Hedge Ratio = Slope of the Option Price Function (Delta)

1. Setup

Let the call option value be:

$$C = C(S, t)$$

where:

- S = stock price
- t = time
- X = strike price

The **hedge ratio (Delta)** is defined as:

$$\Delta = \frac{\partial C}{\partial S}$$

This is the **slope of the option price with respect to the stock price**.

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2. Delta in the Black–Scholes Model

In the Black–Scholes call option pricing formula:

$$C = S\,N(d_1) - Xe^{-rT}N(d_2)$$

it can be shown that:

$$\Delta = \frac{\partial C}{\partial S} = N(d_1)$$

👉 Interpretation:

- $N(d_1) \in (0, 1)$
- It measures how strongly the call option moves with the stock price

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3. Key Idea: Local Linear Approximation

For a small change in stock price dS:

$$dC \approx \frac{\partial C}{\partial S} dS = \Delta\,dS = N(d_1)\,dS$$

👉 Interpretation:

- If the stock increases by dS, the call increases by approximately $N(d_1) \cdot dS$

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4. Constructing the Hedge (Long Stock + Short Calls)

Now consider a portfolio where:

- You **hold 1 share of stock**
- You **write (short) h call options**

Portfolio value:

$$\Pi = S - hC$$

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5. Change in Portfolio Value

The change in the portfolio is:

$$d\Pi = dS - h\,dC$$

Substitute the approximation $dC \approx \Delta dS$:

$$d\Pi \approx dS - h\Delta dS = (1 - h\Delta)\,dS$$

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6. Choosing the Hedge Ratio

To eliminate risk (i.e., remove dependence on dS), set:

$$1 - h\Delta = 0 \quad \Rightarrow \quad h = \frac{1}{\Delta} = \frac{1}{N(d_1)}$$

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7. Intuition: Why This Eliminates Risk

- The stock **gains** dS when price rises
- Each call **gains** approximately ΔdS
- Since you are **short calls**, you lose $h\Delta dS$

With $h = 1/\Delta$:

$$\text{Loss from calls} = \frac{1}{\Delta} \cdot \Delta dS = dS$$

👉 So:

- Gain from stock = dS
- Loss from calls = dS
- Net effect = **0**

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8. Why This Makes the Portfolio Risk-Free

- The only uncertainty comes from stock price changes dS
- By choosing $h = 1/\Delta$, we eliminate this uncertainty

Thus:

$$d\Pi = \text{risk-free (deterministic)}$$

👉 Therefore:

- The portfolio must earn the **risk-free rate**
- This is the core idea behind Black–Scholes

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9. Economic Intuition (Very Important)

Think of Delta as:

👉 "How many shares one call option behaves like"

- One call behaves like $\Delta = N(d_1)$ shares
- Therefore:
  - One share is equivalent to $\frac{1}{\Delta}$ calls

So to hedge:

- Hold **1 share**
- Short $1/\Delta$ call options

This makes the position **insensitive to stock price movements**

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10. Bottom Line

- The hedge ratio is the slope:

$$\Delta = \frac{\partial C}{\partial S} = N(d_1)$$

- In your setup (long stock + short calls), the hedge requires:

$$h = \frac{1}{\Delta}$$

- This ensures:

$$\text{Gain from stock} = \text{Loss from calls}$$

- Hence, the portfolio becomes **locally risk-free**