{n\pi x}{L}\right).$$

Using an even extension for $g(x)$, we get that

$$B_0 = \frac{1}{L}\int_0^L g(x)dx,$$

and

$$\begin{aligned} \frac{n\pi c}{L} B_n &= \frac{2}{L}\int_0^L g(x)\cos\left(\frac{n\pi x}{L}\right)dx \\ \Rightarrow \qquad B_n &= \frac{2}{n\pi c}\int_0^L g(x)\cos\left(\frac{n\pi x}{L}\right)dx. \end{aligned}$$

Note that in a practical problem, you might have to impose that the average of the initial velocity $g(x)$ is zero. If this is not the case, your solution to the string-with-free-ends problem will have a component that is linearly growing!

3.4 The heat equation

Consider the problem

$$\begin{aligned} u_t &= \sigma u_{xx}, && x \in (0,L), t > 0, \\ u(x,0) &= f(x), && x \in (0,L), \\ u(0,t) &= 0, && t > 0, \\ u(L,t) &= 0, && t > 0. \end{aligned}$$

Here $\sigma > 0$ is the heat conductivity coefficient. The heat equation describes heat flow in a medium with heat conductivity σ. We will consider only the one-dimensional heat equation.

As for the wave equation, we begin by looking for solutions of the form

$$u(x,t) = X(x)T(t).$$

We get

$$\begin{aligned} XT' &= \sigma X''T \\ \Rightarrow \qquad \frac{T'}{\sigma T} &= \frac{X''}{X} = \lambda. \end{aligned}$$

Here λ is a separation constant, using the same argument we used for the wave equation: since the left-hand side depends only on t, and the right-hand side depends only on x, they must both be constant. It follows that $X(x)$ satisfies the following problem:

$$\begin{aligned} X'' - \lambda X &= 0, \\ X(0) &= 0, \\ X(L) &= 0. \end{aligned}$$

This is the same exact problem for X as we had for the wave equation with fixed ends (Dirichlet boundary conditions). As a consequence, we know we get solutions only for $\lambda = -r^2 < 0$, and

$$X_n(x) = \sin\left(\frac{n\pi x}{L}\right), \quad n = 1, 2, \ldots,$$

and

$$\lambda_n = -\frac{n^2\pi^2}{L^2}.$$

It follows that T_n satisfies the ordinary differential equation

$$\frac{T_n'}{\sigma T_n} = \frac{n^2\pi^2}{L^2} \quad \Rightarrow \quad T_n = e^{-\sigma n^2\pi^2 t/L^2}.$$

We could have included a multiplicative constant, but this is not necessary, as the next step is to take a linear superposition of the solutions $u_n(x,t) = X_n(x)T_n(t)$ we have just found:

$$u_n(x,t) = e^{-\sigma n^2\pi^2 t/L^2} \sin\left(\frac{n\pi x}{L}\right).$$

The superposition results in the general solution

$$u = \sum_{n=1}^{\infty} c_n e^{-\sigma n^2\pi^2 t/L^2} \sin\left(\frac{n\pi x}{L}\right).$$

It remains to impose the initial condition. At $t = 0$, we get

$$f(x) = \sum_{n=1}^{\infty} c_n \sin\left(\frac{n\pi x}{L}\right),$$

since all the exponentials become 1, when evaluated at $t = 0$. Using an odd extension of $f(x)$, we get that

$$c_n = \frac{2}{L}\int_0^L f(x) \sin\left(\frac{n\pi x}{L}\right) dx, \quad n = 1, 2, \ldots.$$

Thus, we have solved the initial-value problem for the heat equation with prescribed zero temperature at the ends[5].

3.5 Laplace's equation

Consider the problem

$$u_{xx} + u_{yy} = 0, \qquad x \in (0, L), \; y \in (0, M).$$

This is **Laplace's equation**, posed on a rectangle. You could imagine getting to Laplace's equation by wanting to find time-independent (or stationary) solutions of the multi-dimensional

[5] That was quick! We're getting to be good at this.

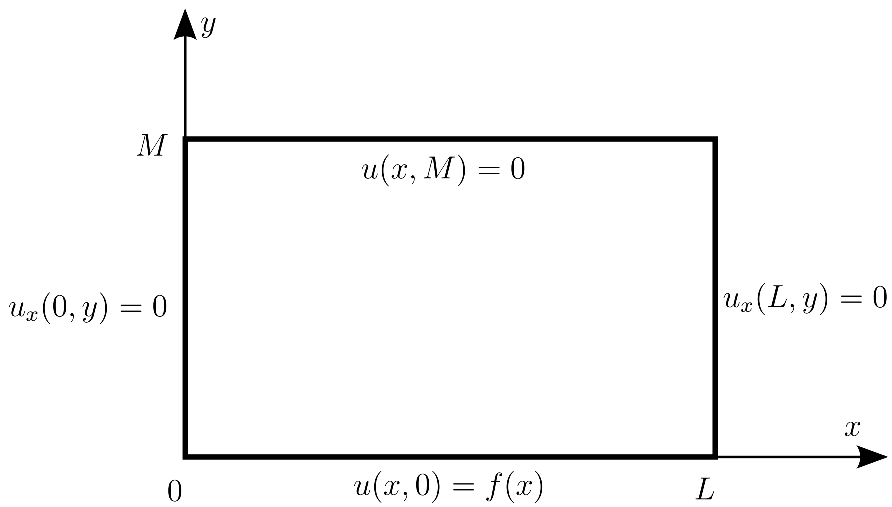

Figure 3.3: The domain for Laplace's equation on the rectangle.

wave equation $u_{tt} = c^2(u_{xx}+u_{yy})$ or of the multi-dimensional heat equation $u_t = \sigma(u_{xx}+u_{yy})$. We impose boundary conditions as follows, see Fig. 3.3:

$$\begin{aligned} u(x,0) &= f(x), \\ u_x(0,y) &= 0, \\ u_x(L,y) &= 0, \\ u(x,M) &= 0. \end{aligned}$$

First, we look for separated solutions:

$$u(x,y) = X(x)Y(y),$$

and we get

$$X''Y + XY'' = 0 \quad \Rightarrow \quad \frac{X''}{X} = -\frac{Y''}{Y} = \lambda.$$

As before, λ is a constant, since X''/X and $-Y''/Y$ are dependent on x and y separately. We have that

$$\begin{aligned} X'' - \lambda X &= 0, \\ X'(0) &= 0, \\ X'(L) &= 0. \end{aligned}$$

As usual, we consider different cases for λ.

- $\lambda > 0$. You should check that this does not result in any solutions for $X(x)$.

- $\lambda = 0$**.** This results in the solution $X_0(x) = 1$. Similarly, $Y_0(y) = A + By$. Imposing $Y(M) = 0$, we find $0 = A + BM$, so that $A = -BM$. Thus $Y_0(y) = B(y - M)$. This gives rise to the constant solution $u_0(x, y) = y - M$, where we have omitted the multiplicative constant[6].

- $\lambda < 0$**.** As before, we get

$$X = c_1 \cos(rx) + c_2 \sin(rx).$$

To impose the boundary conditions, we need

$$X' = -c_1 r \sin(rx) + c_2 r \cos(rx).$$

From the first boundary condition, we get

$$0 = c_2 r \quad \Rightarrow \quad c_2 = 0.$$

The second boundary condition gives

$$0 = -c_1 r \sin(rL) \quad \Rightarrow \quad r = \frac{n\pi}{L}.$$

This gives rise to

$$X_n(x) = \cos\left(\frac{n\pi x}{L}\right), \quad n = 1, 2, \ldots.$$

Next, we have to solve for $Y_n(y)$. We have

$$\begin{aligned} Y_n'' - \frac{n^2\pi^2}{L^2} Y_n &= 0 \\ \Rightarrow \qquad Y_n &= A_n \cosh\left(\frac{n\pi}{L}(y - M)\right) + B_n \sinh\left(\frac{n\pi}{L}(y - M)\right), \end{aligned}$$

where we have opted to write the solution using hyperbolic functions of a shifted argument. Imposing the boundary condition $y_n(M) = 0$, we find that

$$A_n = 0.$$

It follows that

$$Y_n(y) = B_n \sinh\left(\frac{n\pi}{L}(y - M)\right).$$

The linear superposition of all solutions is[7]

$$u(x, y) = B_0(y - M) + \sum_{n=1}^{\infty} B_n \sinh\left(\frac{n\pi}{L}(y - M)\right) \cos\left(\frac{n\pi x}{L}\right).$$

[6] For now. Don't worry. It'll come back.
[7] It's back!

Imposing the one remaining boundary condition, we get

$$u(x,0) = f(x) = -MB_0 - \sum_{n=1}^{\infty} B_n \sinh\left(\frac{n\pi M}{L}\right)\cos\left(\frac{n\pi x}{L}\right).$$

the rest is some simple Fourier series stuff, using an even extension[8] of $f(x)$!

- For $n = 0$,

$$\begin{aligned} -MB_0 &= \frac{1}{L}\int_0^L f(x)dx \\ \Rightarrow \qquad B_0 &= -\frac{1}{ML}\int_0^L f(x)dx. \end{aligned}$$

- For $n = 1, 2, \ldots$,

$$\begin{aligned} -B_n \sinh\left(\frac{n\pi M}{L}\right) &= \frac{2}{L}\int_0^L f(x)\cos\left(\frac{n\pi x}{L}\right)dx \\ \Rightarrow \qquad B_n &= -\frac{2}{L\sinh\left(\frac{n\pi M}{L}\right)}\int_0^L f(x)\cos\left(\frac{n\pi x}{L}\right)dx. \end{aligned}$$

This completely determines the solution of the Laplace problem on the rectangle.

3.6 Laplace's equation on the disc

Consider the problem (see Fig. 3.4)

$$\begin{aligned} u_{xx} + u_{yy} &= 0, & x^2 + y^2 &< R^2, \\ u(x,y) &= f(x,y), & x^2 + y^2 &= R^2. \end{aligned}$$

It is clear that we should reformulate this problem in polar coordinates. Let

$$v(r,\theta) = u(x,y).$$

Then[9]

$$u_x = \frac{\partial u}{\partial x} = \frac{\partial v}{\partial r}\frac{\partial r}{\partial x} + \frac{\partial v}{\partial \theta}\frac{\partial \theta}{\partial x}.$$

From

$$r^2 = x^2 + y^2,$$

[8] Because the right-hand side is a cosine series.

[9] You can see the chain rules coming a mile away!

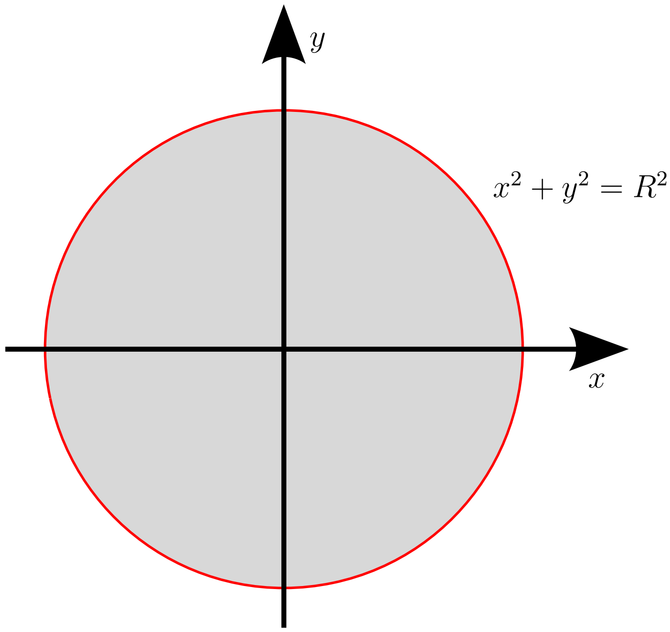

Figure 3.4: The domain for Laplace's equation on the disc.

it follows that

$$2r\frac{\partial r}{\partial x} = 2x \quad \Rightarrow \quad \frac{\partial r}{\partial x} = \frac{x}{r}.$$

similarly, from

$$\tan\theta = \frac{y}{x},$$

we get

$$\sec^2\theta\frac{\partial \theta}{\partial x} = -\frac{y}{x^2} \quad \Rightarrow \quad \frac{r^2}{x^2}\frac{\partial \theta}{\partial x} = -\frac{y}{x^2} \quad \Rightarrow \quad \frac{\partial \theta}{\partial x} = -\frac{y}{r^2}.$$

Thus

$$\begin{aligned} u_x &= v_r\frac{x}{r} - v_\theta\frac{y}{r^2} \\ &= v_r\cos\theta - v_\theta\frac{\sin\theta}{r}. \end{aligned}$$

Next, we apply this same process to get $u_{xx} = (u_x)_x$: in other words, we repeat the above, but with v replaced by the expression we found for u_x. This gives

$$\begin{aligned} u_{xx} &= \left(v_r\cos\theta - v_\theta\frac{\sin\theta}{r}\right)_r\cos\theta - \left(v_r\cos\theta - v_\theta\frac{\sin\theta}{r}\right)_\theta\frac{\sin\theta}{r} \\ &= \left(v_{rr}\cos\theta - v_{r\theta}\frac{\sin\theta}{r} + v_\theta\frac{\sin\theta}{r^2}\right)\cos\theta - \left(v_{r\theta}\cos\theta - v_r\sin\theta - v_{\theta\theta}\frac{\sin\theta}{r} - v_\theta\frac{\cos\theta}{r}\right)_\theta\frac{\sin\theta}{r} \\ &= v_{rr}\cos^2\theta - 2v_{r\theta}\frac{\sin\theta\,\cos\theta}{r} + 2v_\theta\frac{\sin\theta\,\cos\theta}{r^2} + v_r\frac{\sin^2\theta}{r} + v_{\theta\theta}\frac{\sin^2\theta}{r^2}. \end{aligned}$$

Similarly, we find

$$u_{yy} = v_{rr}\sin^2\theta + 2v_{r\theta}\frac{\sin\theta\,\cos\theta}{r} - 2v_\theta\frac{\sin\theta\,\cos\theta}{r^2} + v_r\frac{\sin^2\theta}{r} + v_{\theta\theta}\frac{\cos^2\theta}{r^2}.$$

Adding these expressions, we get amazing simplifications[10]:

$$u_{xx} + u_{yy} = 0 \quad\Rightarrow\quad v_{rr} + \frac{v_r}{r} + \frac{v_{\theta\theta}}{r^2} = 0.$$

We can reformulate our original problem in polar coordinates:

$$\begin{aligned} v_{rr} + \frac{v_r}{r} + \frac{v_{\theta\theta}}{r^2} &= 0, \\ v(R,\theta) &= f(\theta), \quad \theta\in[0,2\pi), \end{aligned}$$

since the boundary condition is given on the circle of radius R.

As usual, we apply separation of variables to look for solutions of the form

$$v(r,\theta) = S(r)T(\theta).$$

We get

$$\begin{aligned} && S''T + \frac{S'T}{r} + \frac{T''S}{r^2} &= 0 \\ \Rightarrow && r^2\frac{S''}{S} + \frac{rS'}{S} + \frac{T''}{T} &= 0 \\ \Rightarrow && r^2\frac{S''}{S} + \frac{rS'}{S} = -\frac{T''}{T} &= \lambda, \end{aligned}$$

where λ is a separation constant. Indeed, in this last line, the left-hand side is a function of r only, while $-T''/T$ depends only on θ. Thus both are constant.

Since the equation for T is the simplest, we solve it first.

- If $\lambda = 0$, then

  $$T'' = 0 \quad\Rightarrow\quad T = a + b\theta.$$

  We want $v(r,\theta)$ to be single-valued as a function of both r and θ. This implies that T should be a periodic function of θ, with period 2π. Thus, $b = 0$. This leaves us with

  $$T_0 = 1,$$

  ignoring the multiplicative constant a.

[10] Always a good sign.

- If $\lambda = \alpha^2 > 0$, with $r > 0$, we get

$$T = a\cos\alpha\theta + b\sin\alpha\theta.$$

Requiring that T is periodic with period 2π, we find that

$$\alpha = n,$$

an integer. Then $\lambda_n = n^2$. This gives

$$T_n = a_n\cos n\theta + b_n\sin n\theta.$$

- If $\lambda = -\alpha^2 < 0$, we find no 2π-periodic solutions[11].

Next, we solve for $S(r)$.

- With $\lambda_0 = 0$, then

$$\begin{aligned} & r^2 S_0'' + r S_0' = 0 \\ \Rightarrow \quad & r S_0'' + S_0' = 0 \\ \Rightarrow \quad & (r S_0')' = 0 \\ \Rightarrow \quad & r S_0' = c_1 \\ \Rightarrow \quad & S_0 = c_1 \ln r + c_2. \end{aligned}$$

This is infinite as $r \to 0$, which we do not want to allow[12]. Thus we require $c_1 = 0$, so that

$$S_0 = 1,$$

where we, once again, have ignored the multiplicative constant. Thus,

$$v_0(r,\theta) = S_0(r)T_0(\theta) = 1.$$

- Next, we examine $\lambda_n = n^2$, $n = 1, 2, \ldots$. We have

$$r^2 S_n'' + r S_n' - n^2 S_n = 0.$$

This is an **Euler** or **equivariant** ordinary differential equation. Thus we look for solutions of the form

$$S = r^p.$$

We get

$$\begin{aligned} & p(p-1) + p - n^2 = 0 \\ \Rightarrow \quad & p^2 = n^2 \\ \Rightarrow \quad & p = \pm n. \end{aligned}$$

[11] Don't make me say it. OK, fine: you should check this.
[12] Nothing goes to infinity on our watch!

This gives

$$S_n = c_n r^n + d_n r^{-n}.$$

Since $n > 0$, the second term $\to \infty$ as $r \to 0$. Thus we require $d_n = 0$. Thus

$$S_n = r^n,$$

up to a multiplicative constant.

The general solution is a linear superposition of all solutions we have found:

$$v(r, \theta) = \frac{a_0}{2} + \sum_{n=1}^{\infty} r^n (a_n \cos n\theta + b_n \sin n\theta).$$

Having found the general solution, we impose the boundary condition

$$v(R, \theta) = f(\theta).$$

This results in

$$f(\theta) = \frac{a_0}{2} + \sum_{n=1}^{\infty} R^n (a_n \cos n\theta + b_n \sin n\theta).$$

Using the Fourier series formulae, we find that

$$a_n = \frac{1}{\pi R^n} \int_0^{2\pi} f(\theta) \cos n\theta d\theta,$$

for $n = 0, 1, \ldots$, and

$$b_n = \frac{1}{\pi R^n} \int_0^{2\pi} f(\theta) \sin n\theta d\theta,$$

for $n = 1, 2, \ldots$. This completely determines the solution to the Laplace equation posed on the inside of a circle.

Chapter 4

The method of characteristics

4.1 Conservation laws

Let's go back to wave behavior. We'll go beyond the wave equation.

Derivation of a general scalar conservation law

A conservation law tells us the way in which a particular quantity can change. The simplest examples you know are that in conservative systems, energy is conserved.

Suppose we have a one-dimensional setting, for convenience we call it the x-axis. Suppose the quantity Q changes with our dynamics.

Example. Q could represent

- the number of cars in a traffic flow problem,
- the number of particles in molecular chemistry problem,
- the energy of a physical system,
- the number of people in a social dynamics problem.

Let $u(x, t)$ denote the density of Q. In other words, Q in S is given by

$$Q = \int_a^b u(x, t) dx.$$

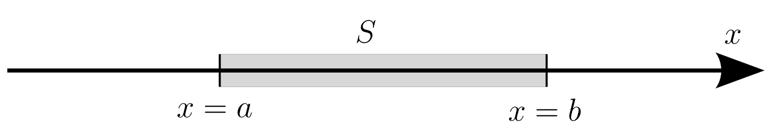

Figure 4.1: A one-dimensional domain, setting up for the derivation of a conservation law.

The quantity Q in S can change in two ways:

1. Q could enter or leave S through a or b, and
2. Q is created or destroyed in S.

It follows that the rate of change of Q, dQ/dt, is given by the rate at which Q enters or leaves at $x = a$, plus the rate at which Q enters or leaves at $x = b$, plus the rate at which Q is created or destroyed in S. In equations,

$$\frac{d}{dt}\int_a^b u(x,t)dx = \phi(a,t) - \phi(b,t) + \int_a^b f(x,t)dx,$$

where $\phi(x,t)$ is the rate at which Q moves past x at time t. If $\phi(x,t) > 0$, then the flow is in the positive x direction, otherwise it is in the negative x direction. Thus, the net rate at which Q enters through the ends of S is

$$\phi(a,t) - \phi(b,t).$$

The $-$ sign for the second term is a consequence of our flow convention: if the flow of Q through $x = b$ is to the right, it is leaving S, thus it results in a **decrease**.

Lastly, if Q is created or destroyed in S, this happens with a source or sink function $f(x,t)$, resulting in an amount of Q that is added equal to

$$\int_a^b f(x,t)dx.$$

Our rate-of-change equation becomes

$$\frac{d}{dt}\int_a^b u(x,t)dx = \phi(a,t) - \phi(b,t) + \int_a^b f(x,t)dx.$$

Suppose that u and ϕ have continuous derivatives, then

$$\begin{aligned} & \int_a^b u_t(x,t)dx = -\int_a^b \phi_x(x,t)dx + \int_a^b f(x,t)dx \\ \Rightarrow \quad & \int_a^b (u_t + \phi_x - f)dx = 0. \end{aligned}$$

Since this is true for all a and b, we get that

$$u_t + \phi_x = f.$$

This is called the *differential form* of the conservation law.

Constitutive relations

Even if we consider $f(x,t)$ as given, we still have one partial differential equation for two quantities u and ϕ. A **constitutitve relation** relates u and ϕ. In many cases such a relation gives ϕ as a function of u. Then $\phi = \phi(u)$, and we get

$$u_t + \phi'(u)u_x = f.$$

Example. The inviscid Burgers equation

$$u_t + uu_x = 0,$$

is a conservation law with $f = 0$, $\phi = u^2/2$. However, there are more possibilities. The same equation can be written as

$$uu_t + u^2u_x = 0 \quad \Rightarrow \quad \left(\frac{1}{2}u^2\right)_t + \left(\frac{1}{3}u^3\right)_x = 0.$$

If we let

$$v = \frac{1}{2}u^2,$$

then

$$u = (2v)^{1/2},$$

and

$$\phi = \frac{u^3}{3} = \frac{1}{3}(2v)^{3/2},$$

and the Burgers equation can be rewritten as

$$v_t + \left(\frac{1}{3}(2v)^{3/2}\right)_x = 0.$$

Which form of the equation we choose will matter in the following lectures. In practice, it is of course dictated by the application we are working on.

4.2 Examples of conservation laws

Diffusion

Consider the undesirable scenario of a pollutant spreading in stagnant water in a horizontal pipe, see Fig. 4.2. Let $u(x,t)$ denote the concentration of pollutant (in mass/length). Then

$$u_t + \phi_x = f$$

is our conservation law. Assume that we have no magical pollutant eating piranhas in the pipe, and no pollutant is destroyed or created, then $f = 0$. Next, we need to relate the flux function ϕ to the concentration u.

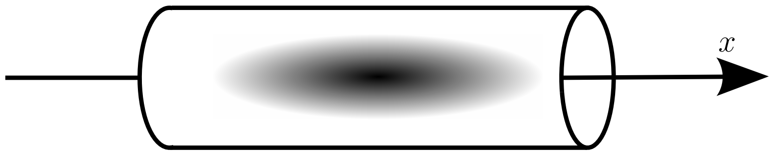

Figure 4.2: Diffusion of a pollutant in stagnant water.

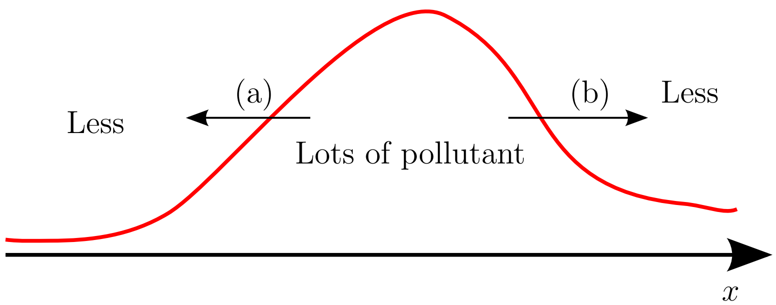

Figure 4.3: Initial concentration of a pollutant in stagnant water.

Suppose we have an initial concentration $u_0(x)$ shown below in Fig. 4.3. We expect that pollutant will flow from areas where there is a lot to areas where there is less. If u_x is positive, as in (a), then ϕ should be negative so that pollutant flows to the left. If, at (b), u_x is negative, then ϕ should be positive and u will flow to the right. The simplest way to make this happen is **Fick's Law**:

$$\phi(x,t) = -Du_x(x,t),$$

where $D > 0$ is the diffusion constant.

Putting all of this together, we get

$$\begin{aligned} & u_t + \phi_x = 0 \\ \Rightarrow \quad & u_t + (-Du_x)_x = 0 \\ \Rightarrow \quad & u_t = Du_{xx}, \end{aligned}$$

the heat or diffusion equation!

Traffic flow

This goes back to studies by Whitham and Lighthill in the 50s. We approximate the number of cars per unit length by a continuous function $u(x,t)$. Assuming there are no exits or entrances, we get

$$f = 0,$$

so that our conservation law becomes

$$u_t + \phi_x = 0.$$

As before, we have to determine a relation between ϕ and u.

$$\begin{aligned}\phi &= \text{cars/time unit}\\ &= \text{Rate at which cars are passing } x \text{ at } t\\ &= u \times v,\end{aligned}$$

where u is the car density and v is the velocity. Thus, we need to relate the velocity v to u, and we will be done. Clearly[1], if u is high then v will be low, and if u is near zero, v should be maximal, the speed limit. The simplest way to do this is

$$v = v_1 - au,$$

where v_1 is the speed limit and a is a positive constant. As $u \to v_1/a$, $v \to 0$, thus $v_1/a = u_1$ is the maximal density possible. We can rewrite the velocity as

$$v = v_1\left(1 - \frac{u}{u_1}\right),$$

for $u \in [0, u_1]$. Our constitutive relation becomes

$$\phi = uv = v_1 u\left(1 - \frac{u}{u_1}\right) = v_1\left(u - \frac{u^2}{u_1}\right).$$

This last form is less instructional, but it is easier to take a derivative. The conservation law becomes

$$\begin{aligned} & & u_t + \phi_x &= 0\\ \Rightarrow & & u_t + v_1\left(u_x - \frac{2uu_x}{u_1}\right) &= 0\\ \Rightarrow & & u_t + v_1\left(1 - \frac{2u}{u_1}\right)u_x &= 0.\end{aligned}$$

4.3 The method of characteristics

We will use the method of characteristics to solve conservation laws of the form

$$\begin{aligned} u_t + \phi_x &= f,\\ u(x,0) &= u_0(x),\end{aligned}$$

[1] That word again!

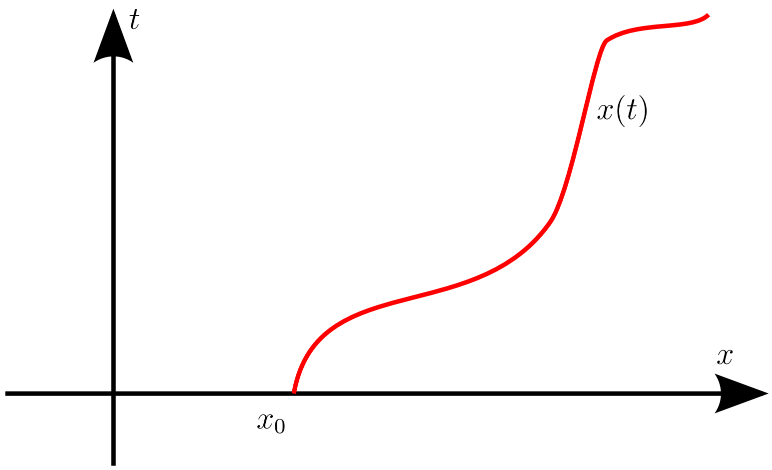

Figure 4.4: A characteristic curve in the (x,t) plane.

where $-\infty < x < \infty$, $t > 0$, unless otherwise stated.

Let's start with the simplest equation of this form, the **advection equation**. Suppose that $\phi = cu$, then we have, with $f = 0$,

$$\begin{aligned} u_t + cu_x &= 0, \\ u(x,0) &= u_0(x). \end{aligned}$$

The method of characteristics looks for special curves in the (x,t) plane along which our PDE becomes an ODE. In other words, we wish to find $x(t)$ in the (x,t) plane such that we only have to solve ODEs along this curve, see Fig. 4.4.

Along these curves $u(x,t)$ becomes a function of t only. Then

$$\frac{d}{dt}u(x(t),t) = u_x\frac{dx}{dt} + u_t.$$

We compare this expression with the PDE we wish to solve

$$u_t + cu_x = 0.$$

We see that if we pick the curves (called the **characteristic curves** or **characteristics**) such that

$$\frac{dx}{dt} = c,$$

then our PDE simply becomes

$$\frac{du}{dt}(x(t),t) = 0.$$

Thus u is constant along characteristic curves in this case. The value of the constant is of course determined by the initial conditions.

Thus, we have to solve

$$\begin{aligned} \frac{du}{dt} &= 0, \quad \text{along curves for which} \\ \frac{dx}{dt} &= c, \quad \text{so that} \\ u(x,0) &= u_0(x), \end{aligned}$$

the given initial condition. We get

$$\frac{dx}{dt} = 0 \quad \Rightarrow \quad x = ct + x_0,$$

where x_0 is the starting point of the characteristic curve at $t = 0$. Next, from

$$\frac{du}{dt} = 0 \quad \Rightarrow \quad u = A(x_0),$$

a constant, which could be different, depending on which characteristic we are solving for u. We know that at $t = 0$

$$u(x(t),t)|_{t=0} = u(x(0),0) = u(x_0,0) = u_0(x_0).$$

We also have that

$$x_0 = x - ct,$$

and thus

$$u = u_0(x - xt),$$

which is the general solution to our advection equation. It shows that the advection equation simply moves the initial condition to the right with velocity c, see Fig. 4.5.

Example. Consider

$$\begin{aligned} u_t + 4u_x &= 0, \\ u(x,0) &= \arctan(x). \end{aligned}$$

We have to solve

$$\frac{du}{dt} = 0,$$

along curves for which

$$\frac{dx}{dt} = 4 \quad \Rightarrow \quad x = 4t + x_0.$$

It follows that

$$u = \arctan(x_0) = \arctan(x - 4t).$$

Next, we consider the case of a nonhomogeneous advection equation. We have

$$u_t + cu_x = f(x,t).$$

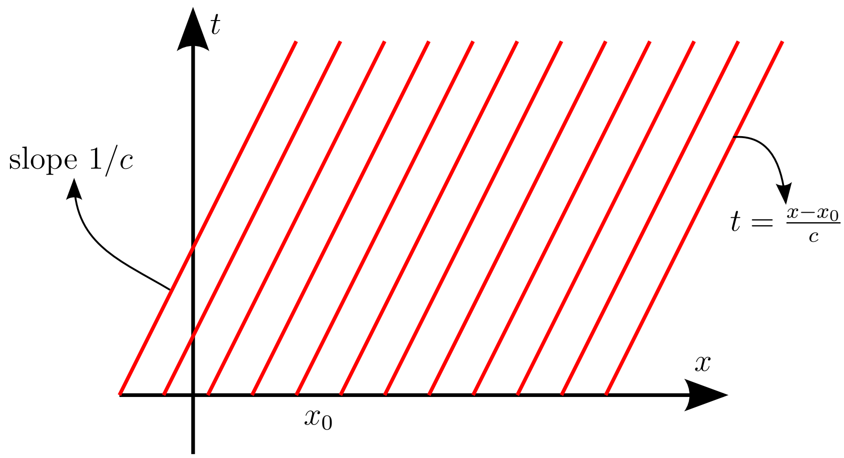

Figure 4.5: The characteristics for the advection equation.

Proceeding as before, we look for curves $x(t)$ so that we get ODEs. Along these curves,

$$\frac{d}{dt}u(x(t),t) = u_t + u_x \frac{dx}{dt}.$$

Thus we solve

$$\begin{aligned} \frac{dx}{dt} &= c, \\ x(0) &= x_0. \end{aligned}$$

The characteristics are the solutions of this system. Along the characteristics, we have to solve

$$\frac{du}{dt} = f(x(t),t),$$

which is easily solved by simply integrating both sides.

Example. Consider the nonhomogeneous problem

$$\begin{aligned} u_t + 4u_x &= 1, \\ u(x,0) &= \arctan(x). \end{aligned}$$

We have to solve

$$\frac{dx}{dt} = 4 \quad \Rightarrow \quad x = 4t + x_0.$$

Along these straight-line characteristics, we solve

$$\frac{du}{dt} = 1 \quad \Rightarrow \quad u = t + A,$$

where A is a constant of integration. Evaluating this at $t = 0$, we get

$$u(x_0, 0) = A = \arctan(x_0).$$

It follows that

$$u = t + \arctan(x - 4t).$$

General linear conservation laws

A general linear conservation law is of the form

$$u_t + c(x,t)u_x = f(x,t),$$

where we assume we have initial conditions

$$u(x, 0) = u_0(x).$$

As before, we have to solve the ODEs

$$\begin{aligned}\frac{du}{dt} &= f(x,t), \quad \text{along curves determined by}\\ \frac{dx}{dt} &= c(x,t), \quad x(0) = x_0.\end{aligned}$$

The second equation is an ODE for $x(t)$, but in general it may be hard to solve. Once it is solved, the first equation gives us u as a function of t, along these characteristics. Note that in this case, the characteristics are typically not straight lines.

Example. Consider the initial-value problem

$$\begin{aligned}u_t + xu_x &= 0,\\ u(x,0) &= \frac{1}{1+x^4}.\end{aligned}$$

We solve for the characteristics first. We have

$$\begin{aligned} & & \frac{dx}{dt} &= x\\ &\Rightarrow & \frac{d}{dt}\ln|x| &= 1\\ &\Rightarrow & \ln\left|\frac{x}{x_0}\right| &= t\\ &\Rightarrow & x &= x_0 e^t.\end{aligned}$$

Next, since

$$\frac{du}{dt} = 0,$$

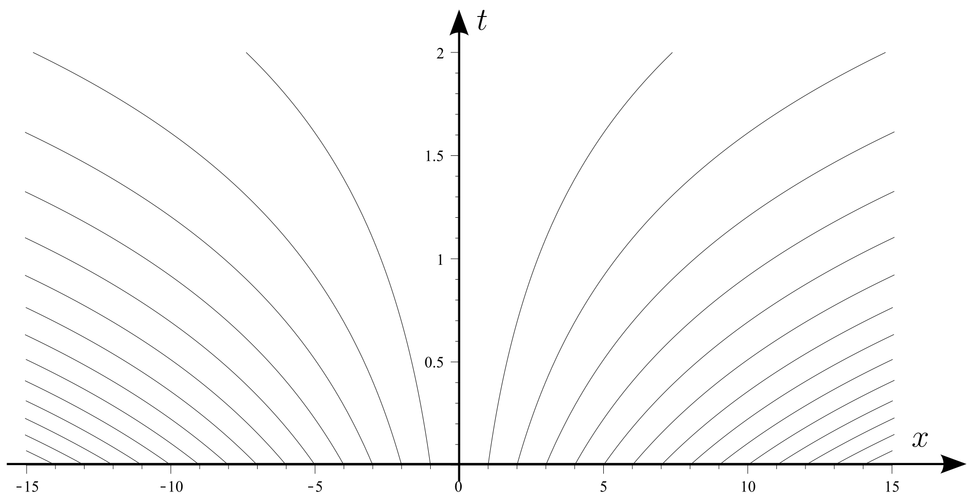

Figure 4.6: The characteristics for different values of x_0 with $c(x, t) = x$.

along the characteristics, with initial condition

$$u(x, 0) = \frac{1}{1 + x^4}.$$

We get

$$u = A,$$

a constant. Imposing the initial condition at $t = 0$,

$$u(x_0, 0) = \frac{1}{1 + x_0^4} = A.$$

It follows that

$$u = \frac{1}{1 + x^4 e^{-4t}},$$

where we have used that $x_0 = x \exp(-t)$. The characteristics are not straight lines in this case. They are shown in Fig. 4.6.

Nonlinear conservation laws

Suppose that $\phi = \phi(u)$, then our conservation law $u_t + \phi_x = f$ becomes

$$u_t + \phi'(u) u_x = f.$$

Define

$$c(u) := \phi'(u).$$

Our PDE is rewritten as

$$u_t + c(u)u_x = f.$$

It follows that we have to solve

$$\frac{du}{dt} = f, \quad u(x_0, 0) = u_0(x_0),$$

along curves determined by

$$\frac{dx}{dt} = c(u), \quad x(0) = x_0.$$

In general this is a coupled, messy system of ODEs. We can make some more progress if $f \equiv 0$. Then the first ODE becomes

$$\frac{du}{dt} = 0 \quad \Rightarrow \quad u = u_0(x_0).$$

Thus this equation tells us that u is constant along characteristics, even though at this point we do not yet know what these characteristics are. We can substitute this result into our ODE for the characteristics:

$$\frac{dx}{dt} = c(u_0(x_0)).$$

Since the right-hand side is constant, we get

$$x = x_0 + tc(u_0(x_0)).$$

It follows that the characteristics are all straight lines, but with varying slopes depending on x_0, as illustrated in Fig. 4.7

The slope depends not only on where (x_0) we start, but also on what the initial condition u_0 is there. In order to get the full solution to the problem, we need to solve the characteristic equation $x = x_0 + tc(u_0(x_0))$ for x_0 as a function of x and t. More often than not, this is not possible. In most cases, we have to be satisfied with an implicit representation of the solution.

Example. Let $\phi = u^2/2$. Then $c(u) = u$, and we consider the initial-value problem

$$\begin{aligned} & u_t + uu_x = 0, \\ & u(x,0) = \begin{cases} 0, & x \leq 0, \\ e^{-1/x}, & x > 0. \end{cases} \end{aligned}$$

The characteristics are given by

$$\begin{aligned} & x = x_0, && x_0 \leq 0, \\ & x = x_0 + te^{-1/x_0}, && x_0 > 0. \end{aligned}$$

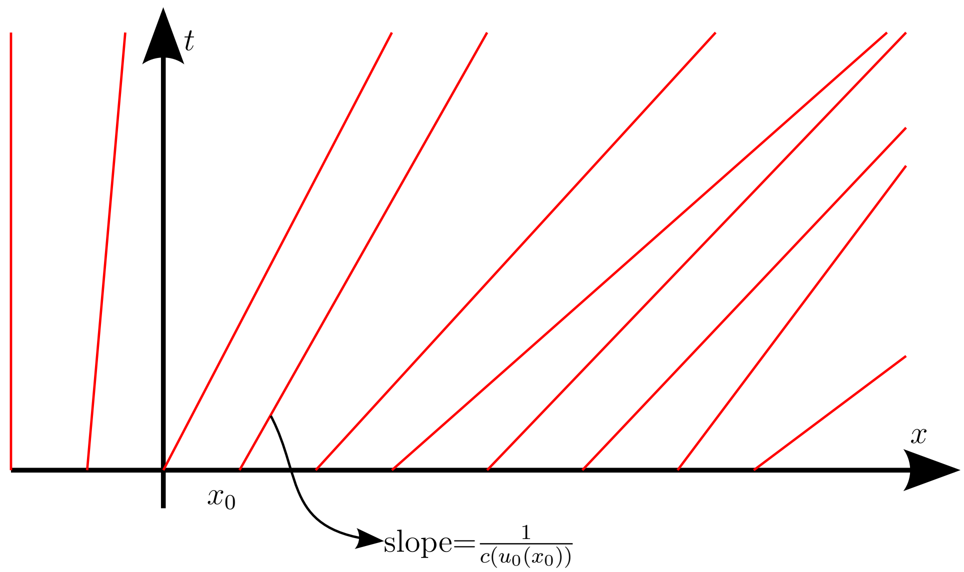

Figure 4.7: The characteristics for different values of x_0 with different slopes $1/c(u_0(x_0))$.

There is no way for us[2] to solve this[3] for x_0 as a function of x and t. However, we can obtain a perfectly fine implicit solution:

$$u(x,t) = \begin{cases} 0, & x \leq 0, \\ e^{-1/x_0}, & x > 0, \end{cases}$$

where x_0 is defined by the equation

$$x = x_0 + te^{-1/x_0}.$$

A plot of the characteristics is shown in Fig. 4.8.

4.4 Breaking and gradient catastrophes

We have seen that the characteristics for the equation

$$u_t + c(u)u_x = 0, \quad u(x,0) = u_0(x),$$

are all straight lines, with slope $1/c(u_0(x_0))$, where x_0 is the starting point of the characteristic at $t = 0$.

Even though the characteristics are just straight lines, things[4] can get very interesting. Let's see what can happen.

[2]Or anyone!

[3]Other than numerically.

[4]Technical term.

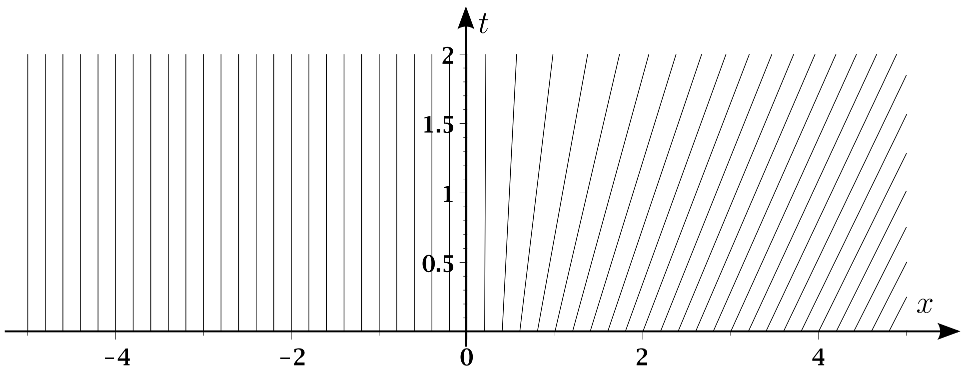

Figure 4.8: The characteristics for the problem with the piecewise defined initial condition.

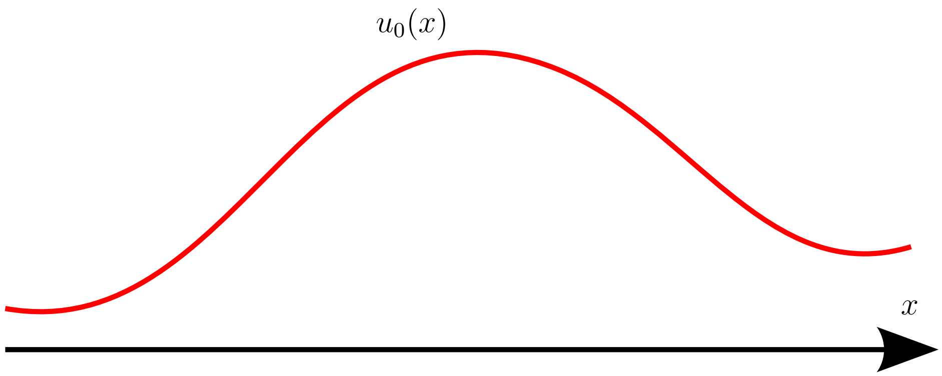

Figure 4.9: The initial profile $u_0(x)$.

1. **The characteristics are parallel**. Our solution is implicitly given by

$$x = x_0 + tc(u_0(x_0)),$$
$$u = u_0(x_0).$$

   Suppose that we start with a profile as shown in Fig. 4.9.

   Following the initial condition along the characteristics, we see that the initial values simply translate as we go forward in time, see Fig. 4.10. Indeed, if $c(u_0(x_0)) = c$, a constant, then $x_0 = x - ct$, and

$$u = u_0(x - ct).$$

2. **The characteristics are spreading out**. Starting from an arctan-like profile, we get the situation depicted in Fig. 4.11. Now the values of the solution get spread out, as the values follow the characteristics. Thus, spreading characteristics lead to smoother solutions.

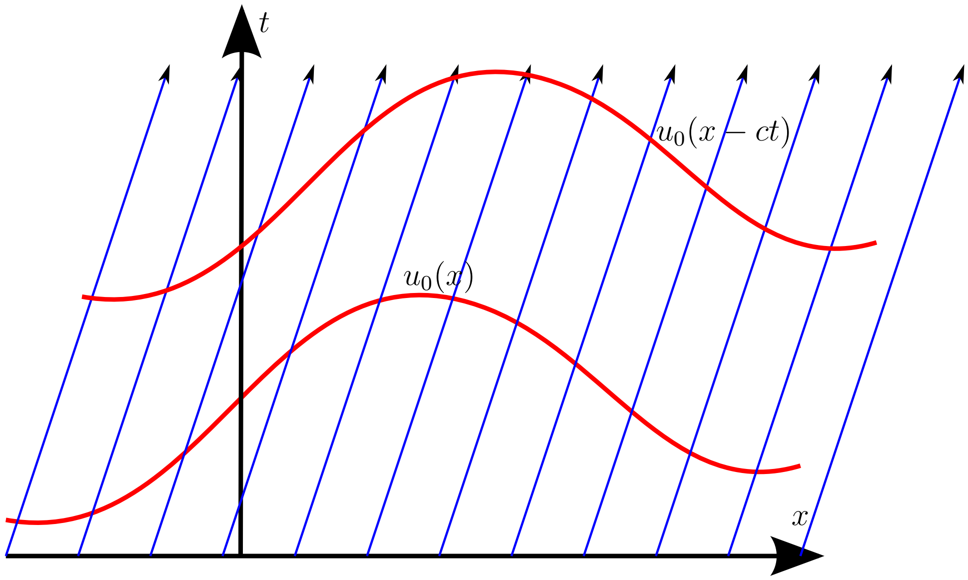

Figure 4.10: Moving the initial condition along the parallel characteristics.

3. **The characteristics are crossing**, as in Fig. 4.12. If the characteristics cross, then the values of $u_0(x)$ between two crossing characteristics starting at a and b get squeezed together as t increases. Thus the solution becomes locally steeper, since rise/run$\rightarrow 0$, since run$\rightarrow 0$ as t approaches the time at which the characteristics cross. At that time, the solution becomes infinitely steep (*i.e.*, it has a vertical tangent), which implies that the **differential form of the conservation law is no longer valid**. Indeed, to derive the differential form, we assumed that all derivatives of u existed and were continuous. If one or all of these derivatives $\rightarrow \infty$, we have to revisit the integral form of the conservation law. Up to the time where we first get the vertical tangent, the differential form works well. Past that time, we have a problem. As is seen in Fig. 4.12, the crossing characteristics give rise to a wedge-like region where any point has multiple characteristics passing through it. Outside of this wedge, we can immediately see what the value of u is by tracing back the unique characteristic going through the point. The value of the initial condition on that characteristic is the value of the solution. For points in the wedge, this does not work, as it is unclear what characteristic to follow back.

   The formation of the solution with a vertical tangent is called a **gradient catastrophe**[5].

Let's do a few examples.

Example. Consider the problem

$$u_t + uu_x = 0,$$

[5]Wow. Sounds serious. It is.

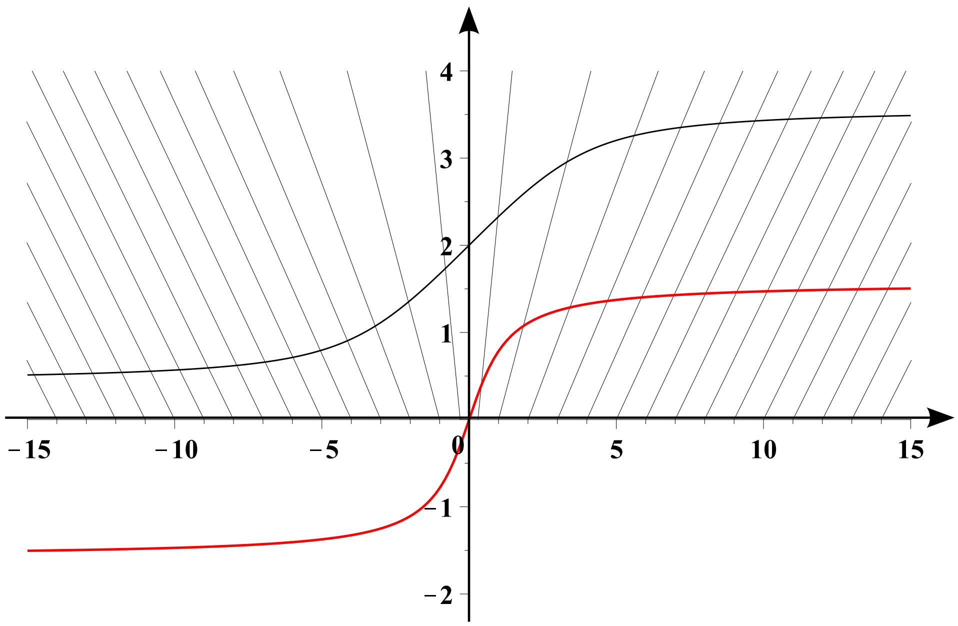

Figure 4.11: Moving the initial condition along the spreading characteristics.

$$u(x,0) = \arctan(x).$$

The implicit solution to this problem is given by

$$\begin{aligned} u(x,t) &= \arctan(x_0), \\ x &= x_0 + tc(u_0(x_0)) \\ &= x_0 + tu_0(x_0) \\ &= x_0 + t\arctan(x_0). \end{aligned}$$

We cannot solve this last equation for x_0 as a function of x and t, and an implicit solution is the best we can do. However, we can plot the characteristics. We have

$$t = \frac{x - x_0}{\arctan(x_0)}.$$

These characteristics, for varying x_0 are plotted in Fig. 4.11. We see that the characteristics are spreading out, and the initial profile of a front-like arctan becomes less steep as time progresses, since the characteristics in the region of the front are fanning out.

Example. Consider the problem

$$\begin{aligned} u_t + uu_x &= 0, \\ u(x,0) &= -\arctan(x). \end{aligned}$$

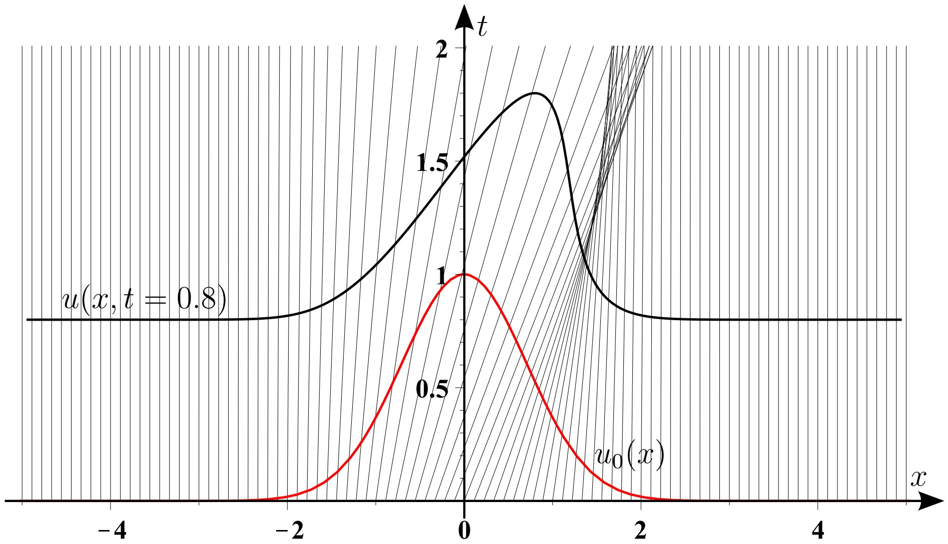

Figure 4.12: Moving the initial condition along the crossing characteristics.

This is almost the same problem as above, but the sign of the initial condition is flipped. The implicit solution to this problem is given by

$$\begin{aligned} u(x,t) &= -\arctan(x_0), \\ x &= x_0 + tc(u_0(x_0)) \\ &= x_0 + tu_0(x_0) \\ &= x_0 - t\arctan(x_0). \end{aligned}$$

As before, we cannot solve this last equation for x_0 as a function of x and t, and an implicit solution is the best we can do. However, we can plot the characteristics. We have

$$t = \frac{-x + x_0}{\arctan(x_0)}.$$

These characteristics, for varying x_0 are plotted in Fig. 4.13 (left). We see that the characteristics are crossing, with an apparent gradient catastrophe occurring at $t = 1$[6]. A wedge region where the characteristics cross is formed. A few time slices of solution profiles are shown in the right panel of Fig. 4.13. We note that the front profile becomes steeper as we approach the gradient catastrophe time, referred to as the **breaking time**.

As discussed, if the characteristics cross, we will get $u_x, u_t \to \infty$, as $t \to t_b$, the so-called breaking time. Now what? The PDE is no longer and we need to rethink what we are doing.

[6]We'll check whether this is correct soon.

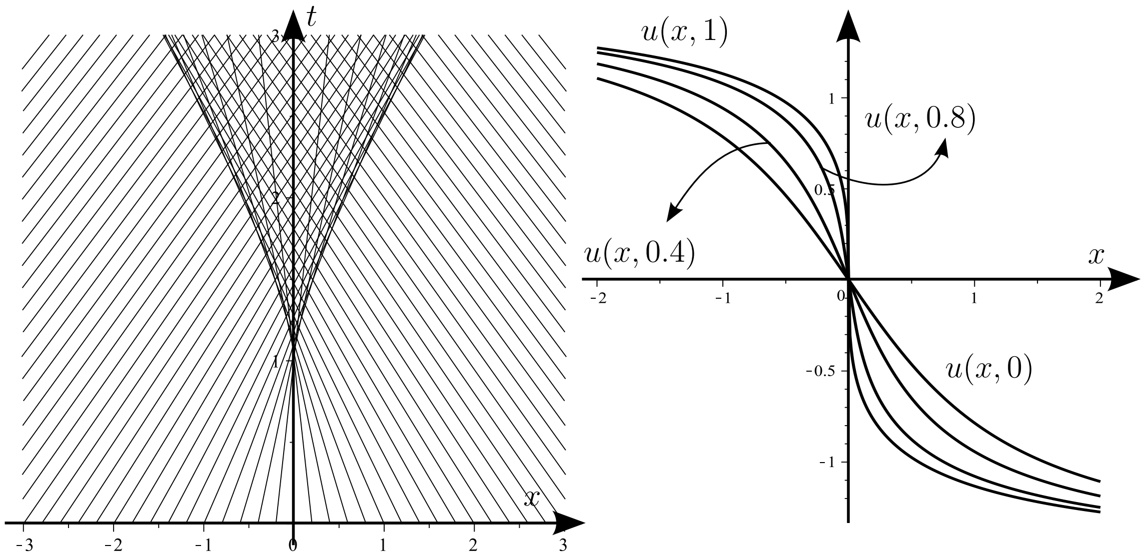

Figure 4.13: Crossing characteristics and the solution profiles that go with them.

We know that the slope of the characteristics is the inverse of the velocity:

$$\text{slope} = \frac{1}{c(u_0(x_0))}.$$

Let's look at the effect this has on different profiles. Let's assume, as in our examples that $c(u)$ is an increasing function of u. In other words, higher values of u will move with higher velocities. Suppose we start with an initial profile $u_0(x)$ that resembles an increasing front, as in Fig. 4.14. Since higher values of u travel faster, the top part of the profile will move more ahead of the bottom part, and the overall effect is that of the spreading out of the solution, *i.e.*, the solution becomes less steep. As shown in the figure, this corresponds to two characteristics where the right one has a lesser slope $1/c_2$ than the left one $1/c_1$, leading to the characteristics fanning out. Thus the values of u in between u_1 and u_2 are being spread out over a larger x interval.

On the other hand, if we start with a downward front, we obtain the situation depicted in Fig. 4.15. Now the higher velocities of the higher values of u lead to the steepening of the profile. The characteristics cross, and the values of u between u_1 and u_2 are condensed in an x interval that shrinks to a point at the crossing of the characteristics, leading to a profile that is infinitely steep: the solution becomes steeper as the interval length decreases, until it becomes vertical.

It might be tempting to guess that the solution behaves as illustrated in Fig. 4.16. **This is incorrect!** The evolution from the first panel to the second one is correct. But at the second panel, the solution has a vertical tangent, and we cannot rely on the characteristics or anything else coming from the PDE anymore. Since the PDE is no longer valid, we cannot

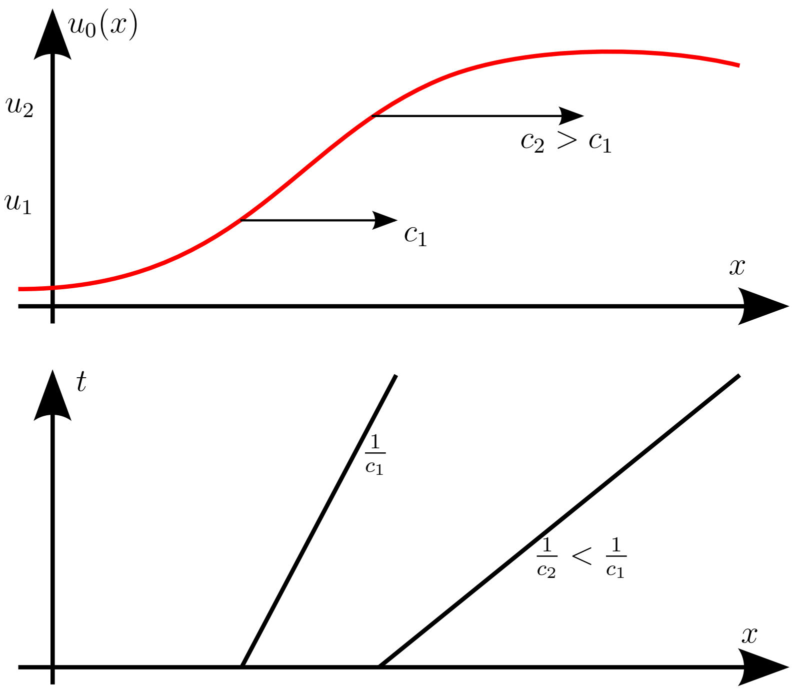

Figure 4.14: The effect of spreading characteristics

use it to move from the second to the last panel. Of course, the solution $u(x, t)$ is supposed to be a single-valued function of x and t. In the third panel, there exists an entire interval of x values for which the solution is tripple valued. Woe!

Before we figure out what happens after a shock (a profile with vertical tangent) forms, we should first determine **when** a shock forms. Thus, we want to determine the so-called breaking time $t_b \geq 0$.

Example. We revisit the example we plotted the characteristics and the solution profiles for in Fig. 4.12. The problem is given by

$$u_t + uu_x = 0,$$
$$u(x, 0) = e^{-x^2}.$$

The characteristics plotted in Fig. 4.12 are given by

$$x = x_0 + te^{-x_0^2} \quad \Rightarrow \quad t = (x - x_0)e^{x_0^2}.$$

The plot seems to indicate that $t_b \approx 1.2$. How can we find this value?

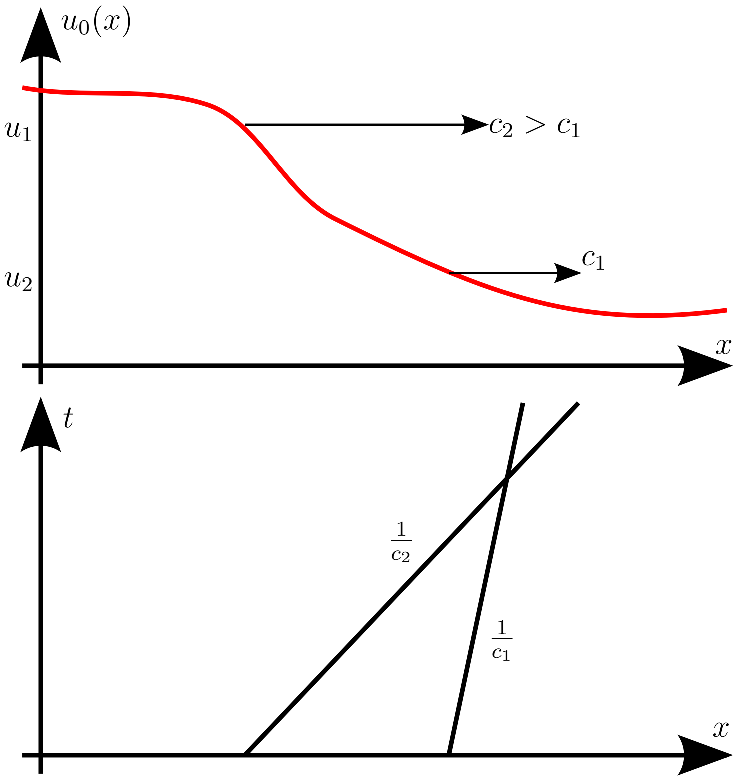

Figure 4.15: The effect of crossing characteristics

The breaking time t_b

The breaking time t_b is the first time for which u_x or u_t become infinite. Let us calculate u_x and u_t. We have

$$\begin{aligned} & u = u_0(x_0), \\ \Rightarrow \quad & u_t = u_0'(x_0)\frac{\partial x_0}{\partial t}, \quad \text{and} \\ \Rightarrow \quad & u_x = u_0'(x_0)\frac{\partial x_0}{\partial x}, \end{aligned}$$

where we have used the chain rule, since x_0 depends implicitly on x and t. Assuming that $u_0(x_0)$ is a nice profile (*i.e.*, no vertical tangents), we see that we need to determine when $\partial x_0/\partial x$ and/or $\partial x_0/\partial t$ are infinite. We have

$$x = x_0 + tc(u_0(x_0)).$$

Taking an x derivative, we get

$$1 = \frac{\partial x_0}{\partial x} + tc'(u_0(x_0))u_0'(x_0)\frac{\partial x_0}{\partial x}$$

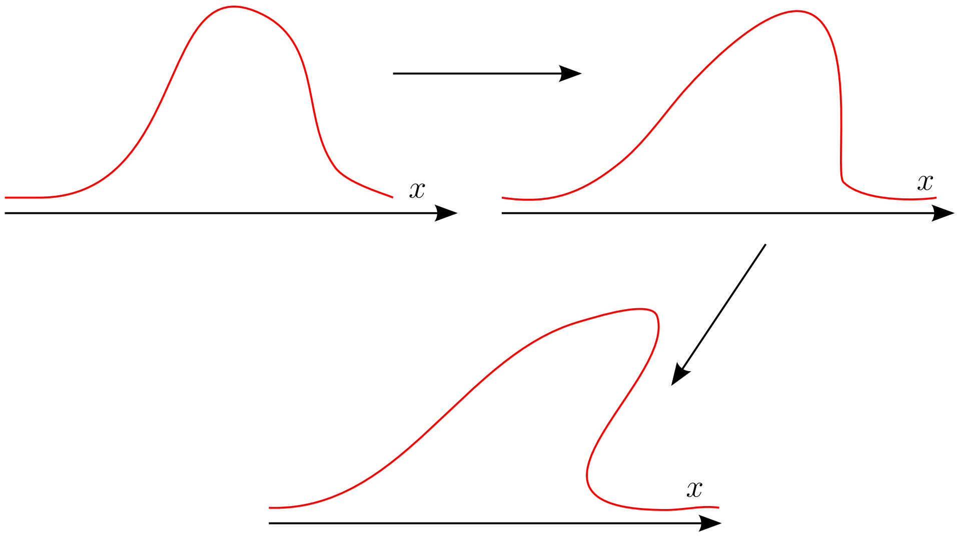

Figure 4.16: The incorrect evolution of the spatial steepening profile

$$\Rightarrow \qquad \frac{\partial x_0}{\partial x} = \frac{1}{1 + tc'(u_0(x_0))u_0'(x_0)}.$$

On the other hand, if we take a t derivative:

$$0 = \frac{\partial x_0}{\partial t} + c(u_0(x_0)) + tc'(u_0(x_0))u_0'(x_0)\frac{\partial x_0}{\partial t}$$
$$\Rightarrow \qquad \frac{\partial x_0}{\partial x} = \frac{-c(u_0(x_0))}{1 + tc'(u_0(x_0))u_0'(x_0)}.$$

In order for one of these expressions to be infinite, their denominator needs to be zero. They have the same denominator, as you might expect. Thus we need

$$1 + tc'(u_0(x_0))u_0'(x_0) = 0$$
$$\Rightarrow \qquad t = \frac{-1}{c'(u_0(x_0))u_0'(x_0)}.$$

We want to find the smallest such time, as long as it is positive. Thus

$$t_b = \min_{x_0} \frac{-1}{c'(u_0(x_0))u_0'(x_0)} \geq 0.$$

The x_0 for which the minimum value t_b is attained gives the characteristic along which the gradient catastrophe will happen.

Example. Revisiting the example above, we have

$$c(u_0(x_0)) = u_0(x_0) \quad \Rightarrow \quad c' = 1,$$

and

$$u_0(x) = e^{-x^2}.$$

The expression for the breaking time becomes

$$\begin{aligned} t_b &= \min_{x_0} \frac{-1}{-2x_0 e^{-x_0^2}} \\ &= \min_{x_0} \frac{1}{2x_0 e^{-x_0^2}}. \end{aligned}$$

Thus, we can find this minimum by maximizing the function

$$F(x) = 2xe^{-x^2}.$$

We have

$$\begin{aligned} F' &= 2e^{-x^2} - 4x^2 e^{-x^2} \\ &= 2e^{-x^2}(1 - 2x^2), \end{aligned}$$

which is zero for

$$x = \frac{\pm}{\sqrt{2}}.$$

Further,

$$F(1/\sqrt{2}) = \sqrt{2}e^{-1/2} > 0, \quad F(-1/\sqrt{2}) = -\sqrt{2}e^{-1/2} < 0.$$

We discard the second possibility, since it leads to a negative breaking time. Thus

$$t_b = \sqrt{\frac{e}{2}} \approx 1.16,$$

which occurs along the characteristic that starts at

$$x_0 = \frac{1}{\sqrt{2}} \approx 0.707,$$

which is in excellent agreement with Fig. 4.12.

You should use this method to find the breaking time in the example with $u_0(x) = -\arctan(x)$.

4.5 Shock waves

When we derived the differential form of the conservation law, we assumed that our functions had continuous derivatives. If they do not, we have to work with the integral form. In other words, once a gradient catastrophe happens and the derivatives of the solution $\to \infty$, the differential form of the equation ceases to be valid and we turn to the integral form. In this section, having determined the breaking time, we figure out how to move beyond it.

Again, let us consider

$$\begin{aligned} u_t + c(u)u_x &= 0, \\ u(x,0) &= u_0(x), \end{aligned}$$

with implicit solution given by

$$\begin{aligned} u(x,t) &= u_0(x_0), \\ x &= x_0 + tc(u_0(x_0)). \end{aligned}$$

As discussed, this solution is valid as long as the derivatives u_x and u_t are finite, *i.e.*, up until $t = t_b$. For $t > t_b$, we have to return to the integral form, which is

$$\frac{d}{dt}\int_a^b u(x,t)dx = \phi(a,t) - \phi(b,t),$$

where we have equated $f \equiv 0$, and

$$\phi_x = c(u)u_x \quad \Rightarrow \quad \phi = \int c(u)du.$$

Once the derivative becomes infinite, we should expect the solution to develop a discontinuity, known as a **shock**. In other words, $u(x,t)$ will have different values u^- (before) and u^+ (after) the shock, where the derivative is vertical. At either side of the shock, the solution satisfies the PDE, but the PDE cannot capture the shock itself.

Suppose we have a wedge-like region of the (x,t) plane where the characteristics cross, as in Fig. 4.17. The idea is to insert a path in the (x,t) plane along which the shock will propagate. Up until the shock path, our previous solution is valid, and we continue to follow the characteristics, as before, until they hit the shock. Such a shock path $x = x_s(t)$ is inserted in red in Fig. 4.17. Once we have inserted the shock path, we can merrily continue the characteristics until the hit the shock, even into the wedge region!

So, how do we find how the shock moves? The integral form is given by

$$\frac{d}{dt}\int_a^b u(x,t)dx = \phi(a,t) - \phi(b,t).$$

Suppose a shock exists at $x = x_s(t)$, in between $x = a$ and $x = b$. Then

$$\frac{d}{dt}\left(\int_a^{x_s} u(x,t)dx + \int_{x_s}^b u(x,t)dx\right) = \phi(a,t) - \phi(b,t).$$

Now we use the Leibniz rule[7], which is really just a big ol' chainrule. Since x_s depends on t, we get

$$\int_a^{x_s} u_t(x,t)dx + u(x_s^-,t)\frac{dx_s}{dt} + \int_{x_s}^b u_t(x,t)dx - u(x_s^+,t)\frac{dx_s}{dt} = \phi(a,t) - \phi(b,t),$$

[7]Recall, $\frac{d}{dt}\int_{a(t)}^{b(t)} f(x,t)dx = \int_{a(t)}^{b(t)} f_t(x,t)dx + f(b(t),t)b'(t) - f(a(t),t)a'(t)$

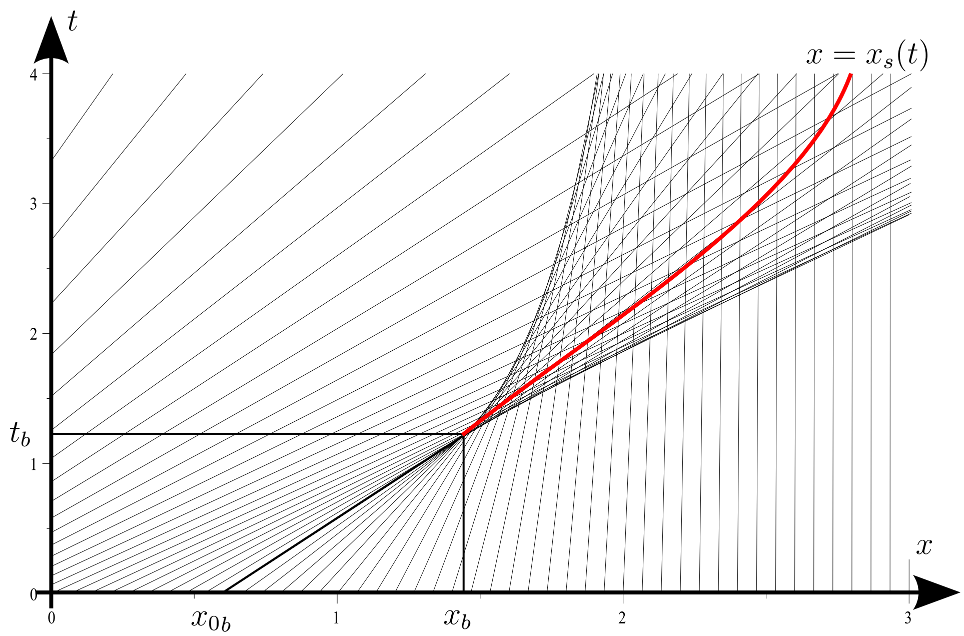

Figure 4.17: The wedge region where characteristics cross, with a shock curve inserted.

where

$$u^- := u(x_s^-, t) = \lim_{x \to x_s^-} u(x,t),$$
$$u^+ := u(x_s^+, t) = \lim_{x \to x_s^+} u(x,t),$$

are the limits as x approaches x_s from the left and right, respectively.

Next, since a and b are completely arbitrary in this process, we now let $a \to x_s^-$ and $b \to x_s^+$. This eliminates the integrals above, since they are integrals of bounded functions over an interval that shrinks to zero. We are left with

$$u^- x_s' - u^+ x_s' = \phi^- - \phi^+$$
$$\Rightarrow \qquad \frac{dx_s}{dt} = \frac{\phi^- - \phi^+}{u^- - u^+} := \frac{\Delta\phi}{\Delta u}.$$

This is known as the **Rankine-Hugoniot condition**. It dictates the speed at which the shock moves. Note that for the PDE

$$u_t + uu_x = 0,$$

we have that $\phi = u^2/2$, so that

$$x_s' = \frac{\phi^- - \phi^+}{u^- - u^+} = \frac{\frac{1}{2}{u^-}^2 - {u^+}^2}{u^- - u^+} = \frac{1}{2}(u^- + u^+),$$

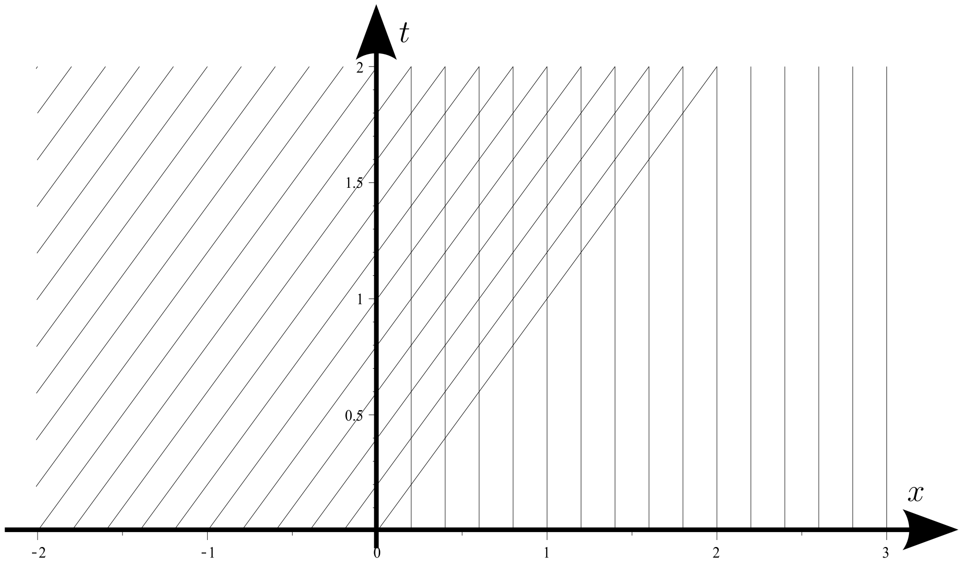

Figure 4.18: The characteristics before a shock path is inserted.

and the shock speed is simply the average value of the solution to the left and right of it. Let's see how the Rankine-Hugoniot condition works.

Example. Consider the problem

$$u_t + uu_x = 0,$$
$$u(x,0) = u_0(x) = \begin{cases} 1, & x \le 0, \\ 0, & x > 0. \end{cases}$$

The implicit solution is given by

$$u(x,t) = u_0(x_0),$$
$$x = x_0 + tu_0(x_0).$$

For $x_0 > 0$, this becomes $u = 0$ and $x = x_0$, while for $x_0 \le 0$, we get $u = 1$ and $x = x_0 + t$. These characteristics are drawn in Fig. 4.18.

We see there is a triangular region where the characteristics cross. We wish to insert a shock path there, starting at $(x,t) = (0,0)$. According to Rankine-Hugoniot,

$$\frac{dx_s}{dt} = \frac{\Delta\phi}{\Delta u}.$$

Here $\phi = u^2/2$. Further, on the left-side of the shock, we will have $u^- = 1$, thus $\phi^- = 1/2$. On the right side of the shock, $u^+ = 0$, so that $\phi^+ = 0$. Thus

$$\frac{dx_s}{dt} = \frac{1}{2} \quad \Rightarrow \quad x_s = \frac{1}{2}t + \alpha,$$

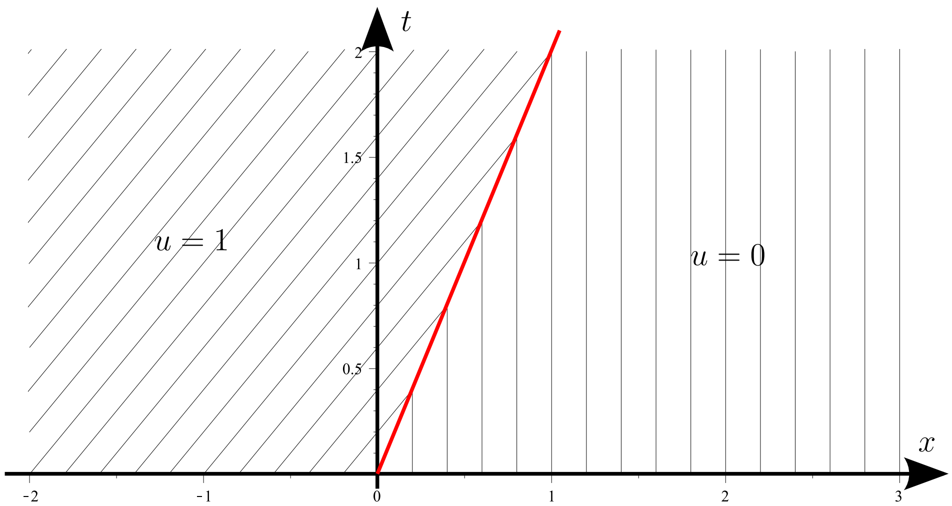

Figure 4.19: The characteristics with the shock path is inserted.

where α is an integration constant. Since $x_s = 0$ at $t = 0$, we see that $\alpha = 0$. Thus the shock path is given by

$$x_s = \frac{t}{2}.$$

We insert this path in the (x, t) plane, which results in Fig. 4.19, where we have continued the characteristics up to the shock line, but not beyond. The solution for our problem is given by

$$u(x,t) = \begin{cases} 1, & x < t/2, \\ 0, & x > t/2. \end{cases}$$

Example. We consider a second example, from traffic flow. Recall that we the traffic flow model is governed by

$$u_t + \phi_x = 0,$$

with

$$\phi = v_1 u \left(1 - \frac{u}{u_1}\right).$$

Let's assume that the speed limit $v_1 = 45$ miles/hour. Similarly, we work with a maximal density of cars of $u_1 = 300$ cars/mile. Thus

$$\phi = 45 \left(u - \frac{u^2}{300}\right).$$

As an initial condition, we have a traffic moving to the right with a velocity of $v =$ 30miles/hour, running into a traffic jam at location $x = 0$. This is plotted in Fig. 4.20.

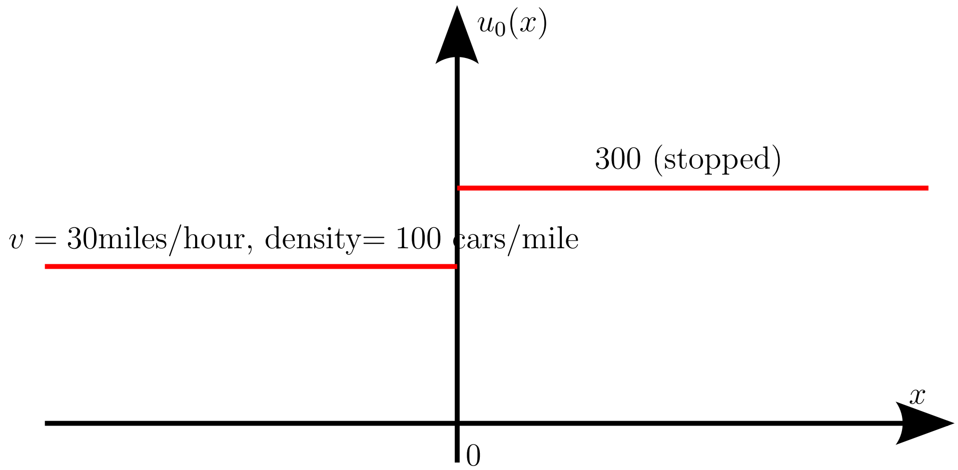

Figure 4.20: The initial condition for the traffic jam problem.

Indeed, if $v = 30$miles/hour, then

$$v = 30 = v_1 \left(1 - \frac{u}{u_1}\right) \quad \Rightarrow \quad u = 100.$$

Thus,

$$u_0(x) = \begin{cases} 100, & x < 0, \\ 300, & x \geq 0. \end{cases}$$

The implicit solution is given by

$$\begin{aligned} x &= x_0 + tc(u_0(x_0)), \\ u &= u_0(x_0). \end{aligned}$$

Here

$$c(u) = \phi'(u) = 45\left(1 - \frac{u}{150}\right).$$

The characteristics and the solution are different, depending on whether $x_0 < 0$ or $x_0 > 0$.

- For $x_0 < 0$, $u_0(x_0) = 100$, so that $c(u_0(x_0)) = c(100) = 15$. Thus the characteristics are given by

  $$x = x_0 + 15t,$$

  i.e., lines of slope $1/15$ in the (x, t) plane. Along these lines, the value of the solution is $u = 100$.

- For $x_0 > 0$, $u_0(x_0) = 300$, so that $c(u_0(x_0)) = -45$. Thus the characteristics are

  $$x = x_0 - 45t,$$

  lines of slope $-1/45$ in the (x, t) plane. Along these lines, the value of the solution is $u = 300$.

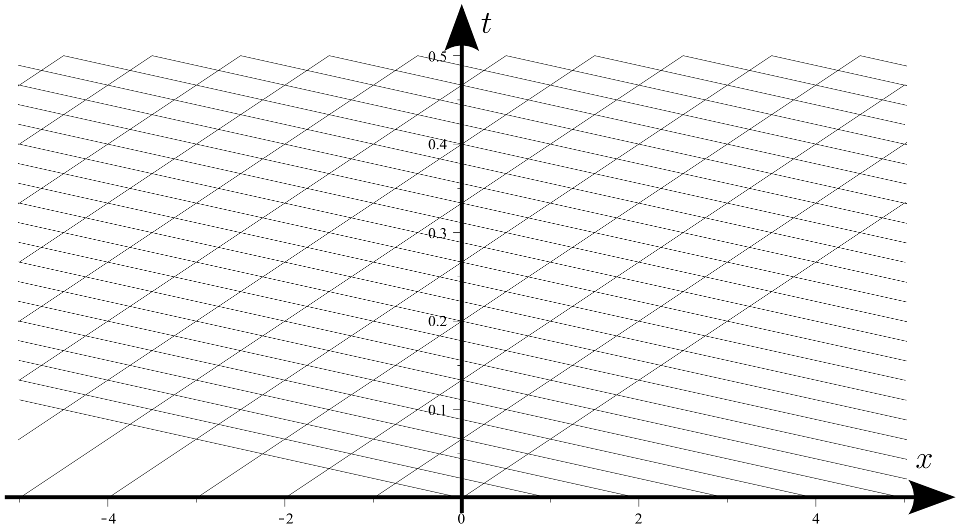

Figure 4.21: The characteristics for the traffic jam problem, without the insertion of a shock path.

Once again, there is a (very large) triangular region where characteristics cross. We use the Rankine-Hugoniot condition, because it's what the cool kids do. We have

$$\frac{dx_s}{dt} = \frac{\Delta\phi}{\Delta u}.$$

On the left-side of the shock, we will have $u^- = 100$, thus $\phi^- = 3000$. On the right side of the shock, $u^+ = 300$, so that $\phi^+ = 0$. Thus

$$x_s' = \frac{3000}{-200} = -15 \quad \Rightarrow \quad x_s = -15t,$$

where we have used that the shock starts at $x_s = 0$ at $t = 0$. The characteristics with this shock line inserted are shown in Fig. 4.22.

Finally, our solution is given by

$$u(x,t) = \begin{cases} 300, & x > -15t, \\ 100, & x < -15t. \end{cases}$$

Furthermore, we learn that the traffic jam backs up at a rate of 15 miles/hour.

4.6 Shock waves and the viscosity method

So far, we have calculated where and when shocks form, and when they form, we have examined how they move, using the Rankine-Hugoniot condition.

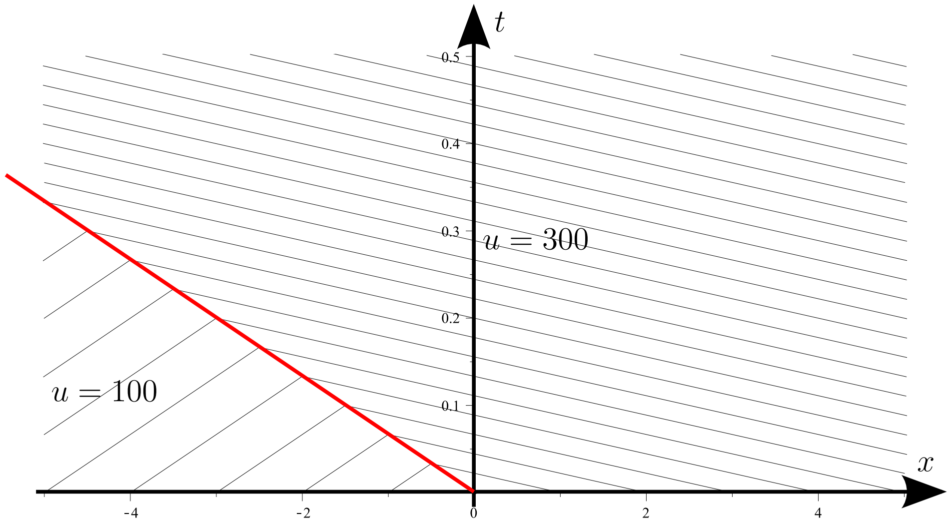

Figure 4.22: The characteristics for the traffic jam problem, with the insertion of a shock path.

In real applications, a shock never forms. Instead, we get a solution with a *very* steep, but not vertical profile. The presence of extra physical effects precludes the formation of actual shocks.

As an example, we consider once again traffic flow. How shall we update our model to take into account extra effects which presumably will prevent shock formation?

We have

$$u_t + \phi_x = 0.$$

So far, we have used $\phi = uv$, with

$$v = v_1 \left(1 - \frac{u}{u_1}\right)$$

as our velocity profile. We have assumed that drivers adjuct their speed based on the car density they observe where they are. Let's give drivers a bit more credit[8]. Let's assume that drivers can adjust their speed based on what they observe ahead of them.

For instance, suppose that drivers observe that the traffic is becoming more dense. In other words, $u_x > 0$. It would seem natural that they would decrease their speed. Further, if u is very small, any chance appears huge and might have a big effect. Similarly, if u is large, and change is not very important, this it is the relative change that matters: we wish to modify our equation for v as a function of u with a term $-ru_x/u$. Here u_x/u is the *relative density change*, and r is a positive proportionality constant. The $-$ sign ensures that the velocity increases as the density increases. Notice that the opposite happens if we see

[8]You can let me know in 30 years whether we should or not.

the density is lighter ahead: the drivers will start to speed up. Thus, our new velocity law becomes

$$v = v_1 \left(1 - \frac{u}{u_1}\right) - r\frac{u_x}{u}.$$

It follows that

$$\begin{aligned} & \phi = uv = v_1 \left(u - \frac{u^2}{u_1}\right) - ru_x \\ \Rightarrow \quad & \phi_x v_1 \left(1 - \frac{2u}{u_1}\right) u_x - ru_{xx}. \end{aligned}$$

Our PDE becomes

$$u_t + \phi_x = 0 \quad \Rightarrow \quad u_t + v_1 \left(1 - \frac{2u}{u_1}\right) u_x = ru_{xx}.$$

We impose the boundary condition

$$\begin{aligned} \lim_{x \to \infty} u(x,t) &= u_1, \\ \lim_{x \to -\infty} u(x,t) &= u_0 < u_1. \end{aligned}$$

In other words, we have gridlock on the right, and some lower-density moving traffic moving into the gridlock. In addition, we have

$$\begin{aligned} \lim_{x \to \infty} u_x(x,t) &= 0, \\ \lim_{x \to -\infty} u(x,t) &= 0. \end{aligned}$$

Traveling wave solutions

If we look for solutions of the form

$$u(x,t) = f(x - ct),$$

we get

$$\begin{aligned} & -cf' + v_1 \left(1 - \frac{2f}{u_1}\right) f' = rf'' \\ \Rightarrow \quad & -cf + v_1 f - \frac{v_1}{u_1} f^2 = rf' + k. \end{aligned}$$

At this point, we can find c and k, using the boundary conditions. Evaluating the above at $+\infty$, we get

$$-cu_1 + v_1 u_1 - v_1 u_1 - r \cdot 0 + k \quad \Rightarrow \quad k = -cu_1.$$

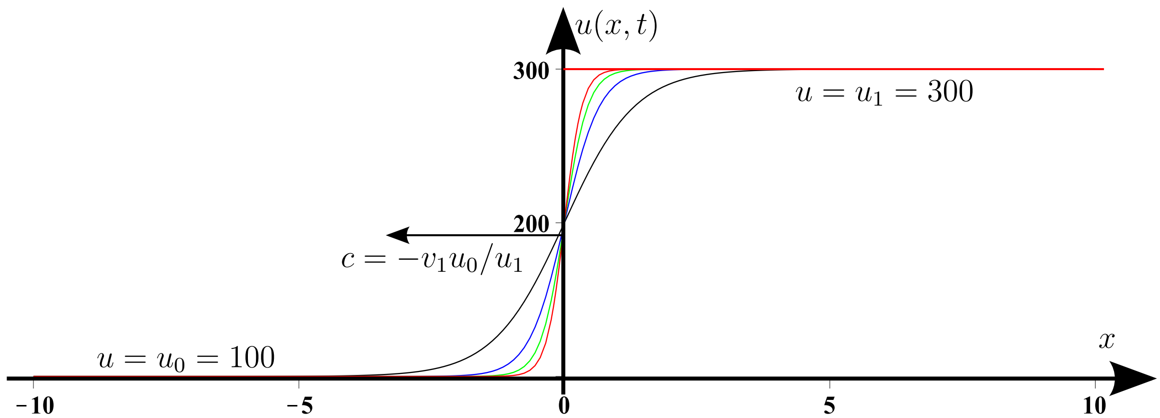

Figure 4.23: The traveling-wave solution for the traffic flow problem with smart drivers. Here $u_1 = 300$, $u_0 = 100$, $v_1 = 45$. Different values of r are used, ranging from 20 to 5.

From an evaluation at $-\infty$, we get

$$-cu_0 + v_1 u_0 - \frac{v_1 u_0^2}{u_1} = r \cdot 0 + k \quad \Rightarrow \quad c = -v_1 \frac{u_0}{u_1}.$$

Solving the ODE, we get[9]

$$u(x,t) = u_1 + \frac{u_0 - u_1}{1 + \exp\left(\frac{v_1(u_1-u_0)}{ru_1}(x + v_1 u_0 t/u_1)\right)}.$$

This solution is drawn in Fig. 4.23, for different values of r. Note that as $r \to 0$,

$$u(x,t) = \begin{cases} u_1, & x - ct > 0, \\ u_0, & x - ct < 0. \end{cases}$$

Thus, the solution becomes steeper and steeper as $r \to 0$, ultimately resulting in the shock solution as $r \to 0$.

Unfortunately, we cannot devote more time to the viscosity method at this point. This example gives a flavor, but it also indicates that the way to introduce extra effects which might arrest shock formation is very problem dependent.

4.7 Rarefaction waves

We have now seen how to follow the solution along characteristics as long as these don't cross. Next, we have learned what to do when they do cross: a shock is formed, and we

[9]Come on, you know the drill. Check this. I'll wait. Done already? OK. Let's move on.

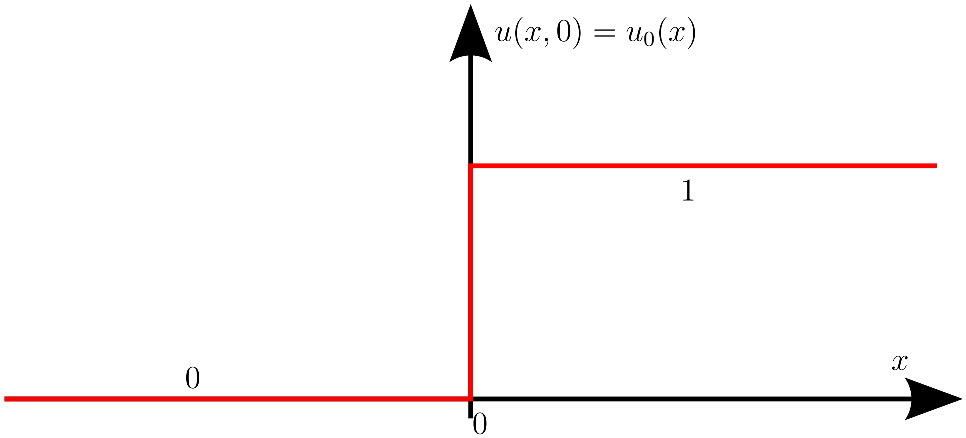

Figure 4.24: The initial condition $u_0(x)$ for the rarefaction example.

know how to propagate it. We know everything!

Example. Consider the problem

$$u_t + uu_x = 0,$$
$$u(x,0) = u_0(x) = \begin{cases} 0, & x < 0, \\ 1, & x > 0. \end{cases}$$

The initial condition is shown in Fig. 4.24.

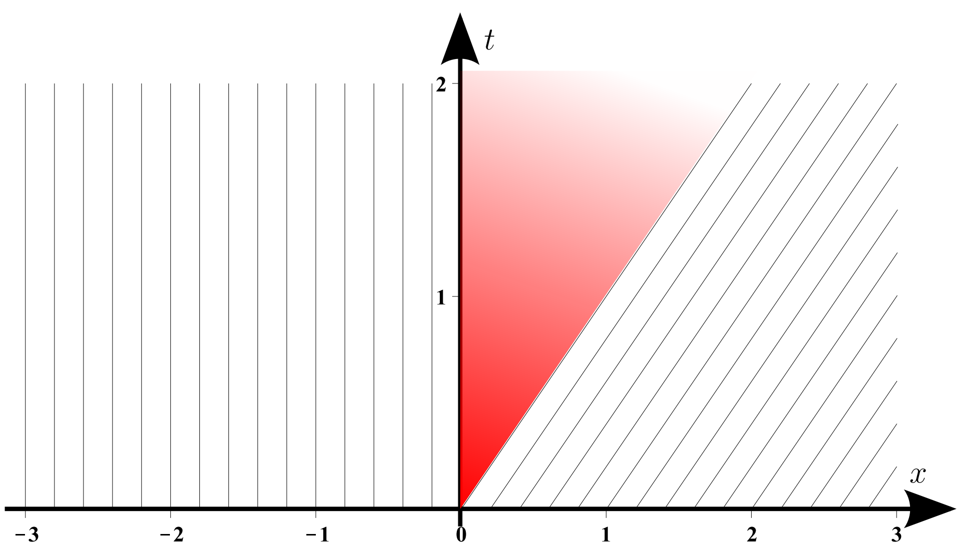

Figure 4.25: The characteristics with the empty region.

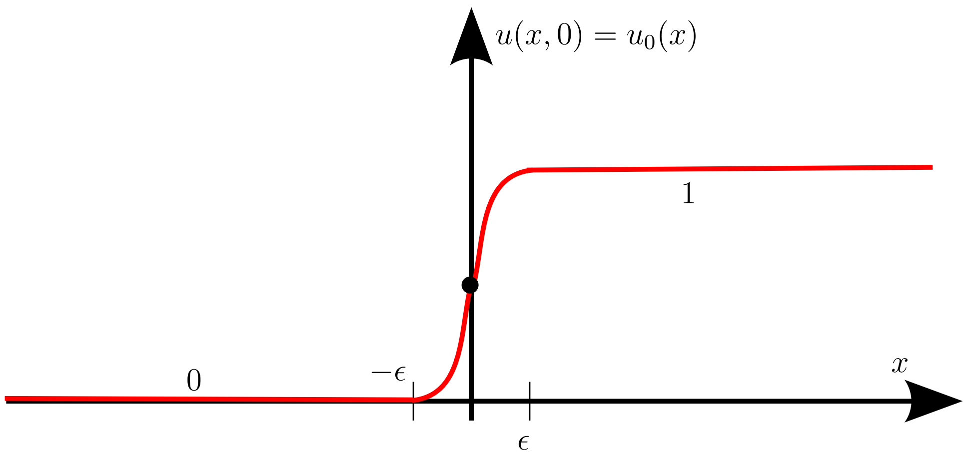

Figure 4.26: A new, improved, smooth initial condition.

The characteristics are given by

$$x = x_0 + u_0(x_0)t,$$

which give two different cases:

$$\begin{aligned} x_0 < 0: &\qquad x = x_0, \\ x_0 > 0: &\qquad x = x_0 + t. \end{aligned}$$

These characteristics are shown in Fig. 4.25. We face the immediate question what the value of the solution is in the red region without characteristics.

To answer this question, we revisit the same problem, but now with an initial condition that is smooth as opposed to discontinuous. Such an initial condition is drawn in Fig. 4.26. It is zero outside of $|x| > \epsilon$, but it changes smoothly from zero to one in $|x| < \epsilon$. The new initial condition is constructed so that it limits to the discontinuous one, $u_0(x)$, as $\epsilon \to 0$.

This gives rise to a new characteristics picture, shown in Fig. 4.27. The characteristics are given by

$$t = \frac{x - x_0}{u_0(x_0)},$$

and it is clear that for x_0 between $-\epsilon$ and ϵ, the characteristics will start at x_0 and have a slope that varies smoothly from 0 for $x_0 = -\epsilon$ to 1 for $x_0 = \epsilon$.

In the limit as $\epsilon \to 0$, we get a fan of characteristics, all starting at $(0,0)$, but with slope ranging from ∞ (on the left) to 1 on the right. This seems like a reasonable way to fill in the empty region: we assume an initial profile with high steepness that is smooth, and we let the steepness $\to \infty$. The limiting characteristic plane is shown in Fig. 4.28. We call the red region in Fig. 4.25 the **rarefaction** region and the characteristics that fill it the **rarefaction fan**.

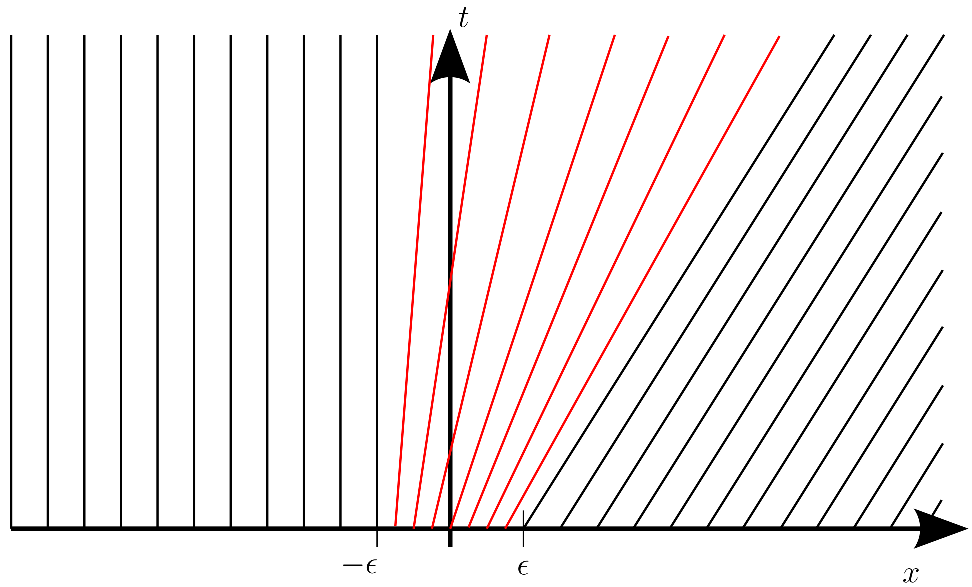

Figure 4.27: The characteristics corresponding to the smooth initial condition.

Now that we have determined the rarefaction characteristics, what is the solution $u(x,t)$ there? We know that $u(x,t)$ should be constant along characteristics. Since these are of the form

$$x = \omega t,$$

it follows that

$$u(x,t) = u(\omega t, t).$$

Since $u(x,t)$ is supposed to be constant along these characteristics, it follows that this last quantity has to be independent of t. Thus

$$u(x,t) = g(\omega) = g(x/t).$$

Now we have to find the function $g(\omega)$. Using the PDE $u_t + uu_x = 0$, we get

$$\begin{aligned} u_t &= g'(\omega)\frac{\partial \omega}{\partial t} = -\frac{x}{t^2}g'(\omega), \\ u_x &= g'(\omega)\frac{\partial \omega}{\partial x} = \frac{1}{t}g'(\omega). \end{aligned}$$

Thus

$$\begin{aligned} & & -\frac{x}{t^2}g'(\omega) + g(\omega)\frac{1}{t}g'(\omega) = 0 \\ \Rightarrow & & -\frac{x}{t}g'(\omega) + g(\omega)g'(\omega) = 0 \\ \Rightarrow & & -\omega g'(\omega) + g(\omega)g'(\omega) = 0 \\ \Rightarrow & & g'(\omega)(g(\omega) - \omega) = 0. \end{aligned}$$

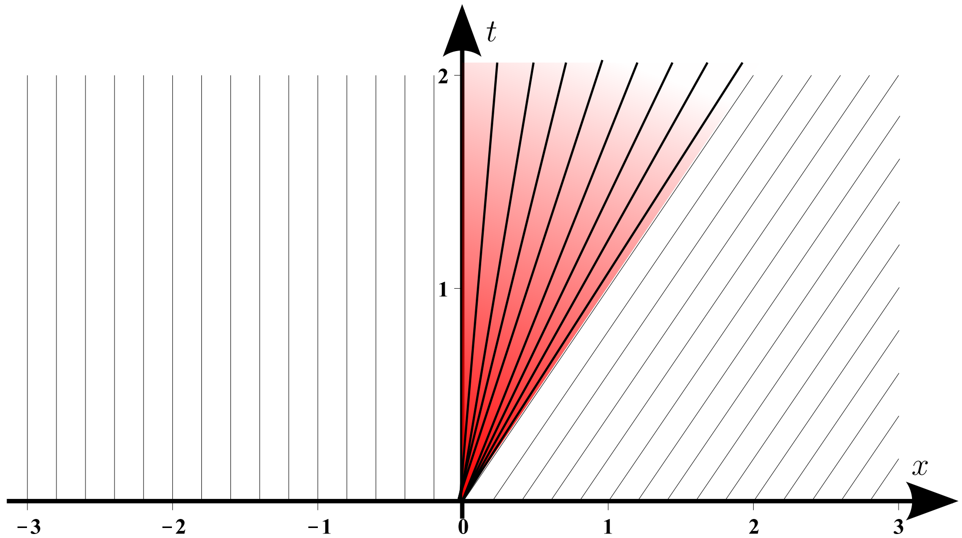

Figure 4.28: The characteristics with the rarefaction fan in place.

There are two possibilities:

1. $g'(\omega) = 0 \;\Rightarrow\; g(\omega)$ =constant. This would mean that $u(x,t)$ is still discontinuous, leading to a shock. But the characteristics do not cross for $t > 0$. Thus this doesn't work. We may ignore this possibility.

2. $g(x) = \omega$, so that

$$u(x,t) = \frac{x}{t},$$

in the rarefaction region.

In summary, our rarefaction problem has the solution

$$u(x,t) = \begin{cases} 0, & x < 0, \\ x/t, & 0 < x < t, \\ 1, & x > t. \end{cases}$$

A few time slices of the solution are shown in Fig. 4.29. The time slices make sense: due to the fanning out of the characteristics in the rarefaction region, the values of u get spread out over a larger region, as t increases.

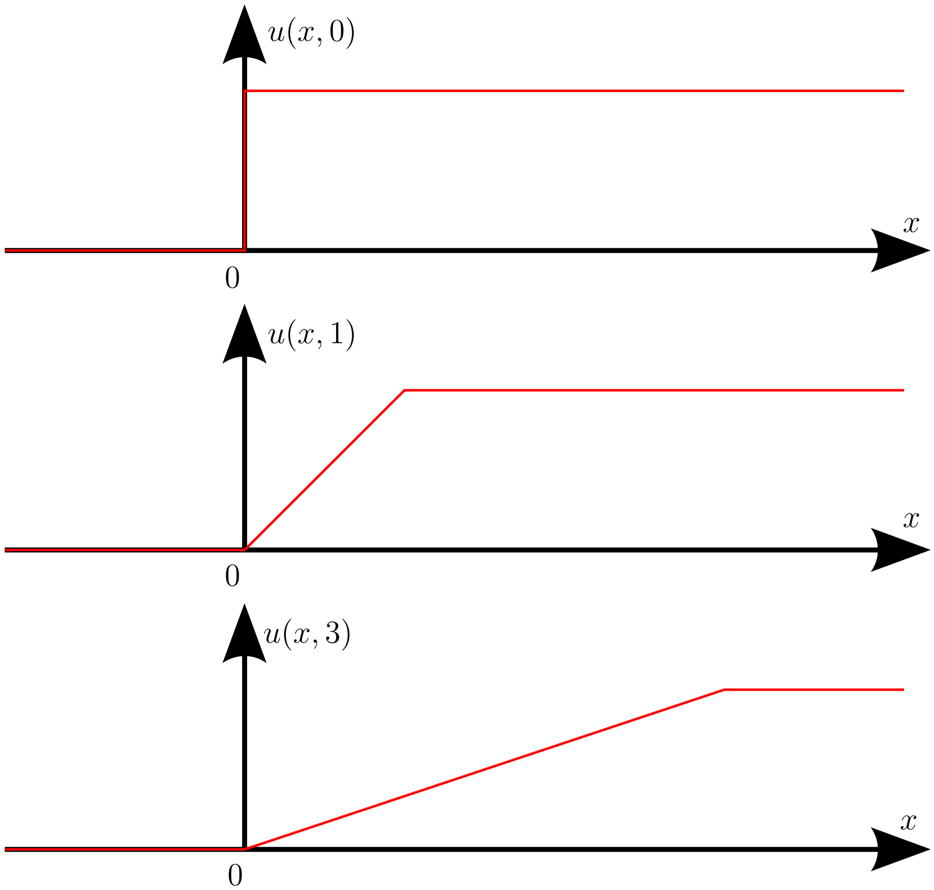

Figure 4.29: Different time slices of the solution.

4.8 Rarefaction and shock waves combined

Consider the problem

$$u_t + uu_x = 0,$$
$$u(x,0) = u_0(x) = \begin{cases} 1, & x \in (0,1), \\ 0, & x \notin (0,1). \end{cases}$$

The initial condition $u_0(x)$ for this problem is shown in Fig. 4.30. The characteristics are given by

$$x = x_0 + tu_0(x_0),$$

or,

$$\begin{aligned} x_0 &\in (0,1): & x &= x_0, \\ x_0 &\notin (0,1): & x &= x_0 + t. \end{aligned}$$

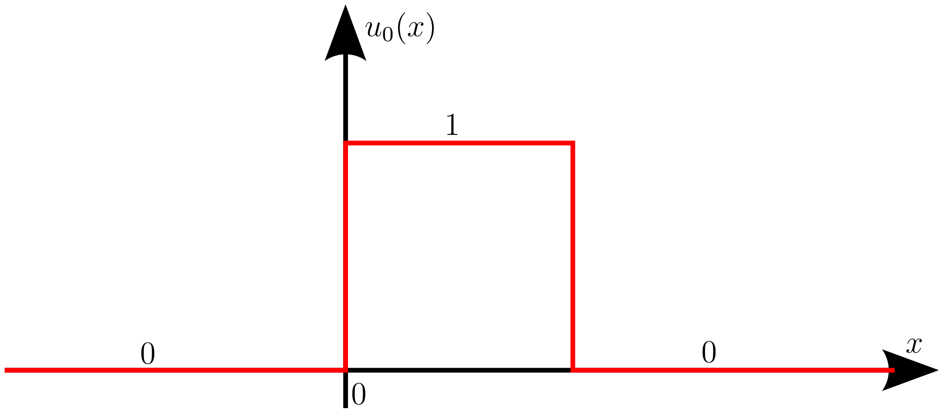

Figure 4.30: Different time slices of the solution.

The characteristics, as given above, are drawn in Fig. 4.31.

Regions A, B, and C present no problems. However, we see there are two immediate problems, already at $t = 0$, in regions D and E. In D, no characteristics are present, and a rarefaction fan is needed, while in E, a shock needs to be inserted. And we have no clue what's going to happen in region F! But, that's a concern for another page. Right now, one thing at a time: let's make sure we can move beyond the initial condition, by solving our problems at $t = 0$.

1. In region D, we insert characteristics $x = \omega t$, $\omega \in (0, 1)$, as in the previous section. Along these characteristics,
$$u(x, t) = g(\omega).$$
Substitution in the PDE gives, as before, $g(\omega) = \omega$, so that
$$u(x, t) = \omega = \frac{x}{t},$$
in region D.

2. In region E, we use the Rankine-Hugoniot condition to insert a shock path. The shock will receive the value $u^- = 1$ from the left (the characteristics from region B) and the value $u^+ = 0$ from the right (the characteristics from region C). Thus
$$x_s' = \frac{1}{2}(u^- + u^+) = \frac{1}{2} \quad \Rightarrow \quad x_s = \frac{1}{2}t + 1,$$
where the integration constant has been chosen to ensure that the shock line starts at $x_0 = 1$.

The characteristic plane with shock and fan inserted is shown in Fig. 4.32. We have resolved all problems at $t = 0$ and we can move the initial condition forward in time. The

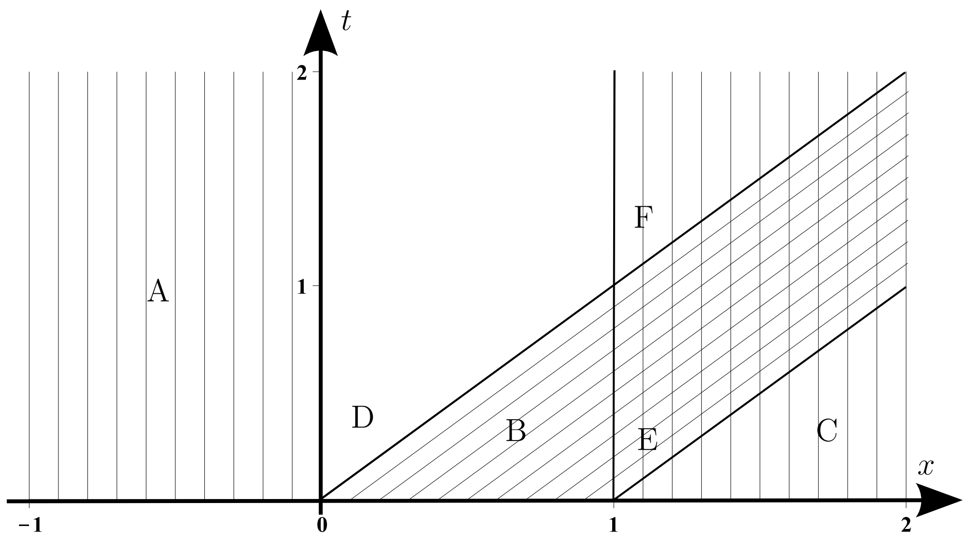

Figure 4.31: The characteristics, pre rarefaction, pre shock.

solution is perfectly well defined in regions A, B, c, D, and we have a beautiful shock moving along shock path E. But, we're not quite done yet! At $(x, t) = (2, 2)$, the shock line x_s (in red) hits the first characteristic (in bold black) of the rarefaction fan in region B. This implies that the left and right values we used in the Rankine-Hugoniot condition will have to be altered. Past $t = 2$, the value of u^+ (coming in from the right) remains at 0, following the characteristics in region C. However, the value u^- will no longer be 1, as now the values come from the rarefaction fan. Thus

$$u^- = \frac{x_s}{t},$$

where we have imposed that $x = x_s$ on the shock line. The problem determining our new shock path is

$$\begin{aligned} \frac{dx_s}{dt} &= \frac{1}{2}(u^- + u^+) = \frac{x_s}{2t}, \\ x_s(2) &= 2, \end{aligned}$$

where the last equation simply state that the new shock path is a continuation of the old one. We get

$$\begin{aligned} && \frac{dx_s}{x_s} &= \frac{1}{2}\frac{dt}{t} \\ \Rightarrow && \ln x_s &= \frac{1}{2}\ln t + c \\ \Rightarrow && \ln x_s &= \ln \sqrt{Ct}, \end{aligned}$$

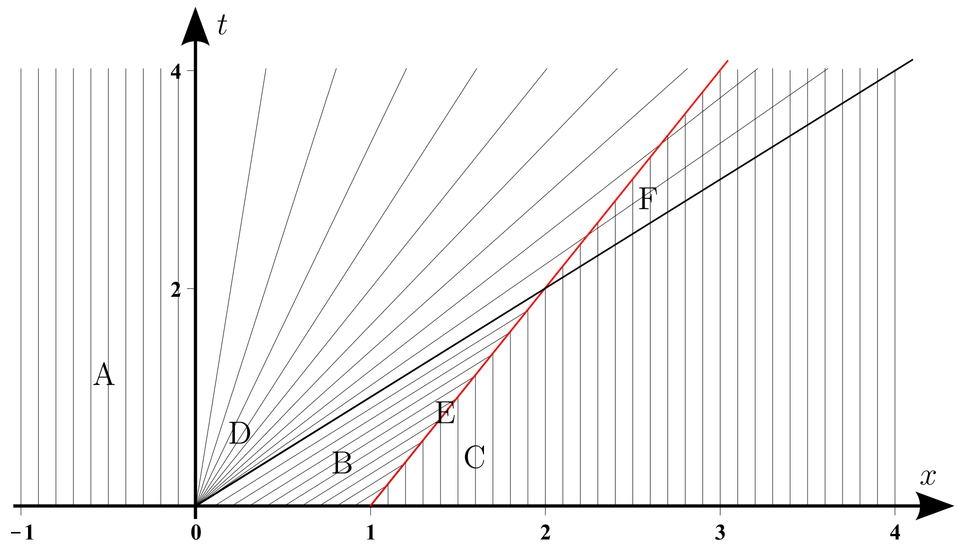

Figure 4.32: The new and improved characteristics, now with shock and fan.

where c and C are constants, related by $c = \ln(C)/2$. Applying the initial condition, we get

$$\ln 2 = \ln \sqrt{2C} \quad \Rightarrow \quad C = 2,$$

and our new shock path becomes

$$x_s = \sqrt{2t} \quad \Rightarrow \quad t = \frac{1}{2} x_s^2.$$

This is an upward parabola starting at $(x, t) = (2, 2)$. Note that at this point

$$\frac{dt}{dx_s} = x_s = 2,$$

so that the shock line starts of tangent to our original shock path.

The complete characteristic plane is drawn in Fig. 4.33, where all problems have been resolved. Region D contains a rarefaction fan, and we have a two-stage shock, following a straight line along E, and a parabolic path along F.

There are six different solution regions.

1. In $x < 0$, $t > 0$ the solution is $u = 0$. This is region A in Fig. 4.33.

2. In $t < x < t/2 + 1$, $t > 0$, the solution is $u = 1$. This corresponds to region B in Fig. 4.33.

3. If $x > t/2 + 1$, $t \in (0, 2)$, then $u = 0$. This corresponds to the part of region C below the red straight-line shock path in Fig. 4.33.

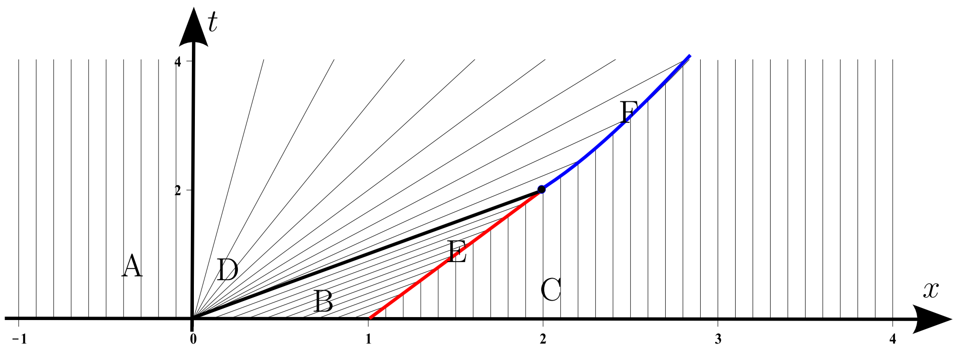

Figure 4.33: The newer and more improved characteristics, now with complete shock and fan.

4. If $x \in (0,t)$, $t \in (0,2)$, then $u = x/t$. This corresponds to the lower part of region D in Fig. 4.33, to the left of the black line.

5. Similarly, if $x \in (0, \sqrt{2t})$, $t > 2$, then $u = x/t$, corresponding to the upper part of region D in Fig. 4.33, to the left of the blue path.

6. Lastly, if $x > \sqrt{2t}$ and $t > 2$, then $u = 2$, corresponding to the part of region C, below the blue path.

The solution profiles at $t = 0$, $t = 1$, $t = 2$ and $t = 3$ are shown in Fig. 4.34. As $t \to \infty$, we have that the shock speed $x_s' = (\sqrt{2t})' = 1/\sqrt{2t} \to 0$, and the shock slows down as we proceed.

Note that

$$
\begin{aligned}
& & u_t + uu_x &= 0 \\
\Rightarrow & & \frac{d}{dt}\int_{-\infty}^{\infty} u dx + \int_{-\infty}^{\infty} uu_x dx &= 0 \\
\Rightarrow & & \frac{d}{dt}\int_{-\infty}^{\infty} u dx + \frac{1}{2}u^2\bigg|_{-\infty}^{\infty} &= 0 \\
\Rightarrow & & \frac{d}{dt}\int_{-\infty}^{\infty} u dx &= 0 \\
\Rightarrow & & \int_{-\infty}^{\infty} u(x,t) dx &= 1.
\end{aligned}
$$

The last line comes from the fact that the area underneath the solution is 1 initially, while the line before it says that this area is conserved. We have also used that $\lim_{x \to \pm\infty} u = 0$. We easily verify this: at $t = 1$, the area is the area of a trapezoid. It is equal to $1 \times (1/2 + 3/2)/2 = 1$. Next, at $t = 2$, we need the area of a triangle, equal to $2 \times 1/2 = 1$. Lastly, at $t = 3$, we still have a triangle, with area $\sqrt{6} \times \sqrt{2/3}/2 = 1$. It also follows from this argument that as

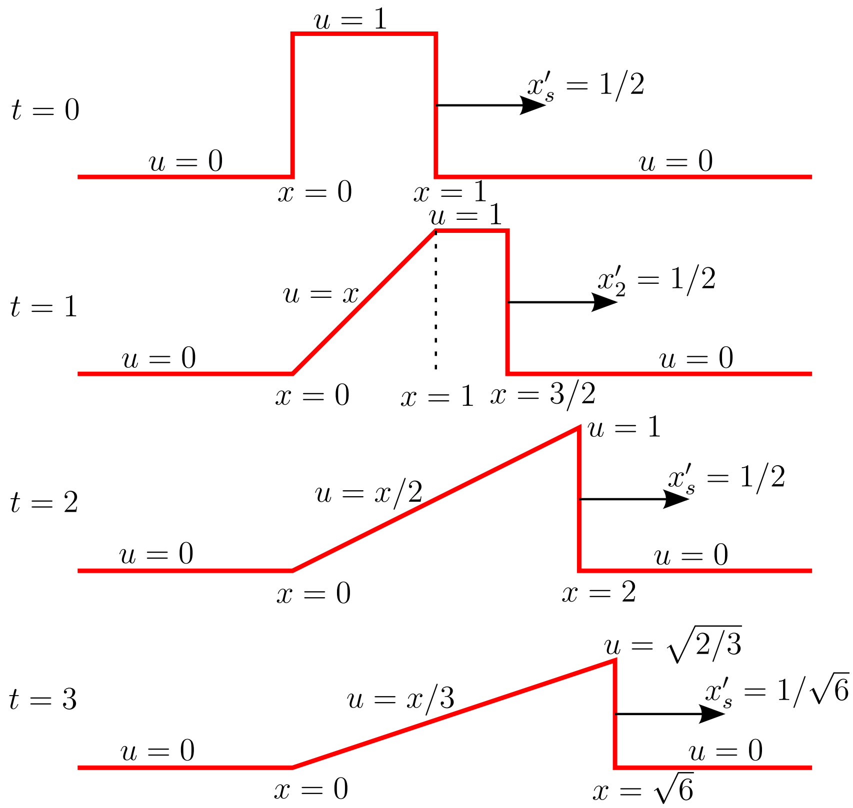

Figure 4.34: Three different solution profiles, at $t = 0$, $t = 1$, $t = 2$ and $t = 3$.

we march on, the top of the triangle is lowered, while its base grows so that its area remains unchanged.

A warning

Let's reconsider the example on page 92. We have

$$u_t + uu_x = 0,$$

$$u(x,0) = \begin{cases} 1, & x < 0, \\ 0, & x > 0. \end{cases}$$

Previously, we used $\phi = u^2/2$. With $u^- = 1$, $u^+ = 0$, $\phi^- = 1/2$, $\phi^+ = 0$, the Rankine-Hugoniot condition gives

$$x_s' = \frac{1}{2},$$

which we used to determine a perfectly fine shock. This condition is based on the conservation of u.

However, consider the same problem, but written as

$$uu_t + u^2 u_x = 0,$$
$$u(x,0) = \begin{cases} 1, & x < 0, \\ 0, & x > 0. \end{cases}$$

Now, the PDE is written

$$\rho_t + \Phi_x = 0,$$

with $\rho = u^2/2$ and $\Phi = u^3/3$. The Rankine-Hugoniot condition becomes

$$x_s' = \frac{\Delta\Phi}{\Delta\rho} = \frac{\frac{1}{3}u^{-3} - \frac{1}{3}u^{+3}}{\frac{1}{2}u^{-2} - \frac{1}{2}u^{+2}} = \frac{2}{3},$$

and we find a **different**, but still equally fine shock. Thus, the same PDE can give rise to two (or many more) conclusions about the shock velocity. How do we choose? Which one is correct?

It turns out the answer is **we** don't choose. We are lowly mathematicians, who solve problems given to us by our experimentalist friends[10]. The application we're dealing with dictates which quantity is conserved, which implies how we should write the PDE. One we know this, we can propagate the shock. But simply knowing the PDE is not enough information to work with the shock, as the PDE breaks down once we have a shock.

4.9 Weak solution of PDEs

Classical solutions of conservation laws

When we are faced with solving a PDE with initial and boundary conditions, we usually want to find a solution that satisfies the equation **everywhere** in our domain of interest. This implies that everywhere in this domain of interest, the solution should have as many continuous derivatives as appear in the equation. Such a solution is referred to as a **classical solution**.

What we have been doing recently does not result in classical solutions: we have often had solutions that are piecewise defined, and that have corners (*i.e.*, discontinuous derivatives) or even shocks (discontinuous function). So, apart from at a few points, such solutions satisfy the equation, just not everywhere. And we know this. After all, we returned to the integral form (the conservation law form) of the PDE to get such solutions. We refer to such piecewise defined solutions as **weak solutions**.

The notion of a weak solution allows us to get away with just a bit more: weak solutions (at least so far) satisfy the equation almost everywhere. In this lecture, we look at a different

[10] Everyone should have at least one experimentalist friend. Get yours today!

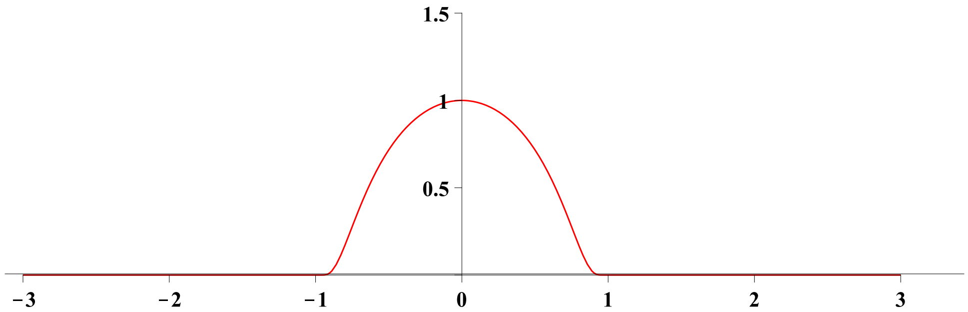

Figure 4.35: The test function $T(x)$.

way to characterize weak solutions of the PDE, through the so-called weak formulation of the PDE.

The weak form of the conservation law. Test functions

Consider the function

$$T(x) = \begin{cases} e^{-x^2/(1-x^2)}, & |x| < 1, \\ 0, & |x| \geq 0. \end{cases}$$

This function is plotted in Fig. 4.35.

Let's investigate this function at $x = 1$. Clearly, as $x \to 1$ from the right, $T(x)$ and all of its derivatives are zero at $x = 1$. How about the left limits?

$$\lim_{x<1} T(x) = e \lim_{x<1} e^{-1/(1-x^2)}.$$

The denominator in the exponent is always positive and approaches zero. Thus $-1/(1-x^2) \to -\infty$ and

$$\lim_{x<1} T(x) = 0.$$

Next, we calculate the derivative of $T(x)$ for $|x| < 1$[11]:

$$T'(x) = \frac{-2x}{(x^2-1)} e^{-x^2/(1-x^2)}.$$

Taking the limit as $x \to 1$ from the left, it is clear that the exponential will dominate and once again,

$$\lim_{x<1} T'(x) = 0.$$

The same argument holds for all derivatives: at all orders, we get the same exponential[12] and some rational function of x. Thus

$$\lim_{x<1} T^{(n)}(x) = 0, \quad \text{for } n \geq 0.$$

[11] No, this is not 1^{11}. Rather, it's the obligatory reminder that you should check this.

[12] Exponentials do that, you know.

Remark. As a side note, this implies that the Taylor series of $T(x)$ around $x = 1$ is given by

$$T(x) = \sum_{n=0}^{\infty} (x-1)^2 \frac{T^{(n)}(1)}{n!} \equiv 0.$$

Quite a weird result!

We call $T(x)$ a test function or, sometimes, a Schwarz function. It is a function that is infinitely differentiable but has **compact support**. This means that it is nonzero only in a finite region. By shifting x and rescaling, we can control where the support of the test function is located and how large it is.

Similarly, consider

$$T(x,t) = \begin{cases} e^{-(x^2+t^2)/(1-(x^2+t^2))}, & x^2+t^2 < 1, \\ 0, & x^2+y^2 \geq 0. \end{cases}$$

This is a test function in the (x, t) plane. Basically, it looks like a hat with compact support.

Suppose we wish to solve

$$\begin{aligned} u_t + \phi_x &= 0, \\ u(x,0) &= u_0(x), \end{aligned}$$

for $x \in \mathbb{R}$, t¿0. Let $T(x,t)$ be a test function. Then $T(x,t)u(x,t)$ is zero at every point outside the circle $x^2 + y^2 = 1$, so we have isolated a small part of $u(x,t)$. Then

$$\begin{aligned} & T(x,t)u_t + T(x,t)\phi_x = 0 \\ \Rightarrow \quad & \int_0^\infty \int_{-\infty}^\infty (T(x,t)u_t + T(x,t)\phi_x)\,dx\,dt = 0 \\ \Rightarrow \quad & \int_0^\infty dt \int_{-\infty}^\infty dx\,(T(x,t)u_t + T(x,t)\phi_x) = 0. \end{aligned}$$

Here the last line is just a less confusing version of the previous line.

Now we integrate by parts[13].

$$\begin{aligned} & \int_{-\infty}^\infty dx \left(u(x,t)T(x,t)|_{t=0}^{t=\infty} - \int_0^\infty dt u(x,t)T_t(x,t) \right) + \\ & \int_0^\infty dt \left(\phi(x,t)T(x,t)|_{x=-\infty}^{x=\infty} - \int_{-\infty}^\infty dx \phi(x,t)T_x(x,t) \right) = 0 \\ \Rightarrow \quad & \int_{-\infty}^\infty dx \left(0 - u_0(x)T(x,0) - \int_0^\infty u(x,t)T_t(x,t) \right) + \end{aligned}$$

[13]Secret: integration by parts is the main weapon of a good applied mathematician.

$$\int_0^\infty dt \left(0 - \int_{-\infty}^\infty dx \phi(x,t) T_x(x,t)\right) = 0$$

$$\Rightarrow \qquad -\int_{-\infty}^\infty u_0(x) T(x,0) - \int_0^\infty dt \int_{-\infty}^\infty dx \left(u(x,t)T_t(x,t) + \phi(x,t)T_x(x,t)\right) = 0.$$

This is known as the **weak form** of the PDE. It involves derivatives only of the test function, and just the value of u and ϕ, which are defined everywhere, even for weak solutions. We should remark that the weak form of the equation gives us back the PDE, provided that the solutions are classical solutions. In that case, we can start from the weak form, use integration by parts and get the PDE.

There are many reasons for wanting to work with the weak form of an equation. Here are two.

1. **Larger function spaces.** The modern theory of PDEs likes to talk about solutions is specific function spaces. Examples are $C(\mathbb{R})$, the space of continuous functions, defined for all $x \in \mathbb{R}$, $C^1(\mathbb{R})$, the space of differentiable functions on $\mathbb{R}$, *etc.* Clearly, $C^1(\mathbb{R}) \subset C(\mathbb{R})$, and so on. In general, we like to solve PDEs in the largest function space possible: a larger space implies fewer conditions on the solution, so the solution is more general. Since the weak form of the PDE has fewer conditions on the solution $u(x,t)$, by only needing its value, and not its derivatives, it allows us to work in larger function spaces.

2. **Numerical solutions.** An important example here is the finite element method: we approximate functions using basis elements, like piecewise constant functions, or piecewise linear functions, *etc.* Thus, we're giving up on classical solutions when we do this. This is not a problem at all for the weak formulation. Thus finite element methods are applied to the weak formulation, not to the PDE itself.