Problem 1

Find the Fourier series expansion for the function,

$$f(x) = \cot \frac{x - ia}{2},$$

where $a > 0$, and $x \in [0, 2\pi]$.
There are multiple ways to solve this, one of them relies on contour integration.

Problem 2

Consider generating function for Legendre polynomials,

$$G(x,p) = \frac{1}{\sqrt{1 - 2px + x^2}} = \sum_{n=0}^{\infty} p^n P_n(x),$$

where $P_n(x)$ denotes nth Legendre polynomial. One can view this as an example of an expansion of $G(x,p)$ in series of Legendre polynomials with coefficients, $b_n = p^n$.

Let $G(x,p)$ be also expanded in a Chebyshev series,

$$G(x,p) = \sum_{n=0}^{\infty} a_n^{(Cheb)} T_n(x),$$

where $T_n(x)$ is nth Chebyshev polynomial, and $a_n^{(Cheb)}$ are the expansion coefficients.

Show that for large n the Chebyshev expansion coefficients are given asymptotically by,

$$a_n^{(Cheb)} \sim \frac{p^n}{\sqrt{n}}$$

for $n \to \infty$.

Problem 3

Consider KdV equation on the line $-\infty < x < \infty$,

$$u_t + 6uu_x + u_{xxx} = 0,$$

- Determine the soliton solution by looking for a traveling solution, $u(x,t) = f(x - ct)$, that satisfies $f(x) \to 0$ as $x \to \pm\infty$.
- Linearize the KdV equation around the solution you found, and determine find the equation satisfied by small perturbations around the soliton.
- Find the eigenvalue problem that determines the linear stability of a soliton.
- Can you describe the method you would use to solve the eigenvalue problem, and why you have chosen it.