Continuum Mechanics

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Contents

1 Ch1

1.1 Index notation

1.1.1 Summation convention and dummy indices

A dummy index is a repeated index.

$$s = a_i x_i = a_m x_m = \sum_{i=1}^{n} a_i x_i \tag{1}$$

$$a_{ij} x_i y_j = a_{11} x_1 y_1 + a_{12} x_1 y_2 + a_{13} x_1 y_3 + \ldots + a_{32} x_3 y_2 + a_{33} x_3 y_3 = \sum_i \sum_j a_{ij} x_i y_j \tag{2}$$

1.1.2 Free indices

A free index appears once in each product term of an equation. i, kl are free in

$$a_{ij} x_j = b_i, \ A_{km} A_{lm} = T_{kl}. \tag{3}$$

The first equation represents 3 equations for each b_i. The second equation represents 9 equations for each T_{kl}.

1.1.3 Kronecker delta

$$\delta_{ij} = 0, \ i \neq j; \quad 1, \ i = j \tag{4}$$

$$\delta_{ii} = 3 \tag{5}$$

$$\delta_{ij} = \delta_{ji} \tag{6}$$

$$\delta_{im} a_m = a_m \delta_{im} = a_m \delta_{mi} = \delta_{mi} a_m \tag{7}$$

$$\delta_{im} T_{mj} = T_{mj} \delta_{im} = T_{ij} \tag{8}$$

$$T_{ij} \delta_{ij} = T_{ij} \delta_{ji} = T_{ii} \tag{9}$$

1.1.4 Levi-Civita (permutation) symbol

$$\epsilon_{ijk} = \underbrace{1, \ ijk \to 123, \ 231, \ 312}_{1\to2\to3\to \ +}; \quad \underbrace{-1, ijk \to 321, \ 213, \ 132}_{- \ \leftarrow1\leftarrow2\leftarrow3}; \quad \underbrace{0, \ \text{otherwise}}_{1\to1\to2\to \ 0} \tag{10}$$

$$\delta_{ij} \epsilon_{ijk} = \delta_{ji} \epsilon_{ijk} = \epsilon_{jjk} = 0 \tag{11}$$

$$\epsilon_{ijk} \epsilon_{mnk} = \delta_{im} \delta_{jn} - \delta_{in} \delta_{jm} \tag{12}$$

$$\epsilon_{ijk} \epsilon_{mjk} = 2\delta_{im} \tag{13}$$

$$\epsilon_{ijk} \epsilon_{ijk} = 6 \tag{14}$$

1.1.5 Substitution

$$\underbrace{a_i = U_{im}(b_m),\ b_i = V_{im}c_m}_{\text{given}} \rightarrow \underbrace{b_m = V_{mn}c_n}_{\text{reindex}} \rightarrow \underbrace{a_i = U_{im}(V_{mn}c_n).}_{\text{substitute}} \tag{15}$$

1.1.6 Multiplication

$$p = a_m b_m,\ q = c_m d_m \rightarrow pq = (a_m b_m)(c_n d_n) \neq (a_m b_m)(c_m d_m). \tag{16}$$

1.1.7 Factoring

$$T_{ij}n_j - \lambda \underbrace{n_i}_{\mathbf{I.}} = T_{ij}n_j - \lambda \underbrace{\delta_{ij}n_j}_{\mathbf{I.}} = (T_{ij} - \lambda\delta_{ij})n_j. \tag{17}$$

1.1.8 Contracting

Contracting is the act of

$$T_{ij} \rightarrow T_{ii}. \tag{18}$$

it is true by contraction that

$$A_{ij} = B_{ij} + C_{ij} \rightarrow A_{ii} = B_{ii} + C_{ii}. \tag{19}$$

Trace

$$\alpha = S_{ii}. \tag{20}$$

Contracted multiplication term

$$u_i = A_{ij}v_j. \tag{21}$$

Multiplication of second order tensors is too a contraction:

$$C_{ij} = A_{ik}B_{kj}. \tag{22}$$

1.2 Tensors

1.2.1 Dot (inner, scalar) product and norm

Dot product $\mathbf{u} \cdot \mathbf{v} \Longleftrightarrow \mathbf{u}^T\mathbf{v}$ maps two vectors to a scalar.

$$\mathbf{u} \cdot \mathbf{v} = u_i\mathbf{e}_i \cdot v_j\mathbf{e}_j = u_i v_j \underbrace{(\mathbf{e}_i \cdot \mathbf{e}_j)}_{\mathbf{I.}} = u_i v_j \underbrace{\overbrace{\delta_{ij}}^{\mathbf{II.}}}_{\mathbf{I.}} = u_i \underbrace{v_j \overbrace{\delta_{ji}}^{\mathbf{II.}}}_{\mathbf{III.}} = u_i \underbrace{v_i.}_{\mathbf{III.}} \tag{23}$$

Euclidean norm

$$||\mathbf{u}|| = \sqrt{\mathbf{u} \cdot \mathbf{u}} = \sqrt{u_i^2} = \sqrt{u_1^2 + u_2^2 + u_3^2}. \tag{24}$$

Scalar

$$a = \mathbf{u} \cdot \mathbf{v} = ||\mathbf{u}||||\mathbf{v}|| \cos\theta. \tag{25}$$

1.2.2 Tensor (outer, dyadic) product

$$\mathbf{D} = \mathbf{A} \otimes \mathbf{v} \Longleftrightarrow D_{ijk} = A_{ij} v_k; \quad S_{ij} = u_i v_j \to \begin{bmatrix} S_{11} & S_{12} & S_{13} \\ S_{21} & S_{22} & S_{23} \\ S_{31} & S_{22} & S_{33} \end{bmatrix} = \begin{bmatrix} u_1 v_1 & u_1 v_2 & u_1 v_3 \\ u_2 v_1 & u_2 v_2 & u_2 v_3 \\ u_3 v_1 & u_3 v_2 & u_3 v_3 \end{bmatrix}. \tag{26}$$

Note that

$$(\mathbf{u} \otimes \mathbf{v})\mathbf{c} = \mathbf{u}(\mathbf{v} \cdot \mathbf{c}) = u_i v_j c_j. \tag{27}$$

$$\mathbf{S} = S_{ijkl\ldots}(\mathbf{e}_i \otimes \mathbf{e}_j \otimes \mathbf{e}_k \otimes \mathbf{e}_l \otimes \ldots) \tag{28}$$

1.2.3 Cross product

$$\mathbf{u} \times \mathbf{v} = (u_i \mathbf{e}_i) \times (v_j \mathbf{e}_j) = u_i v_j \underbrace{(\mathbf{e}_i \times \mathbf{e}_j)}_{\mathbf{I}.} = u_i v_j \underbrace{(\epsilon_{ijk} \mathbf{e}_k).}_{\mathbf{I}.} \tag{29}$$

$$\mathbf{w} = \mathbf{u} \times \mathbf{v} = (||\mathbf{u}||||\mathbf{v}|| \sin\theta)\mathbf{n} = A\mathbf{n} \tag{30}$$

where A is area spanned by the two vectors and $\mathbf{n}$ is unit normal to $\mathbf{u}$, $\mathbf{v}$. Volume of three vectors $\mathbf{u}$, $\mathbf{v}$, $\mathbf{w}$

$$\begin{aligned} V = \mathbf{u} \cdot (\mathbf{v} \times \mathbf{w}) &= u_i \mathbf{e}_i \cdot (v_j \mathbf{e}_j \times w_k \mathbf{e}_k) = u_i \underbrace{\mathbf{e}_i \cdot}_{\mathbf{I}.} (v_j w_k \epsilon_{jkm} \underbrace{\mathbf{e}_m)}_{\mathbf{I}.} \\ &= u_i v_j w_k \epsilon_{jkm} \underbrace{\delta_{im}}_{\mathbf{I}.} = \underbrace{u_m v_j w_k \epsilon_{mjk}}_{\text{change } u} = \underbrace{u_i v_j w_k \epsilon_{ijk}}_{\text{reindex } m \to i.}. \end{aligned} \tag{31}$$

Moreover

$$(\mathbf{u} \times \mathbf{v}) \cdot \mathbf{w} = (\mathbf{v} \times \mathbf{w}) \cdot \mathbf{u} = (\mathbf{w} \times \mathbf{u}) \cdot \mathbf{v} = V \tag{32}$$

measures volume of a parallelepiped spanned by three vectors.

1.2.4 Double contraction operation

Decreasing rank by two,

$$\beta = A_{ij} B_{ij} \Longleftrightarrow \beta = \mathbf{A} : \mathbf{B}. \tag{33}$$

$$A_{ij} = E_{ijkl} B_{kl} \Longleftrightarrow \mathbf{A} = \mathbf{E} : \mathbf{B}. \tag{34}$$

1.2.5 Tensor algebra

Tensors $\mathbf{S}$ are required by definition to fulfill

$$\mathbf{S}(\mathbf{u} + \mathbf{v}) = \mathbf{S}\mathbf{u} + \mathbf{S}\mathbf{v}, \quad \mathbf{S}(\alpha \mathbf{u}) = \alpha \mathbf{S}\mathbf{u}. \tag{35}$$

1.2.6 Tensor components

With $\mathbf{e}_i = \{\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3\}$,

$$\mathbf{S}\mathbf{e}_i = S_{ji}\mathbf{e}_j \to \mathbf{S}\mathbf{e}_j = S_{ij}\mathbf{e}_i \to S_{ij} = \mathbf{e}_i \cdot \mathbf{S}\mathbf{e}_j. \tag{36}$$

For instance, noting $\mathbf{u} \cdot \mathbf{v} \Longleftrightarrow \mathbf{u}^T\mathbf{v}$,

$$\begin{Bmatrix} 1 \\ 0 \\ 0 \end{Bmatrix}^T \begin{bmatrix} S_{11} & S_{12} & S_{13} \\ S_{21} & S_{22} & S_{23} \\ S_{31} & S_{22} & S_{33} \end{bmatrix} \begin{Bmatrix} 0 \\ 0 \\ 1 \end{Bmatrix} = \begin{Bmatrix} 1 \\ 0 \\ 0 \end{Bmatrix}^T \begin{Bmatrix} S_{13} \\ S_{23} \\ S_{33} \end{Bmatrix} = S_{13}. \tag{37}$$

1.2.7 Transpose

The definition

$$\mathbf{u} \cdot (\mathbf{T}\mathbf{v}) = \mathbf{v} \cdot (\mathbf{T}^T\mathbf{u}) \tag{38}$$

informs, letting $\mathbf{u} = \mathbf{e}_i,\ \mathbf{v} = \mathbf{e}_j$ and Eq. 36,

$$(\mathbf{T}^T)_{ij} = T_{ji},\ \mathbf{T}^T = T_{ji}(\mathbf{e}_i \otimes \mathbf{e}_j). \tag{39}$$

Transpose rules are

$$(ST)^T = \mathbf{T}^T\mathbf{S}^T,\ (\mathbf{S} + \mathbf{T})^T = \mathbf{S}^T + \mathbf{T}^T,\ (\mathbf{S}^T)^T = \mathbf{S}. \tag{40}$$

Transpose affects multiplication in that, if $\mathbf{C} = \mathbf{A}\mathbf{B} \Longleftrightarrow C_{ij} = A_{ik}B_{kj}$,

$$\mathbf{C} = \mathbf{A}\mathbf{B}^T \Longleftrightarrow C_{ij} = A_{ik}B_{jk} \tag{41}$$

$$\mathbf{C} = \mathbf{A}^T\mathbf{B} \Longleftrightarrow C_{ij} = A_{ki}B_{kj}. \tag{42}$$

$$\mathbf{C} = \mathbf{A}^T\mathbf{B}^T \Longleftrightarrow C_{ij} = A_{ki}B_{jk}. \tag{43}$$

Any tensor $\mathbf{S}$ can be decomposed into symmetric and skew parts

$$\text{sym}\mathbf{S} + \text{skw}\mathbf{S} = \frac{1}{2}(\mathbf{S} + \mathbf{S}^T) + \frac{1}{2}(\mathbf{S} - \mathbf{S}^T) \Longleftrightarrow \frac{1}{2}(S_{ij} + S_{ji}) + \frac{1}{2}(S_{ij} - S_{ji}). \tag{44}$$

1.2.8 Determinant and inverse

The determinant of tensor $\mathbf{S}$

$$\det \mathbf{S} = \frac{\mathbf{S}\mathbf{u} \cdot (\mathbf{S}\mathbf{v} \times \mathbf{S}\mathbf{w})}{\mathbf{u} \cdot (\mathbf{v} \times \mathbf{w})} \Longleftrightarrow \epsilon_{ijk}S_{1i}S_{2j}S_{3k} \tag{45}$$

informs the ratio between the volume of the parallelepiped spanned by $\mathbf{Su}$, $\mathbf{Sv}$, and $\mathbf{Sw}$ with respect to $\mathbf{u}$, $\mathbf{v}$, and $\mathbf{w}$, serving as a sort of norm. Then inverse of $\mathbf{S}$

$$S^{-1}_{ij} = \frac{1}{2}(\det \mathbf{S})^{-1}\epsilon_{ikl}\epsilon_{jmn}S_{mk}S_{nl}. \tag{46}$$

Determinant rules are

$$\det \mathbf{S}^T = \det \mathbf{S}, \quad \det(\mathbf{S}\mathbf{T}) = (\det \mathbf{S})(\det \mathbf{T}), \quad \det \mathbf{S}^{-1} = (\det \mathbf{S})^{-1}. \tag{47}$$

Inverse rules are

$$(\mathbf{S}\mathbf{T})^{-1} = \mathbf{T}^{-1}\mathbf{S}^{-1} \to (\mathbf{S}\mathbf{T})(\mathbf{T}^{-1}\mathbf{S}^{-1}) \Longleftrightarrow (S_{ik}\underbrace{T_{km})(T^{-1}_{mn}}_{\mathbf{I}.}S^{-1}_{nj}) = S_{ik}\underbrace{\delta_{kn}}_{\mathbf{I}.}S^{-1}_{nj}$$

$$= S_{in}S^{-1}_{nj} = \delta_{ij}. \tag{48}$$

1.2.9 Orthogonal tensors

Any orthogonal tensor $\mathbf{Q}$ is defined by

$$\mathbf{Q}\mathbf{u} \cdot \mathbf{Q}\mathbf{v} = (\mathbf{u}^T\mathbf{Q}^T)(\mathbf{Q}\mathbf{v}) = \mathbf{u}^T\mathbf{v} = \mathbf{u} \cdot \mathbf{v} \to \mathbf{Q}^T\mathbf{Q} = \mathbf{I} \Longleftrightarrow (Q_{ki}Q_{kj} = \delta_{ij})$$

$$\to \mathbf{Q}^T = \mathbf{Q}^{-1}, \tag{49}$$

in which it preserves the magnitude and angle between the two vectors $\mathbf{u}$ and $\mathbf{v}$. Accepting Eq. 47,

$$1 = \det \mathbf{I} = \det(\mathbf{Q}\mathbf{Q}^T) = \det \mathbf{Q} \det \mathbf{Q}^T = (\det \mathbf{Q})^2 \to \det \mathbf{Q} = \pm 1. \tag{50}$$

The sign of the determinant of $\mathbf{Q}$ signifies its physical meaning.

$$\det \mathbf{Q} = \begin{cases} +1, \quad \mathbf{Q} \text{ is proper, a rotation, e.g.} \begin{bmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}; \\ \\ -1, \quad \mathbf{Q} \text{ is improper, a rotation and reflection, e.g.} \begin{bmatrix} -1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}. \end{cases} \tag{51}$$

1.2.10 Transformation laws for vectors

$\mathbf{Q}$ transforms coordinate system $\mathbf{e}_i$ into

$$[\mathbf{e}'_i] = [\mathbf{Q}]^T[\mathbf{e}_i] \Longleftrightarrow \begin{Bmatrix} \mathbf{e}'_1 \\ \mathbf{e}'_2 \\ \mathbf{e}'_3 \end{Bmatrix} = \underbrace{\begin{bmatrix} Q_{11} & Q_{21} & Q_{31} \\ Q_{12} & Q_{22} & Q_{32} \\ Q_{13} & Q_{23} & Q_{33} \end{bmatrix}}_{\mathbf{Q}^T} \begin{Bmatrix} \mathbf{e}_1 \\ \mathbf{e}_2 \\ \mathbf{e}_3 \end{Bmatrix} \Longleftrightarrow \mathbf{e}'_i = Q_{mi}\mathbf{e}_m. \tag{52}$$

A counter clockwise rotation about the $\mathbf{e}_3$ axis by θ is given by

$$[\mathbf{Q}] = [\mathbf{R}] = \begin{bmatrix} \cos\theta & -\sin\theta & 0 \\ \sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{bmatrix} \tag{53}$$

because

$$\mathbf{e}_1' = \mathbf{e}_1 \cos\theta + \mathbf{e}_2 \sin\theta, \quad \mathbf{e}_2' = \mathbf{e}_2 \cos\theta - \mathbf{e}_1 \sin\theta, \quad \mathbf{e}_3' = \mathbf{e}_3 \tag{54}$$

and

$$[\mathbf{Q}]^T[\mathbf{e}_i] = \begin{bmatrix} \cos\theta & -\sin\theta & 0 \\ \sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{bmatrix}^T \begin{Bmatrix} \mathbf{e}_1 \\ \mathbf{e}_2 \\ \mathbf{e}_3 \end{Bmatrix} = \begin{bmatrix} \cos\theta & \sin\theta & 0 \\ -\sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{Bmatrix} \mathbf{e}_1 \\ \mathbf{e}_2 \\ \mathbf{e}_3 \end{Bmatrix} = \begin{Bmatrix} \mathbf{e}_1' \\ \mathbf{e}_2' \\ \mathbf{e}_3' \end{Bmatrix} = [\mathbf{e}_i']. \tag{55}$$

1.2.11 Transformation laws for tensors

Any tensor $\mathbf{T}$ has components $T_{ij} = \mathbf{e}_i \cdot \mathbf{T}\mathbf{e}_j$ or $T_{ij}' = \mathbf{e}_i' \cdot \mathbf{T}\mathbf{e}_j'$ depending on reference frame. Then because of Eq. 52 ($\mathbf{e}_i' = Q_{mi}\mathbf{e}_m$),

$$\begin{aligned} T_{ij}' = \mathbf{e}_i' \cdot \mathbf{T}\mathbf{e}_j' &= (Q_{mi}\mathbf{e}_m) \cdot \mathbf{T}(Q_{nj}\mathbf{e}_n) = Q_{mi}Q_{nj}(\mathbf{e}_m \cdot \mathbf{T}\mathbf{e}_n) \\ &= Q_{mi}Q_{nj}(T_{mn}) \iff [\mathbf{T}'] = [\mathbf{Q}]^T[\mathbf{T}][\mathbf{Q}]. \end{aligned} \tag{56}$$

For all tensors we have

$$\begin{aligned} \alpha' = \alpha, \quad a_i' = Q_{mi}a_m, \quad T_{ij}' = Q_{mi}Q_{nj}T_{mn}, \\ S_{ijk}' = Q_{mi}Q_{nj}Q_{rk}S_{mnr}, \quad C_{ijkl}' = Q_{mi}Q_{nj}Q_{rk}Q_{sl}C_{mnrs}. \end{aligned} \tag{57}$$

Now, because

$$T_{ij}' = Q_{mi}Q_{nj}T_{mn} \to T_{ii}' = \underbrace{Q_{mi}Q_{ni}}_{\mathbf{I}.\ (\text{Eq. } 49)} T_{mn} = \underbrace{\delta_{mn}}_{\mathbf{I}.} T_{mn} = \delta_{nm}T_{mn} = T_{mm}, \tag{58}$$

the traces of two tensors which could be orthogonal transformations of one another must be equal.

1.2.12 The eigenproblem

If $\mathbf{T}$ transforms $\mathbf{n}$ into a parallel $\lambda\mathbf{n}$, then $\mathbf{n}$ is an eigenvector and λ is an eigenvalue in

$$\mathbf{T}\mathbf{a} = \lambda\mathbf{n} \to (\mathbf{T} - \lambda\mathbf{I})\mathbf{n} = 0, \ \ \mathbf{n}\cdot\mathbf{n} = 1 \iff (T_{ij} - \lambda\delta_{ij})n_j = 0, \ \ n_jn_j = n_1^2 + n_2^2 + n_3^2 = 1. \tag{59}$$

The roots $\lambda = \lambda_i$ of the characteristic equation

$$\det(\mathbf{T} - \lambda\mathbf{I}) = 0 \tag{60}$$

are the eigenvalues of $\mathbf{T}$. Then to admit nontrivial solutions to $\mathbf{n}$, plug λ_i into

$$\begin{bmatrix} T_{11} - \lambda_i & T_{12} & T_{13} \\ T_{21} & T_{22} - \lambda_i & T_{23} \\ T_{31} & T_{32} & T_{33} - \lambda_i \end{bmatrix} \begin{Bmatrix} n_{1,i} \\ n_{2,i} \\ n_{3,i} \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \\ 0 \end{Bmatrix} \iff \begin{cases} (T_{11} - \lambda_i)n_{1,i} + T_{12}n_{2,i} + T_{13}n_{3,i} = 0 \\ T_{21}n_{1,i} + (T_{22} - \lambda_i)n_{2,i} + T_{23}n_{3,i} = 0 \\ T_{31}n_{1,i} + T_{32}n_{2,i} + (T_{33} - \lambda_i)n_{3,i} = 0 \end{cases} \tag{61}$$

assuming T_{ij} are known.

1.2.13 Principal values and directions

Given symmetric system matrix $\mathbf{T}$, eigenvectors $\mathbf{n}_i$, eigenvalues λ_i,

$$\underbrace{\mathbf{T}\mathbf{n}_1}_{\mathbf{I.}} = \underbrace{\lambda_1 \mathbf{n}_1}_{\mathbf{I.}}, \quad \mathbf{T}\mathbf{n}_2 = \lambda_2 \mathbf{n}_2 \tag{62}$$

implies

$$\mathbf{n}_2 \cdot \underbrace{\mathbf{T}\mathbf{n}_1}_{\mathbf{I.}} = \overbrace{\mathbf{n}_2 \cdot \underbrace{\lambda_1 \mathbf{n}_1}_{\mathbf{I.}}}^{\mathbf{II.}} = \overbrace{\lambda_1(\mathbf{n}_2 \cdot \mathbf{n}_1)}^{\mathbf{II.}}, \quad \mathbf{n}_1 \cdot \mathbf{T}\mathbf{n}_2 = \mathbf{n}_1 \cdot \lambda_2 \mathbf{n}_2 = \lambda_2(\mathbf{n}_1 \cdot \mathbf{n}_2). \tag{63}$$

Because of the definition of transpose that is Eq. 38 ($\mathbf{n}_1 \cdot \mathbf{T}\mathbf{n}_2 = \mathbf{n}_2 \cdot \mathbf{T}^T \mathbf{n}_1$), and given $\mathbf{T} = \mathbf{T}^T$,

$$\underbrace{\mathbf{n}_2 \cdot \mathbf{T}\mathbf{n}_1}_{\mathbf{III.}\ \mathbf{T}=\mathbf{T}^T} - \overbrace{\mathbf{n}_1 \cdot \mathbf{T}\mathbf{n}_2}^{\mathbf{IV}.\ \text{Eq. 38}} = \underbrace{\mathbf{n}_2 \cdot \mathbf{T}^T \mathbf{n}_1}_{\mathbf{III.}\ \mathbf{T}=\mathbf{T}^T} - \overbrace{\mathbf{n}_2 \cdot \mathbf{T}^T \mathbf{n}_1}^{\mathbf{IV}.\ \text{Eq. 38}} = 0 = (\lambda_1 - \lambda_2)(\mathbf{n}_1 \cdot \mathbf{n}_2). \tag{64}$$

For distinct λ, this means $\mathbf{n}_1 \cdot \mathbf{n}_2 = 0$, meaning they are perpendicular and form a basis for a coordinate system. Now, given Eq. 36 ($T_{ij} = \mathbf{n}_i \cdot \mathbf{T}\mathbf{n}_j$), and given the eigenproblem Eq. 62 ($\mathbf{T}\mathbf{n}_j = \lambda_j \mathbf{n}_j$),

$$T_{ij} = \mathbf{n}_i \cdot \lambda_j \mathbf{n}_j \tag{65}$$

which implies

$$\begin{bmatrix} T_{11} & T_{12} & T_{13} \\ T_{21} & T_{22} & T_{23} \\ T_{31} & T_{32} & T_{33} \end{bmatrix} = \begin{bmatrix} \lambda_1(\mathbf{n}_1 \cdot \mathbf{n}_1) & \lambda_2(\mathbf{n}_1 \cdot \mathbf{n}_2) & \lambda_3(\mathbf{n}_1 \cdot \mathbf{n}_3) \\ \lambda_1(\mathbf{n}_2 \cdot \mathbf{n}_1) & \lambda_2(\mathbf{n}_2 \cdot \mathbf{n}_2) & \lambda_3(\mathbf{n}_2 \cdot \mathbf{n}_3) \\ \lambda_1(\mathbf{n}_3 \cdot \mathbf{n}_1) & \lambda_2(\mathbf{n}_3 \cdot \mathbf{n}_2) & \lambda_3(\mathbf{n}_3 \cdot \mathbf{n}_3) \end{bmatrix} = \begin{bmatrix} \lambda_1 & 0 & 0 \\ 0 & \lambda_2 & 0 \\ 0 & 0 & \lambda_3 \end{bmatrix} \tag{66}$$

which implies spectral decomposition of $\mathbf{T}$ with respect to coordinate system $\mathbf{n}_i$

$$[\mathbf{T}]_{\mathbf{n}_i} = \sum_i \lambda_i (\mathbf{n}_i \otimes \mathbf{n}_i). \tag{67}$$

1.2.14 Principal scalar invariants

The characteristic equation of the eigenproblem (from $\det(\mathbf{T} - \lambda \mathbf{I}) = 0$) is

$$\lambda^3 - I_1 \lambda^2 + I_2 \lambda - I_3 = 0 \tag{68}$$

where principal scalar invariants

$$\begin{gathered} I_1 = \mathrm{tr}\mathbf{T} = T_{ii}, \\ I_2 = \det \begin{bmatrix} T_{11} & T_{12} \\ T_{21} & T_{22} \end{bmatrix} + \det \begin{bmatrix} T_{22} & T_{23} \\ T_{32} & T_{33} \end{bmatrix} + \det \begin{bmatrix} T_{11} & T_{13} \\ T_{31} & T_{33} \end{bmatrix}, \\ I_3 = \det \mathbf{T}. \end{gathered} \tag{69}$$

Here is a proof of Cayley Hamilton.

1.3 Tensor fields

1.3.1 Temporal derivative

Some important rules of the temporal derivative of a time dependent scalar function, vector, or tensor are

$$\frac{d}{dt}(\mathbf{TS}) = \frac{d\mathbf{T}}{dt}\mathbf{S} + \mathbf{T}\frac{d\mathbf{S}}{dt}, \quad \frac{d}{dt}(\mathbf{T}) = \frac{d(T_{ij})}{dt}(\mathbf{e}_i \otimes \mathbf{e}_j), \quad \frac{d}{dt}(\mathbf{T}^T) = \left(\frac{d\mathbf{T}}{dt}\right)^T. \tag{70}$$

1.3.2 Differentiation

The notation for differentiation of scalars, vectors, and tensors respectively is

$$\phi_{,i} = \frac{\partial \phi}{\partial x_i}, \quad v_{i,j} = \frac{\partial v_i}{\partial x_j}, \quad T_{ij,k} = \frac{\partial T_{ij}}{\partial x_k}; \quad \phi_{,ij} = \frac{\partial^2 \phi}{\partial x_i \partial x_j}. \tag{71}$$

Some identities are

$$x_{i,j} = \frac{x_i}{x_j} = \delta_{ij}, \tag{72}$$

$$(A_{ij}x_j)_{,i} = A_{ij}\underbrace{x_{j,i}} + \cancelto{0}{A_{ij,i}x_j} = A_{ij}\underbrace{\delta_{ji}} = A_{ii}, \quad \mathbf{A} \neq \mathbf{A}(\mathbf{x}) \tag{73}$$

$$(T_{ij}x_j)_{,i} = T_{ij}x_{j,i} + T_{ij,i}x_j = T_{ij}\delta_{ji} + T_{ij,i}x_j = T_{ii} + T_{ij,i}x_j, \quad \mathbf{T} = \mathbf{T}(\mathbf{x}). \tag{74}$$

1.3.3 Gradient

Differential operator

$$\nabla(*) = \frac{\partial}{\partial x_i}(*)\mathbf{e}_i. \tag{75}$$

Gradient (maximum directional rate of change, perpendicular to surface) increases rank by 1. Gradient of a scalar $\rightarrow$ vector

$$\nabla \phi = \frac{\partial \phi}{\partial x_i}\mathbf{e}_i = \phi_{,i}\mathbf{e}_i. \tag{76}$$

Gradient of a vector $\rightarrow$ second order tensor

$$\nabla \mathbf{v} = \frac{\partial \mathbf{v}}{\partial x_j} \otimes \mathbf{e}_j = \frac{\partial v_i}{\partial x_j}(\mathbf{e}_i \otimes \mathbf{e}_j) = v_{i,j}(\mathbf{e}_i \otimes \mathbf{e}_j). \tag{77}$$

Gradient of a second order tensor $\rightarrow$ third order tensor

$$\nabla \mathbf{T} = \frac{\partial \mathbf{T}}{\partial x_k} \otimes \mathbf{e}_k = T_{ij,k}(\mathbf{e}_i \otimes \mathbf{e}_j \otimes \mathbf{e}_k). \tag{78}$$

1.3.4 Curl

Curl (local rate of rotation) maintains rank. There is no curl of a scalar. Curl of vector → vector

$$\nabla \times \mathbf{v} = -\frac{\partial \mathbf{v}}{\partial x_j} \times \mathbf{e}_j = -\frac{\partial v_i}{\partial x_j}(\mathbf{e}_i \times \mathbf{e}_j) = -v_{i,j}\epsilon_{ijk}\mathbf{e}_k = v_{i,j}\epsilon_{kji}\mathbf{e}_k \tag{79}$$

making components

$$(\nabla \times \mathbf{v})_k = v_{i,j}\epsilon_{kji} \Longleftrightarrow \underbrace{(\nabla \times \mathbf{v})_i = v_{k,j}\epsilon_{ijk}.}_{\text{reindex } k \leftrightarrow i} \tag{80}$$

1.3.5 Divergence

Divergence (outgoingness) decreases rank. There is no divergence of a scalar. Divergence of vector → scalar

$$\nabla \cdot \mathbf{v} = \frac{\partial v_i}{\partial x_i} = v_{i,i}. \tag{81}$$

Divergence of tensor → vector

$$\nabla \cdot \mathbf{T} = \frac{\partial T_{ij}}{\partial x_j}\mathbf{e}_i = T_{ij,j}\mathbf{e}_i = \left(\frac{\partial T_{11}}{\partial x_1} + \frac{\partial T_{12}}{\partial x_2} + \frac{\partial T_{13}}{\partial x_3}\right)\mathbf{e}_1 + \left(\dots\right)\mathbf{e}_2 + \left(\dots\right)\mathbf{e}_3. \tag{82}$$

1.3.6 Laplacian

Laplacian maintains rank. It is the divergence of the gradient. Laplacian of scalar → scalar

$$\nabla \cdot \nabla\phi = \nabla^2\phi = \phi_{,ii} = \frac{\partial^2\phi}{\partial x_1 \partial x_1} + \frac{\partial^2\phi}{\partial x_2 \partial x_2} + \frac{\partial^2\phi}{\partial x_3 \partial x_3}. \tag{83}$$

Laplacian of vector → vector

$$\nabla^2\mathbf{v} = \frac{\partial^2 v_i}{\partial x_j \partial x_j}\mathbf{e}_i = v_{i,jj}\mathbf{e}_i = \left(\frac{\partial^2 v_1}{\partial x_1 \partial x_1} + \frac{\partial^2 v_1}{\partial x_2 \partial x_2} + \frac{\partial^2 v_1}{\partial x_3 \partial x_3}\right)\mathbf{e}_1 + \left(\dots\right)\mathbf{e}_2 + \left(\dots\right)\mathbf{e}_3. \tag{84}$$

1.3.7 Divergence theorem

For a vector,

$$\underbrace{\int_{\partial\Omega} w_i n_i dA}_{\textbf{I}.} = \underbrace{\int_{\Omega} w_{i,i} dV}_{\textbf{II}.} \Longleftrightarrow \underbrace{\int_{\partial\Omega} \mathbf{w}\cdot\mathbf{n} dA}_{\textbf{I}.} = \underbrace{\int_{\Omega} (\nabla\cdot\mathbf{w}) dV.}_{\textbf{II}.} \tag{85}$$

An interpretation of this statement is:

- (**I**) flux of **w** out of the surface = (**II**) sinks and sources of **w** inside body.

2 Ch2

2.1 Body motion

Continuum mechanics studies deformation subject to an external load. Kinematics characterize the set of possible deformations.
A body occupies a region of Euclidean point space ϵ. A reference configuration B_0 can deform in to a subsequent configuration B. A material point $\mathbf{X} \in B_0$; a motion of B_0 is a smooth function $\mathbf{\Phi}$ that maps each $\mathbf{X} \in B_0$ to an $\mathbf{x} \in B$, so that

$$\mathbf{x} = \mathbf{\Phi}(\mathbf{X}, t) \Longleftrightarrow x_i = \Phi_i(X_i, t). \tag{86}$$

For fixed t, $\mathbf{\Phi}(\mathbf{X})$ is called the deformation map. $\mathbf{\Phi}(\mathbf{X})$ is one to one in $\mathbf{X}$. That is, no two material points occupy the same spatial point. Displacement

$$\mathbf{u}(\mathbf{X}) = \mathbf{x} - \mathbf{X} = \mathbf{\Phi}(\mathbf{X}) - \mathbf{X}. \tag{87}$$

The first example is uniform stretch. This is

$$x_1 = \alpha_1 X_1,\ x_2 = \alpha_2 X_2,\ x_3 = \alpha_3 X_3. \tag{88}$$

Another is simple shear, or

$$x_1 = X_1 + \gamma X_2,\ x_2 = X_2,\ x_3 = X_3. \tag{89}$$

In general for simple shear in direction $\mathbf{S}$ with normal $\mathbf{n}$,

$$\mathbf{x} = (\mathbf{I} + \gamma \mathbf{S} \otimes \mathbf{n})\mathbf{X}. \tag{90}$$

Figure 1: Simple shear and uniform stretch.

In this case the shearing direction is $\{1\ 0\ 0\}^{\mathrm{T}}$ with normal $\{0\ 1\ 0\}^{\mathrm{T}}$ giving

$$\begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} = S_i n_j. \tag{91}$$

If

$$\mathbf{x} = \mathbf{\Phi}(\mathbf{X}) \tag{92}$$

then easily

$$\mathbf{\Phi}^{-1}(\mathbf{x}) = \mathbf{\Phi}^{-1}(\mathbf{\Phi}(\mathbf{X})) = \mathbf{X}. \tag{93}$$

The reference configuration/material description

$$X_I;\ A_{IJ};\ \text{Grad}(*),\ \nabla_0(*);\ \text{Curl}(*),\ \nabla_0 \times (x);\ \text{Div}(*),\ \nabla_0(*) \tag{94}$$

is called Lagrangian and the deformed configuration/spatial description

$$x_i;\ a_{ij};\ \text{grad}(*),\ \nabla(*);\ \text{curl}(*),\ \nabla \times (x);\ \text{div}(*),\ \nabla(*) \tag{95}$$

is called Eulerian. For example,

$$\nabla_0 \Phi = \frac{\partial \Phi}{\partial X_I}\mathbf{e}_I = \text{Grad}(\Phi) = \Phi_{,I}\mathbf{e}_I, \tag{96}$$

$$\nabla_0 \mathbf{v} = \nabla_0 v_I \mathbf{e}_I = \frac{\partial v_I}{\partial X_J}(\mathbf{e}_I \otimes \mathbf{e}_J), \tag{97}$$

$$(\nabla_0 \mathbf{v})_{IJ} = \frac{\partial v_I}{X_J}, \tag{98}$$

$$\nabla_0 \times \mathbf{v} = \nabla_0 \times v_I \mathbf{e}_I := -\frac{\partial v_I \mathbf{e}_I}{\partial X_J} \times \mathbf{e}_J$$

$$= -\frac{\partial v_I}{\partial X_J}(\mathbf{e}_I \times \mathbf{e}_J) = -\frac{\partial v_I}{\partial X_J}\epsilon_{IJK}\mathbf{e}_K = \frac{\partial v_I}{\partial X_J}\epsilon_{KJI}\mathbf{e}_K = v_{I,J}\epsilon_{KJI}\mathbf{e}_K. \tag{99}$$

2.2 Description of local deformation

$\mathbf{\Phi} = \mathbf{\Phi}(\mathbf{X})$ informs movement of a point in a body. Let dV_0 be a sphere or local neighborhood around $\mathbf{X}$. Let d$\mathbf{X}$, an infinitesimal material vector, be the radius of this sphere.

2.2.1 Deformation gradient

The relationship

$$dx_i = F_{iJ} dX_J \Longrightarrow F_{iJ} = \frac{\partial x_i}{\partial X_J} \Longleftrightarrow \nabla_0(\mathbf{x}) \tag{100}$$

defines the deformation gradient $\mathbf{F}$. It maps material vectors to spatial vectors. For uniform stretch,

$$x_1 = \alpha_1 X_1,\ x_2 = \alpha_2 X_2,\ x_3 = \alpha_3 X_3 \Longrightarrow [F_{ij}] = \begin{bmatrix} \alpha_1 & 0 & 0 \\ 0 & \alpha_2 & 0 \\ 0 & 0 & \alpha_3 \end{bmatrix}. \tag{101}$$

For simple shear,

$$x_1 = X_1 + \gamma X_2,\ x_2 = X_2,\ x_3 = X_3 \Longrightarrow [F_{ij}] = \begin{bmatrix} 1 & \gamma & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}. \tag{102}$$

Recall $\mathbf{u} = \mathbf{x} - \mathbf{X}$. Then

$$\nabla_0(\mathbf{u}) = \nabla_0(\mathbf{x} - \mathbf{X}) = \frac{\partial x_i}{\partial X_J} - \frac{\partial X_I}{\partial X_J} = F_{iJ} - \delta_{IJ} \Longrightarrow \mathbf{F} = \nabla_0 \mathbf{u} + \mathbf{I}. \tag{103}$$

2.2.2 Deformation of volume

Suppose there is a cube with dimensions $l\mathbf{e}_1, l\mathbf{e}_2, l\mathbf{e}_3$. Suppose it undergoes deformation such

Figure 2: Some deformation.

that

$$l\mathbf{e}_1 \to \delta_1, \quad l\mathbf{e}_2 \to \delta_2, \quad l\mathbf{e}_3 \to \delta_3, \tag{104}$$

making

$$d\mathbf{X} = \begin{Bmatrix} l\mathbf{e}_1 \\ l\mathbf{e}_2 \\ l\mathbf{e}_3 \end{Bmatrix} \Longleftrightarrow l\mathbf{e}_i, \quad d\mathbf{x} = \begin{Bmatrix} \delta_1 \\ \delta_2 \\ \delta_3 \end{Bmatrix} \Longleftrightarrow \delta_i. \tag{105}$$

We know Eq. 100 ($d\mathbf{x} = \mathbf{F}d\mathbf{X}$), so

$$\delta_i = l\mathbf{F}\mathbf{e}_i \tag{106}$$

Recall Eq. 32 ($V = \mathbf{r}_1 \cdot (\mathbf{r}_2 \times \mathbf{r}_3)$), the volume of a parallelepiped spanned by three vectors. This means

$$dV_0 = l^3 \mathbf{e}_1 \cdot (\mathbf{e}_2 \times \mathbf{e}_3) = l^3, \quad dV = \delta_1 \cdot (\delta_2 \times \delta_3). \tag{107}$$

Substituting Eq. 106 into 107 we find

$$dV = l^3 \mathbf{F}\mathbf{e}_1 \cdot (\mathbf{F}\mathbf{e}_2 \times \mathbf{F}\mathbf{e}_3) = dV_0 \frac{\mathbf{F}\mathbf{e}_1 \cdot (\mathbf{F}\mathbf{e}_2 \times \mathbf{F}\mathbf{e}_3)}{\cancelto{1}{\mathbf{e}_1 \cdot (\mathbf{e}_2 \times \mathbf{e}_3)}} = dV_0 \det \mathbf{F}. \tag{108}$$

Therefore we introduce J such that

$$\frac{dV}{dV_0} = \det \mathbf{F} = J = \begin{cases} 1, & \text{volume preserving deformation,} \\ \neq 1, & \text{volume change.} \end{cases} \tag{109}$$

For uniform expansion Eq. 101, $J = \alpha_1 \alpha_2 \alpha_3 \neq 1$. For simple shear Eq. 102, $J = 1$.

2.2.3 Deformation of area

Suppose area element in the reference configuration

$$d\mathbf{A}_0 = \mathbf{N} dA_0 = \mathbf{N}\Bigg(||d\mathbf{X}||\ ||d\mathbf{Y}|| \sin\theta_{\mathbf{XY}} \Bigg) = d\mathbf{X} \times d\mathbf{Y} = dX_I dY_J \epsilon_{IJK} \mathbf{e}_K. \tag{110}$$

where $\mathbf{N}$ is normal to surface. Then area element in the spatial configration

$$\begin{aligned} d\mathbf{a} = d\mathbf{x} \times d\mathbf{y} = dx_i dy_j \epsilon_{ijk} \mathbf{e}_k \Longrightarrow \underbrace{dx_j dy_k \overbrace{\epsilon_{jki}}^{\mathbf{I.}} \mathbf{e}_i}_{\text{reindex } i\to j,\ j\to k,\ k\to i} &= \underbrace{dx_j dy_k}_{\mathbf{II.}} \overbrace{\epsilon_{ijk}}^{\mathbf{I.}} \mathbf{e}_i \\ &= \underbrace{(F_{jJ} dX_J)(F_{kK} dY_K)}_{\mathbf{II.}} \epsilon_{ijk} \mathbf{e}_i = d\mathbf{a}. \end{aligned} \tag{111}$$

Elementwise

$$(F_{jJ} dX_J)(F_{kK} dY_K) \epsilon_{ijk} = (d\mathbf{a})_i = n_i da. \tag{112}$$

Then

$$(F_{jJ} dX_J)(F_{kK} dY_K) \epsilon_{ijk} (F_{iI}) = n_i da (F_{iI}) \tag{113}$$

implies

$$\begin{aligned} n_i da F_{iI} = \underbrace{(F_{iI} F_{jJ} F_{kK} \epsilon_{ijk})}_{\mathbf{III.}} dX_J dY_K = \underbrace{(\det F \epsilon_{IJK})}_{\mathbf{III.}} dX_J dY_K = (J \underbrace{\epsilon_{IJK}) dX_J dY_K}_{\mathbf{IV.}} \\ = J \underbrace{(d\mathbf{X} \times d\mathbf{Y})}_{\mathbf{IV.}} = J N_I dA_0 \end{aligned} \tag{114}$$

which implies Nanson's formula

$$n_i da = J F^{-1}_{Ii} N_I dA_0 \Longrightarrow da = J F^{-1}_{Ii} dA_0 \Longleftrightarrow d\mathbf{a} = J \mathbf{F}^{-T} d\mathbf{A}_0. \tag{115}$$

2.2.4 Symmetric positive definite tensors

Positive definiteness of $\mathbf{C}$ requires for all $\mathbf{x}$

$$\mathbf{x} \cdot \mathbf{C}\mathbf{x} > 0 \tag{116}$$

and implies that

$$\det \mathbf{C} > 0; \tag{117}$$

that $\mathbf{RCR}^T$ is symmetric; that there exists $\mathbf{U}$ such that

$$\mathbf{U}^2 = \mathbf{C} \to \mathbf{U} = \sqrt{\mathbf{C}}; \tag{118}$$

and that $\mathbf{C}$ admits the spectral decomposition

$$\mathbf{C} = \sum_i \lambda_i \overbrace{\mathbf{r}_i \otimes \mathbf{r}_i}^{\mathbf{I.}} = \sum_i \lambda_i \overbrace{\mathbf{r}_i \mathbf{r}_i^T}^{\mathbf{I.}} \tag{119}$$

as in Eq. 67. Solving for eigenvectors λ_i,

$$\sum_i \mathbf{r}_i^T \mathbf{C} \mathbf{r}_i = \sum_i \mathbf{r}_i^T (\lambda_i \mathbf{r}_i \overbrace{\mathbf{r}_i^T)}^{\mathbf{II.}} \mathbf{r}_i = \sum_i \lambda_i \overbrace{(\mathbf{r}_i \cdot \mathbf{r}_i)}^{\mathbf{II.}} (\mathbf{r}_i \cdot \mathbf{r}_i) = \sum_i \lambda_i. \tag{120}$$

2.2.5 Polar decomposition theorem

Any $\mathbf{F}$ admits the decompositions

$$\mathbf{F} = \mathbf{RU} = \mathbf{VR} \Longleftrightarrow F_{iJ} = R_{iI} U_{IJ} = V_{ij} R_{jJ}. \tag{121}$$

where right- and left-stretch tensors $\mathbf{U} \Longleftrightarrow U_{IJ}$ and $\mathbf{V} \Longleftrightarrow V_{ij}$ are symmetric and positive definite, guaranteeing of them positive eigenvalues; and $\mathbf{R} \Longleftrightarrow R_{iJ}$ is an orthogonal ($\mathbf{Q}^T = \mathbf{Q}^{-1}$) rotation. Then

$$\mathbf{F}^T \mathbf{F} = (\mathbf{U}^T \mathbf{R}^T)(\mathbf{RU}) = \mathbf{U}^T \mathbf{U} = \mathbf{U}^2 \Longleftrightarrow F_{iI} F_{iJ} = U_{IJ} U_{IJ}, \tag{122}$$

$$\mathbf{F}\mathbf{F}^T = (\mathbf{VR})(\mathbf{R}^T \mathbf{V}^T) = \mathbf{V}\mathbf{V}^T = \mathbf{V}^2 \Longleftrightarrow F_{iJ} F_{Jj} = V_{ij} V_{ij}, \tag{123}$$

$$\mathbf{V} = \mathbf{RUR}^T \Longleftrightarrow V_{ij} = R_{iI} U_{IJ} R_{Jj}, \tag{124}$$

$$\mathbf{U} = \mathbf{R}^T \mathbf{VR} \Longleftrightarrow U_{IJ} = R_{Ii} V_{ij} R_{jJ}. \tag{125}$$

The above components permit a transformation from dX_I to dx_i by

- stretch by U_{IJ}, rotation by $R_{iI} \to F_{iJ} = R_{iI} U_{IJ}$, or
- rotation by R_{jJ}, stretch by $V_{ij} \to F_{iJ} = V_{ij} R_{jJ}$.

2.2.6 Right Cauchy Green

Let infinitesimal vector magnitudes

$$dS_0 = ||d\mathbf{X}|| = \sqrt{X_1^2 + X_2^2 + X_3^2} = \sqrt{d\mathbf{X} \cdot d\mathbf{X}} = \sqrt{d\mathbf{X}^T d\mathbf{X}} = \sqrt{dX_I dX_I}, \tag{126}$$

$$dS = ||d\mathbf{x}|| = \sqrt{x_1^2 + x_2^2 + x_3^2} = \sqrt{d\mathbf{x} \cdot d\mathbf{x}} = \sqrt{d\mathbf{x}^T d\mathbf{x}} = \sqrt{dx_i dx_i}. \tag{127}$$

Notice d$\mathbf{x} = \mathbf{F}$d$\mathbf{X}$, so we can define some $\mathbf{C}$ as

$$dS^2 = d\mathbf{x}^T d\mathbf{x} = (d\mathbf{X}^T \underbrace{\mathbf{F}^T)(\mathbf{F}}_{\mathbf{I}.} d\mathbf{X}) = d\mathbf{X}^T \underbrace{\mathbf{C}}_{\mathbf{I}.} d\mathbf{X} \tag{128}$$

which we call right Cauchy Green deformation tensor

$$\mathbf{C} = \mathbf{F}^T\mathbf{F} \Longleftrightarrow C_{IJ} = F_{iI}F_{iJ} = \frac{\partial x_i}{\partial X_I}\frac{\partial x_i}{\partial X_J}, \quad dS^2 = dX_I C_{IJ} dX_J. \tag{129}$$

$\mathbf{C}$ is symmetric. Notice also

$$J = \det \mathbf{F} = \sqrt{\det \mathbf{F}^T \det \mathbf{F}} = \sqrt{\det \mathbf{C}} \tag{130}$$

and, as in Eq. 122,

$$\mathbf{C} = \mathbf{F}^T\mathbf{F} = \mathbf{U}^T\mathbf{R}^T\mathbf{R}\mathbf{U} = \mathbf{U}^T\mathbf{U} = \mathbf{U}^2 \rightarrow \mathbf{U} = \sqrt{\mathbf{C}}. \tag{131}$$

If we are given a stretch ratio

$$\alpha = \frac{dS}{dS_0} = \sqrt{\frac{dX_I C_{IJ} dX_J}{dX_I dX_I}} \tag{132}$$

we deduce that on the diagonals ($J = I \leftrightarrow$ stretch in the $\mathbf{e}_I$ direction),

$$\alpha = \sqrt{C_{IJ}} = \sqrt{C_{II}} \rightarrow C_{II} = \alpha^2. \tag{133}$$

Otherwise ($J \neq I$, stretch in a particular direction), if we are given stretch ratio $\alpha(\leftrightarrow \mathbf{U}) = \lambda^2(\leftrightarrow \mathbf{C})$ and that direction $\mathbf{r}$, we can use Eq. 120 ($\mathbf{r}_i^T\mathbf{C}\mathbf{r}_i = \lambda_i^2$) to solve for that $C_{IJ} = C_{JI}$, recalling symmetry of $\mathbf{C}$. Another relation is

$$C_{IJ} = \cos\theta \sqrt{C_{II}}\sqrt{C_{JJ}}, \quad I \neq J, \tag{134}$$

where θ is the angle between the deformed $d\mathbf{x}$ and $d\mathbf{y}$. θ can be found in

$$d\mathbf{x} \cdot d\mathbf{y} = ||d\mathbf{x}||\ ||d\mathbf{y}|| \cos\theta \rightarrow \cos\theta = \frac{dx_1dy_1 + dx_2dy_2 + dx_3dy_3}{\sqrt{dx_1^2 + dx_2^2 + dx_3^2}\sqrt{dy_1^2 + dy_2^2 + dy_3^2}} \tag{135}$$

if given d$\mathbf{x}$ and d$\mathbf{y}$.

2.2.7 Left Cauchy Green

There is a right Cauchy Green deformation tensor $\mathbf{C} = \mathbf{F}^T\mathbf{F} \Longleftrightarrow C_{IJ} = F_{iI}F_{jJ}$, and there is also a left Cauchy Green deformation tensor

$$\mathbf{B} = \mathbf{F}\mathbf{F}^T \Longleftrightarrow B_{ij} = F_{iJ}F_{jJ} \tag{136}$$

which also must be symmetric positive definite. Notice that

$$\det \mathbf{B} = \det \mathbf{F}\mathbf{F}^T = \det \mathbf{F}^2 \rightarrow J = \det \mathbf{F} = \sqrt{\det \mathbf{B}} \tag{137}$$

and, because of Eq. 123,

$$\mathbf{B} = \mathbf{V}^2 \rightarrow \mathbf{V} = \sqrt{\mathbf{B}}. \tag{138}$$

2.2.8 Lagrangian strain tensor

Strain measures how different dS is from dS_0. A good deformation measure is, starting with Eq. 129,

$$dS^2 - dS_0^2 = dX_I C_{IJ} dX_J - dX_I \underbrace{(dX_I)}_{\textbf{I.}} = dX_I C_{IJ} dX_J - dX_I \underbrace{(\delta_{IJ} dX_J)}_{\textbf{I.}}$$
$$= dX_I dX_J (C_{IJ} - \delta_{IJ}) = dX_I dX_J (2E_{IJ}). \tag{139}$$

This representation defines the material Lagrangian strain tensor $\mathbf{E}$ as

$$E_{IJ} = \frac{1}{2}(\underbrace{C_{IJ}}_{\textbf{II.}} - \delta_{IJ}) = \frac{1}{2}(\underbrace{F_{iI}F_{iJ}}_{\textbf{II.}\ \text{Eq. 129.}} - \delta_{IJ}) \Longleftrightarrow \mathbf{E} = \frac{1}{2}(\mathbf{C} - \mathbf{I})$$
$$= \frac{1}{2}(\mathbf{F}^T\mathbf{F} - \mathbf{I}) = \frac{1}{2}((\nabla_0 \mathbf{u}^T + \mathbf{I}^T)(\nabla_0 \mathbf{u} + \mathbf{I}) - \mathbf{I})$$
$$= \frac{1}{2}(\nabla_0 \mathbf{u}^T \nabla_0 \mathbf{u} + \nabla_0 \mathbf{u}^T + \nabla_0 \mathbf{u} \cancel{+ \mathbf{I} - \mathbf{I}}) = \frac{1}{2}(\nabla_0 \mathbf{u}^T \nabla_0 \mathbf{u} + \nabla_0 \mathbf{u}^T + \nabla_0 \mathbf{u}) = \mathbf{E} \tag{140}$$

provided Eq. 103 ($\mathbf{F} = \nabla_0 \mathbf{u} + \mathbf{I}$).

2.2.9 Euler-Almansi strain tensor

Now, suppose we look at how different dS_0 is from dS, which is the reverse way. Since $dx_i = F_{iJ} dX_J$, then

$$dX_J = F_{iJ}^{-1} dx_i \tag{141}$$

and, in acknowledging the multiplication rule that is Eq. 16 and the definition of $\mathbf{B}$ in Eq. 136, such that

$$dS^2 - dS_0^2 = dx_i dx_i - dX_I dX_I = dx_i dx_i - (F_{iJ}^{-1} dx_i)(F_{jJ}^{-1} dx_j)$$
$$= dx_i dx_i - (F_{iJ}F_{jJ})^{-1} dx_i dx_j = dx_i \delta_{ij} dx_j - (F_{iJ}F_{jJ})^{-1} dx_i dx_j$$
$$= dx_i dx_j (\delta_{ij} - B_{ij}^{-1}) = dx_i dx_j (2e_{ij}), \tag{142}$$

we define Euler-Almansi strain tensor $\mathbf{e}$ as

$$e_{ij} = \frac{1}{2}(\delta_{ij} - B_{ij}^{-1}) \Longleftrightarrow \mathbf{e} = \frac{1}{2}(\mathbf{I} - \mathbf{F}^{-T}\mathbf{F}^{-1}). \tag{143}$$

Provided

$$\mathbf{u} = \mathbf{x} - \mathbf{X} \rightarrow \nabla \mathbf{u} = \frac{\partial x_i}{\partial x_j} - \frac{\partial X_I}{\partial x_j} = \delta_{ij} - F_{jI}^{-1} \Longleftrightarrow \mathbf{F}^{-1} = \mathbf{I} - \nabla \mathbf{u}, \quad \mathbf{F}^{-T} = \mathbf{I} - \nabla \mathbf{u}^T, \tag{144}$$

we reduce Eq. 143 to

$$\mathbf{e} = \frac{1}{2}(\mathbf{I} - (\mathbf{I} - \nabla \mathbf{u}^T)(\mathbf{I} - \nabla \mathbf{u}))$$
$$= \frac{1}{2}(\mathbf{I} - \mathbf{I} + \nabla \mathbf{u} + \nabla \mathbf{u}^T - \nabla \mathbf{u}^T \nabla \mathbf{u})$$
$$= \frac{1}{2}(\nabla \mathbf{u} + \nabla \mathbf{u}^T - \nabla \mathbf{u}^T \nabla \mathbf{u}) = \mathbf{e}. \tag{145}$$

2.2.10 Principal directions and invariants of deformations and strains

Recall the spectral decomposition of symmetric, positive definite left Cauchy green deformation tensor that is

$$\mathbf{C} = \sum_i \lambda_i^C \mathbf{r}_i \otimes \mathbf{r}_i = \sum_i \lambda_i^C \mathbf{r}_i \mathbf{r}_i^T \tag{146}$$

which is also Eq. 119. Then

$$\mathbf{U} = \sqrt{\mathbf{C}} = \sum_i \sqrt{\lambda_i^C} \mathbf{r}_i \otimes \mathbf{r}_i = \sum_i \sqrt{\lambda_i^C} \mathbf{r}_i \mathbf{r}_i^T. \tag{147}$$

Similarly for $\mathbf{B}$ and $\mathbf{V}$,

$$\mathbf{B} = \sum_i \lambda_i^B \mathbf{l}_i \otimes \mathbf{l}_i = \sum_i \lambda_i^B \mathbf{l}_i \mathbf{l}_i^T \tag{148}$$

$$\mathbf{V} = \sum_i \sqrt{\lambda_i^B} \mathbf{l}_i \otimes \mathbf{l}_i = \sum_i \sqrt{\lambda_i^B} \mathbf{l}_i \mathbf{l}_i^T = \sqrt{\mathbf{B}}. \tag{149}$$

Because according to Eq. 124 $\mathbf{V}= \mathbf{R}\mathbf{U}\mathbf{R}^T$,

$$\left(\sum_i \sqrt{\lambda_i^B} \mathbf{l}_i \mathbf{l}_i^T\right) = \mathbf{R}^T \left(\sum_i \sqrt{\lambda_i^C} \mathbf{r}_i \mathbf{r}_i^T\right) \mathbf{R} \tag{150}$$

implies

$$\lambda_i^B = \lambda_i^C, \quad \mathbf{l}_i = \mathbf{R}\mathbf{r}_i. \tag{151}$$

That is, the eigenvectors of $\mathbf{C}$ and $\mathbf{B}$ ($\mathbf{U}$ and $\mathbf{V}$) are the same, and the eigenvectors are related through the rotation matrix $\mathbf{R}$ in $\mathbf{RU} = \mathbf{VR} = \mathbf{F}$. Now,

$$\mathbf{F} = \mathbf{RU} = \sum_i \sqrt{\lambda_i^C} \mathbf{R}\mathbf{r}_i \otimes \mathbf{r}_i = \sum_i \sqrt{\lambda_i^C} \mathbf{l}_i \otimes \mathbf{r}_i, \tag{152}$$

and

$$\mathbf{E} = \frac{1}{2}(\mathbf{C} - \mathbf{I}) = \sum_i (\lambda_i^C - 1)(\mathbf{r}_i \otimes \mathbf{r}_i). \tag{153}$$

Recalling Eq. 69, principal scalar invariants

$$I_1 = \mathrm{tr}\mathbf{C} = C_{ii}, \tag{154}$$

$$I_2 = \det \begin{bmatrix} C_{11} & C_{12} \\ C_{21} & C_{22} \end{bmatrix} + \det \begin{bmatrix} C_{22} & C_{23} \\ C_{32} & C_{33} \end{bmatrix} + \det \begin{bmatrix} C_{11} & C_{13} \\ C_{31} & C_{33} \end{bmatrix}, \tag{155}$$

$$I_3 = \det \mathbf{C} = \det(\mathbf{F}^T \mathbf{F}) = J^2. \tag{156}$$

For

$$[\mathbf{C}]_{\mathbf{r}_i} = \begin{bmatrix} \lambda_1^C & 0 & 0 \\ 0 & \lambda_2^C & 0 \\ 0 & 0 & \lambda_3^C \end{bmatrix}, \tag{157}$$

we get

$$I_1(\mathbf{C}) = \lambda_1^C + \lambda_2^C + \lambda_3^C, \tag{158}$$

$$I_2(\mathbf{C}) = \lambda_1^C\lambda_2^C + \lambda_2^C\lambda_3^C + \lambda_1^C\lambda_3^C, \tag{159}$$

$$I_3(\mathbf{C}) = \lambda_1^C\lambda_2^C\lambda_3^C. \tag{160}$$

2.2.11 Small strain theory

In small strain theory,

$$\mathbf{F} = \mathbf{I} \Longleftrightarrow F_{iJ} = \delta_{iJ}. \tag{161}$$

Also, the distinction between $\mathbf{E}$ and $\mathbf{e}$, formerly

$$\mathbf{E} = \frac{1}{2}(\nabla_0\mathbf{u}^T\nabla_0\mathbf{u} + \nabla_0\mathbf{u}^T + \nabla_0\mathbf{u})$$

and

$$\mathbf{e} = \frac{1}{2}(\nabla\mathbf{u} + \nabla\mathbf{u}^T - \nabla\mathbf{u}^T\nabla\mathbf{u}),$$

diminishes as $\nabla = \nabla_0 = \text{Grad} = \text{grad}$. Then strain simply reduces to small engineering strain tensor

$$\boldsymbol{\epsilon} = \frac{1}{2}(\nabla\mathbf{u} + \nabla\mathbf{u}^T) \Longleftrightarrow \epsilon_{ij} = \frac{1}{2}(u_{i,j} + u_{j,i}). \tag{162}$$

Substituting Eq. 162 into $\mathbf{E}$ (Eq. 140) and then into $\mathbf{e}$ (Eq. 145),

$$\mathbf{E} = \boldsymbol{\epsilon} + \frac{1}{2}(\nabla_0\mathbf{u}^T\nabla_0\mathbf{u}), \tag{163}$$

$$\mathbf{e} = \boldsymbol{\epsilon} - \frac{1}{2}(\nabla\mathbf{u}^T\nabla\mathbf{u}). \tag{164}$$

It is also clear that, because of Eq. 103 ($\mathbf{F} = \nabla_0\mathbf{u} + \mathbf{I}$),

$$\boldsymbol{\epsilon} = \frac{1}{2}(\mathbf{F}^T + \mathbf{F}) - \mathbf{I} = \frac{1}{2}(\nabla_0\mathbf{u}^T + \mathbf{I}^T + \nabla_0\mathbf{u} + \mathbf{I}) - \mathbf{I} = \frac{1}{2}(\nabla\mathbf{u}^T + \nabla\mathbf{u}). \tag{165}$$

In matrix form

$$[\epsilon_{ij}] = [\frac{1}{2}(u_{i,j} + u_{j,i})] = \begin{bmatrix} \frac{1}{2}\left(\frac{\partial u_1}{\partial u_1} + \frac{\partial u_1}{\partial u_1}\right) & \frac{1}{2}\left(\frac{\partial u_1}{\partial u_2} + \frac{\partial u_2}{\partial u_1}\right) & \frac{1}{2}\left(\frac{\partial u_1}{\partial u_3} + \frac{\partial u_3}{\partial u_1}\right) \\ \frac{1}{2}\left(\frac{\partial u_2}{\partial u_1} + \frac{\partial u_2}{\partial u_1}\right) & \frac{1}{2}\left(\frac{\partial u_2}{\partial u_2} + \frac{\partial u_2}{\partial u_2}\right) & \frac{1}{2}\left(\frac{\partial u_2}{\partial u_3} + \frac{\partial u_3}{\partial u_2}\right) \\ \frac{1}{2}\left(\frac{\partial u_3}{\partial u_1} + \frac{\partial u_1}{\partial u_3}\right) & \frac{1}{2}\left(\frac{\partial u_3}{\partial u_2} + \frac{\partial u_2}{\partial u_3}\right) & \frac{1}{2}\left(\frac{\partial u_3}{\partial u_3} + \frac{\partial u_3}{\partial u_3}\right) \end{bmatrix}, \tag{166}$$

where clearly

$$\epsilon_{ii} = u_{i,i}, \quad \epsilon_{ij} = \epsilon_{ji}. \tag{167}$$

Physically, elements of ϵ can be thought of as

$$\begin{cases} \epsilon_{ii} & \text{change in length per unit length} \\ \epsilon_{ij} & \text{change in angle between material lines in } \mathbf{e}_1 \text{ and } \mathbf{e}_2 \text{ directions.} \end{cases} \tag{168}$$

2.3 Kinematics rates

2.3.1 Material and spatial time derivatives

Recall that $\mathbf{\Phi}$ is a smooth function that takes $\mathbf{X}$ to $\mathbf{x}$, so that $\mathbf{x} = \mathbf{\Phi}(\mathbf{X}, t)$. Then, although seemingly backwards at first, material time derivative

$$\underbrace{\dot{\mathbf{x}} = \dot{\mathbf{\Phi}}(\mathbf{X}, t) = \frac{d\mathbf{\Phi}}{dt} = \begin{cases} \frac{\partial \mathbf{\Phi}}{\partial t}, & \mathbf{X} \text{ is fixed,} \\ \frac{\partial \mathbf{\Phi}}{\partial t} + \frac{\partial \mathbf{\Phi}}{\partial X_I}\frac{\partial X_I}{\partial t}, & \end{cases}}_{\mathbf{I.}} \tag{169}$$

and spatial time derivative

$$\underbrace{\dot{\mathbf{X}} = \dot{\mathbf{\Phi}}^{-1}(\mathbf{x}, t) = \frac{d\mathbf{\Phi}^{-1}}{dt} = \begin{cases} \frac{\partial \mathbf{\Phi}^{-1}}{\partial t}, & \mathbf{x} \text{ is fixed,} \\ \frac{\partial \mathbf{\Phi}^{-1}}{\partial t} + \frac{\partial \mathbf{\Phi}^{-1}}{\partial x_i}\frac{\partial x_i}{\partial t}, & \end{cases}}_{\mathbf{II.}} \tag{170}$$

2.3.2 Velocity and acceleration fields

Material velocity

$$\underbrace{\mathbf{V}(\mathbf{X}, t) = \dot{\mathbf{x}} = \dot{\mathbf{\Phi}}(\mathbf{X}, t) = \frac{\partial \mathbf{\Phi}}{\partial t},\ \mathbf{X} \text{ fixed}}_{\mathbf{I.}} \tag{171}$$

and spatial description of material velocity

$$\underbrace{\mathbf{v}(\mathbf{x}, t) = \dot{\mathbf{X}} = \dot{\mathbf{\Phi}}^{-1}(\mathbf{x}, t) = \frac{\partial \mathbf{\Phi}^{-1}}{\partial t},\ \mathbf{x} \text{ fixed.}}_{\mathbf{II.}} \tag{172}$$

Material acceleration

$$\underbrace{\mathbf{A}(\mathbf{X}, t) = \ddot{\mathbf{x}} = \ddot{\mathbf{\Phi}}(\mathbf{X}, t) = \frac{\partial^2 \mathbf{\Phi}}{\partial^2 t},\ \mathbf{X} \text{ fixed}}_{\mathbf{I.}} \tag{173}$$

and spatial description of material acceleration

$$\underbrace{\mathbf{a}(\mathbf{x}, t) = \ddot{\mathbf{X}} = \ddot{\mathbf{\Phi}}^{-1}(\mathbf{x}, t) = \frac{\partial^2 \mathbf{\Phi}^{-1}}{\partial^2 t},\ \mathbf{x} \text{ fixed.}}_{\mathbf{II.}} \tag{174}$$

Notice that, because we are describing movement at a material point in both cases,

$$\mathbf{v}(\mathbf{x}, t) = \mathbf{V}(\mathbf{X}, t), \quad \mathbf{a}(\mathbf{x}, t) = \mathbf{A}(\mathbf{X}, t). \tag{175}$$

Then, using $\mathbf{\Phi}^{-1}(\mathbf{x}, t) = \mathbf{X}$,

$$\mathbf{v}(\mathbf{x}, t) = \mathbf{V}(\mathbf{\Phi}^{-1}(\mathbf{x}, t), t), \quad \mathbf{a}(\mathbf{x}, t) = \mathbf{A}(\mathbf{\Phi}^{-1}(\mathbf{x}, t), t). \tag{176}$$

If $\underbrace{\textbf{X} \text{ (the material point) is fixed}}_{\textbf{I.}}$, the velocity is the speed and direction of that point as the material moves along its trajectory. If $\underbrace{\textbf{x} \text{ (the point in space) is fixed}}_{\textbf{II.}}$, the velocity at that fixed place $\mathbf{x}$ is the speed and direction of particles flowing through that point. Now, suppose there is some scalar field $\phi(x_i, t)$ and some vector field $\boldsymbol{\omega}(x_j, t)$. The relationship between the material time derivative and the spatial time derivative for each of these is

$$\frac{d}{dt}\phi = \frac{\partial \phi}{\partial t} + \frac{\partial \phi}{\partial x_i}\frac{\partial x_i}{\partial t} = \frac{\partial \phi}{\partial t} + \underbrace{\nabla\phi \cdot \mathbf{v}}_{\frac{\partial \phi}{\partial x_i}\mathbf{e}_i \cdot v_j \mathbf{e}_j}, \tag{177}$$

$$\frac{d}{dt}(\omega_i \mathbf{e}_i) = \frac{\partial \omega_i}{\partial t} + \frac{\partial \omega_i}{\partial x_j}\frac{\partial x_j}{\partial t} = \frac{\partial \omega_i}{\partial t} + (\nabla\omega)_{ij} v_j. \tag{178}$$

From these relations we obtain velocity on the right hand side. Then acceleration

$$a_i = \frac{\partial v_i}{\partial t} + v_{i,j} v_j \Longleftrightarrow \mathbf{a} = \frac{\partial \mathbf{v}}{\partial t} + (\nabla \mathbf{v})\mathbf{v}. \tag{179}$$

Consider the example

$$x_1 = (1+t)X_1, \quad x_2 = (1+t)^2 X_2 = (1+2t+t^2)X_2, \quad x_3 = (1+t^2)X_3$$
$$\Longrightarrow X_1 = \frac{x_1}{1+t}, \quad X_2 = \frac{x_2}{(1+t)^2}, \quad X_3 = \frac{x_3}{(1+t^2)}.$$

Then

$$\mathbf{V} = \dot{\mathbf{x}} = \begin{Bmatrix} X_1 \\ 2(1+t)X_2 \\ 2tX_3 \end{Bmatrix} \Longrightarrow \mathbf{v} = \begin{Bmatrix} x_1/(1+t) \\ 2x_2/(1+t) \\ 2tx_3/(1+t^2) \end{Bmatrix}$$

and

$$\mathbf{A} = \begin{Bmatrix} 0 \\ 2X_2 \\ 2X_3 \end{Bmatrix} \Longrightarrow \mathbf{a} = \begin{Bmatrix} 0 \\ 2x_2/(1+t)^2 \\ 2x_3/(1+t^2) \end{Bmatrix} = \frac{\partial \mathbf{v}}{\partial t} + (\nabla \mathbf{v})_{ij}(\mathbf{v})_j.$$

2.3.3 Rate of change of deformation and strains

Spatial gradient of velocity

$$\mathbf{L} = \nabla \mathbf{v} \Longleftrightarrow L_{ij} = \frac{\partial v_i}{\partial x_j} = v_{i,j}. \tag{180}$$

Then rate of change of deformation gradient

$$\dot{\mathbf{F}} = \mathbf{L}\mathbf{F} \Longleftrightarrow \dot{F_{iJ}} = L_{ij} F_{jJ} \tag{181}$$

and

$$\mathbf{L} = \dot{\mathbf{F}}\mathbf{F}^{-1} \Longleftrightarrow L_{ij} = \dot{F_{iJ}} F^{-1}_{Jj}. \tag{182}$$

We can decompose $\mathbf{L}$ into rate of deformation tensor $\mathbf{D}$ and spin tensor $\mathbf{W}$ such that

$$\mathbf{D} = \mathbf{W} = \text{sym}\mathbf{L} + \text{skw}\mathbf{L} \Longleftrightarrow D_{ij} + W_{ij} = \frac{1}{2}(L_{ij} + L_{ji}) + \frac{1}{2}(L_{ij} - L_{ji}). \tag{183}$$

The rate of change of Lagrangian strain

$$\begin{aligned}\dot{\mathbf{E}} &= \frac{\partial}{\partial t}\left[\frac{1}{2}(\mathbf{F}^T\mathbf{F} - \mathbf{I})\right] = \frac{1}{2}(\dot{\mathbf{F}}^T\mathbf{F} + \mathbf{F}^T\dot{\mathbf{F}}) = \frac{1}{2}(\mathbf{F}^T\mathbf{L}^T\mathbf{F} + \mathbf{F}^T\mathbf{L}\mathbf{F}) \\ &= \frac{1}{2}\mathbf{F}^T(\mathbf{L} + \mathbf{L}^T)\mathbf{F} = \mathbf{F}^T\mathbf{D}\mathbf{F} \Longleftrightarrow \dot{E}_{IJ} = F_{Ii}D_{ij}F_{jJ}. \end{aligned} \tag{184}$$

The rate of chnge of Euler Almansi strain

$$\dot{\mathbf{e}} = \frac{1}{2}(\mathbf{L}^T\mathbf{B}^{-1} + \mathbf{B}^{-1}\mathbf{L}) \Longleftrightarrow \frac{1}{2}(L_{ki}B^{-1}_{kj} + B^{-1}_{ik}L_{kj}) \tag{185}$$

or

$$\dot{\mathbf{e}} = \mathbf{D} - \mathbf{L}^T\mathbf{e} - \mathbf{e}\mathbf{L}. \tag{186}$$

Consider that for an invertible tensor $\mathbf{S}$

$$\dot{\overline{\det \mathbf{S}}} = \det \mathbf{S}\ \text{tr}(\dot{\mathbf{S}}\mathbf{S}^{-1}). \tag{187}$$

Then, from Eq. 182 ($\mathbf{L} = \dot{\mathbf{F}}\mathbf{F}^{-1}$),

$$\dot{J} = \dot{\overline{\det \mathbf{F}}} = \det \mathbf{F}\ \text{tr}(\dot{\mathbf{F}}\mathbf{F}^{-1}) = J\ \text{tr}\mathbf{L} = J\ \text{tr}(v_{i,j}) = Jv_{i,i} = J\ \text{div}\mathbf{v}. \tag{188}$$

Isochoric motion is volume-preserving such that $\dot{J} = 0 \leftarrow v_{i,i} = 0$.

2.3.4 Reynolds transport theorem

The time rate of change of an integral of ϕ over some subbody E $\subseteq$ B

$$\dot{I} = \frac{d}{dt}\int_E \phi(\mathbf{x}, t)dV = \int_E \left(\dot{\phi} + \phi\text{div}\mathbf{v}\right)dV. \tag{189}$$

From Eq. 177 we know

$$\dot{\phi} = \frac{d}{dt}\phi = \frac{\partial\phi}{\partial t} + \nabla\phi \cdot v,$$

so

$$\dot{\phi} + \phi\ \text{div}\ \mathbf{v} = \frac{\partial\phi}{\partial t} + \nabla\phi \cdot v + \phi\ \text{div}\ \mathbf{v} = \frac{\partial\phi}{\partial t} + \ \text{div}(\phi\mathbf{v}). \tag{190}$$

Therefore, because of the divergence theorem $\int_E v_i n_i dV = \int_{\partial E} v_{i,i} dA$,

$$\dot{I} = \int_E \left(\frac{\partial\phi}{\partial t} + \ \text{div}(\phi\mathbf{v})\right)dV = \int_E \left(\frac{\partial\phi}{\partial t}\right)dV + \phi\underbrace{\int_E \left(\ \text{div}(\mathbf{v})\right)dV}_{\mathbf{I}.} \tag{191}$$

$$= \int_E \left(\frac{\partial\phi}{\partial t}\right)dV + \phi\underbrace{\int_{\partial E} \left(\mathbf{v} \cdot \mathbf{n}\right)dA.}_{\mathbf{I},} \tag{192}$$

which can be thought of as (the production of ϕ inside E) + (the net transport of ϕ across ∂E).

3 Ch3

3.1 Conservation of mass

3.2 Force and stress in deformable bodies

3.2.1 Body forces

3.2.2 Surface forces

3.2.3 Stress

3.3 Balance of linear momentum

3.3.1 Cauchy stress tensor

3.3.2 Local form of linear momentum balance

3.4 Balance of angular momentum

3.5 Lagrangian description of momentum balances

3.5.1 Material form of linear momentum balance

3.5.2 Material form of angular momentum balance

3.6 Power balance

3.6.1 Principle of virtual power

3.6.2 Alternative formulations of force and moment balances

4 Ch4

4.1 Thermodynamics introduction

4.1.1 Thermodynamic equilibrium and state variables

4.1.2 Energy and entropy

4.2 Continuum thermodynamics

4.2.1 First law, energy balance

4.2.2 Second law, non-negative entropy production

5 Ch5

5.1 Developing physically meaningful constitutive theories

5.2 Compatibility with thermodynamics

5.2.1 Coleman Noll procedure

5.2.2 Alternative thermodynamic potentials

5.3 Material frame indifference

5.3.1 Transformation rule for kinematic fields

5.3.2 Transformation rule for stress

5.3.3 Constraints on constitutive relations

6 Ch6

6.1 Basic laws

6.2 General constitutive equations

6.3 Coleman Noll procedure

6.4 Material frame indifference

6.5 Fourier Law

6.6 Initial/boundary value problem in heat transfer theory

7 Ch7

7.1 Brief review

7.1.1 Kinematic relations

Recall Eq. 121, the polar decomposition of deformation gradient

$$\mathbf{F} = \mathbf{R}\mathbf{U} = \mathbf{V}\mathbf{R} \tag{193}$$

into rotation $\mathbf{R}$ and either right stretch tensor $\mathbf{U}$ or left stretch tensor $\mathbf{V}$, with

$$\mathbf{U} = \sqrt{\mathbf{F}^T\mathbf{F}} \tag{194}$$

7.1.2 Basic laws

7.1.3 Transformation law under frame change

7.2 Constitutive theory

7.2.1 Consequence of frame indifference

7.2.2 Thermodynamic restriction

7.3 Initial/boundary value problem

7.4 Isotropic solids

An isotropic tensor $\mathbf{T}$ satisfies

$$\mathbf{Q}\mathbf{T}(\mathbf{A})\mathbf{Q}^T = \mathbf{T}(\mathbf{Q}\mathbf{A}\mathbf{Q}^T) \tag{195}$$

with rotation $\mathbf{Q}$. An isotropic scalar ϕ satisfies

$$\phi(\mathbf{A}) = \phi(\mathbf{Q}\mathbf{A}\mathbf{Q}^T). \tag{196}$$

The condition of isotropy imposes severe functional restrictions. An isotropic material is one whose properties are the same in all directions. For isotropic materials, every rotation is a symmetry transformation such that

$$\Psi(\mathbf{Q}^T\mathbf{C}\mathbf{Q}) = \Psi(\mathbf{C}) \tag{197}$$

for all rotations $\mathbf{Q}$ and for all symmetric $\mathbf{C}$. Let us choose $\mathbf{Q} = \mathbf{R}^T$ where $\mathbf{R}$ is the rotation in the polar decomposition $\mathbf{F} = \mathbf{R}\mathbf{U} = \mathbf{V}\mathbf{R}$.

7.5 Hyperelastic isotropic solid

A hyperelastic solid possesses strain energy density $W(\mathbf{F})$ from which stress is obtained. Free energy per unit volume

$$W = \rho_0\Psi_0. \tag{198}$$

7.5.1 Blatz Ko materials

7.5.2 Incompressible materials

Incompressible materials by definition require

$$J = 1 = \det \mathbf{F}. \tag{199}$$

7.6 Linear theory of elasticity

7.6.1 Small deformation

7.6.2 Constitutive equation for small deformation

7.6.3 Summary and further assumptions

8 Ch8

8.1 Brief review

8.1.1 Kinematic relations

8.1.2 Basic laws

8.1.3 Transformation rules

8.2 Elastic fluids

8.2.1 Constitutive theory

8.2.2 Consequence of frame indifference

8.2.3 Consequence of thermodynamics

8.3 Compressible viscous fluids

8.3.1 General constitutive equaitons

8.3.2 Consequence of frame indifference

8.3.3 Consequence of thermodynamics

8.3.4 Linear Newtonian viscous fluid

8.3.5 Nonlinear non-Newtonian viscous fluid

8.3.6 Compressible Navier Stokes equation

8.4 Incompressible fluids

8.4.1 Free energy imbalance for incompressible body

8.4.2 Incompressible viscious fluids

8.4.3 Incompressible Navior Stokes equation