(Based on J. Boyd, "Chebyshev and Fourier Spectral Methods")

Convergence of Chebyshev series

Given $f(z) = \sum_{n=0}^{\infty} a_n T_n(z)$,

$T_n(z)$ - Chebyshev polynomial, $T_n(\cos\theta) = \cos n\theta$

is equivalent to a cosine series, since

$$f(\cos\theta) = \sum_{n=0}^{\infty} a_n \cos n\theta$$

hence, one may call Chebyshev series a Fourier cosine series "in disguise".

$f(z)$ is infinitely differential for $-1 \le z \le 1$, then convergence of series is determined by singularities of $f(z)$ in the complex plane.

Let $z = x + iy = \cos(\eta + i\mu) = \cosh\mu\cos\eta - i\sinh\mu\sin\eta$

$$\left.\begin{aligned} x(\mu,\eta) &= \cosh\mu\cos\eta \\ y(\mu,\eta) &= -\sinh\mu\sin\eta \end{aligned}\right) \text{ - elliptic coordinates, } \begin{aligned} &\mu \in [0, +\infty) \\ &\eta \in [0, 2\pi) \end{aligned}$$

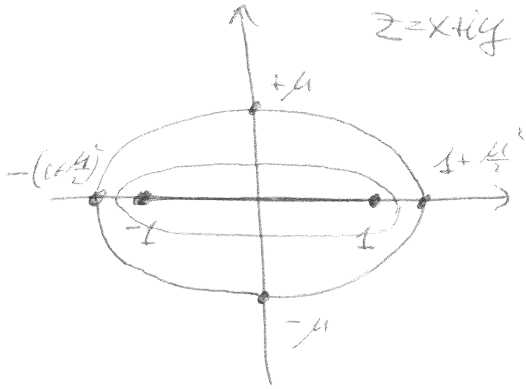
Clearly,

$$1 = \cos^2\eta + \sin^2\eta = \frac{x^2}{\cosh^2\mu} + \frac{y^2}{\sinh^2\mu}.$$

$\frac{x^2}{\cosh^2\mu} + \frac{y^2}{\sinh^2\mu} = 1$ — is an ellipse for each value of μ.

— ellipse foci $(\pm c, 0)$, where $c = \sqrt{\cosh^2\mu - \sinh^2\mu} = 1$

Ellipse foci are fixed, independent of μ



for small μ, $\cosh^2\mu \sim 1 + \frac{\mu^2}{2} +$

$\sinh\mu \sim \mu +$

$$\frac{x^2}{(1+\frac{\mu^2}{2})^2} + \frac{y^2}{\mu^2} = 1$$

at $y=0$

$x = \pm(1+\frac{\mu^2}{2})$

at $x=0$

$y = \pm\mu$

In the limit $\mu \to 0$, the ellipse shrinks to $x \in [-1,1]$, $y=0$

for large μ,

$\cosh\mu = \frac{1}{2}(e^{\mu} + e^{-\mu}) \sim \frac{1}{2}e^{\mu}$

$\sinh\mu = \frac{1}{2}(e^{\mu} - e^{-\mu}) \sim \frac{1}{2}e^{\mu}$

$$\frac{x^2}{(\frac{1}{2}e^{\mu})^2} + \frac{y^2}{(\frac{1}{2}e^{\mu})^2} = 1$$

$$x^2 + y^2 = (\tfrac{1}{2}e^{\mu})^2 = r^2$$

circle of radius $\frac{1}{2}e^{\mu}$ as $\mu \to \infty$

Observations:

— ellipses are narrow and long for small μ.

— Singularities of $f(z)$ that are very close to $z = \pm 1$ are no worse than singularities on the imaginary axis. Hence Chebyshev polynomials are much better at resolving singularities or boundary layers near endpoints at ± 1

— If $f(z)$ has a pole (branch point, etc) on an ellipse with $\mu = \mu_0$, then Chebyshev series converges in all ellipses with $\mu < \mu_0$

and diverges for $\mu > \mu_0$

- Also note, convergence on $[-1,1]$ does not imply convergence on $x \in (-\infty, +\infty)$!

Legendre Polynomials

Legendre polynomials are found from generating function

$$(*) \quad \frac{1}{\sqrt{1-2px+p^2}} = \sum_{n=0}^{\infty} p^n P_n(x)$$ – Taylor expansion in p, coefficients are Legendre polynomials

$$P_n(x) = \frac{1}{n!} \frac{\partial^n}{\partial p^n}\left(\frac{1}{\sqrt{1-2px+p^2}}\right)\Bigg|_{p=0}$$ – not very practical unless n is small

Recursion formula

$$(n+1) P_{n+1}(x) = (2n+1) x P_n(x) - n P_{n-1}(x), \quad P_0 = 1, \; P_1 = x$$

{ Try to derive the recursion formula from $(*)$ as a "HW" exercise }

Legendre polynomial convergence rate

The domain of convergence for Legendre series is the interior of the largest ellipse, with foci at ± 1, which is free of singularities of the function being expanded. This is identical to ellipses we found for Chebyshev convergence.

In the limit $N \to \infty$, the maximum error of a Legendre series is larger than that of the Chebyshev series, that is, if we compare approximation by N terms of Legendre series vs Chebyshev series the error is larger by $O(\sqrt{N})$

The generating function itself provides an example,

$$\frac{1}{\sqrt{1-2px+p^2}} = \sum_{n=0}^{\infty} p^n P_n(x)$$

on the other hand $$\frac{1}{\sqrt{1-2px+p^2}} = \sum_{n=0}^{\infty} a_n^{(\text{Chebyshev})} T_n(x)$$

One can then show, $a_n^{(\text{Cheb})} \sim \frac{p^n}{\sqrt{n}}, \quad n \to \infty$

{ As a 'HW' exercise, show the asymptotic $a_n \sim \frac{p^n}{\sqrt{n}}$ for $n \to \infty$
Hint: You can reduce this problem to Fourier series convergence }