$\texttt{MTH 839}$ Nonlinear Waves
Homework 1

Due: Nov 7, 2025

Dispersion relation

Consider Benjamin-Ono equation, linearize it and find its dispersion relation for small amplitude waves $u \ll 1$,

$$u_t + uu_x = \hat{H}u_{xx}$$

where $\hat{H}$ denotes the Hilbert transform defined as,

$$\hat{H}f(x) = \frac{1}{\pi}\text{p.v.}\int\limits_{-\infty}^{\infty} \frac{f(y)dy}{y-x},$$

and p.v. denotes principal value.

Method of Stationary Phase

Consider the linear Schrödinger equation on the line,

$$i\psi_t + \psi_{xx} = 0,$$

with decaying boundary conditions. We consider the initial value problem with data

$$\psi(x,0) = \psi_0(x),$$

where $\int_{-\infty}^{\infty} |\psi_0(x)|^2\, dx$ is finite.

1. Write down the solution $\psi(x,t)$ in terms of the Fourier transform of $\psi_0(x)$.
2. Using the method of stationary phase, determine the leading asymptotic behavior of $\psi(x,t)$ as $t \to \infty$ while keeping the ratio x/t fixed. Identify the point of stationary phase and describe qualitatively how the phase of the oscillatory integral contributes to the asymptotics.

3. For the Gaussian initial condition

$$\psi_0(x;a) = \frac{1}{\sqrt{\pi a}}\, e^{-x^2/a},$$

evaluate the integral exactly and compare the exact solution with the stationary-phase approximation along the lines $x/t = 1$ and $x/t = 2$. Verify that the amplitude of ψ decays as $|\psi(x,t)| \sim t^{-1/2}$ for large t.

Averaged Lagrangian ⇒ Envelope equation

Start from the Klein–Gordon Lagrangian with $V(\phi) = \frac{1}{2}\phi^2$. Consider a weakly nonlinear, slowly modulated wavetrain

$$\phi(x,t) \approx \epsilon^{1/2}\Big(A(X,T)e^{i(kx-\omega t)} + \overline{A}(X,T)e^{-i(kx-\omega t)}\Big), \quad (X,T) = (\epsilon x, \epsilon t),$$

1. Insert the ansatz into $\mathcal{L}$, average over the fast phase, and obtain an *averaged Lagrangian* $\langle \mathcal{L} \rangle(A, A_X, A_T)$ to $O(\epsilon^2)$.
2. Treat A and $\overline{A}$ as independent fields and derive the Euler–Lagrange equation for A.
3. Show that A satisfies the linear Schrödinger equation,

$$i(A_T + \omega'(k)A_X) + \tfrac{1}{2}\omega''(k)A_{XX} = 0,$$

and compute $\omega''(k)$ from the averaging.