MAE 529 - Finite Element Structural Analysis

Joseph Marziale March 14, 2023

Contents

1 1D bar

2 1D EB beam

3 2D CST

3.1 Strong form

Consider Fig. 1, a small section of a 2D plane subject to a certain distribution of force.

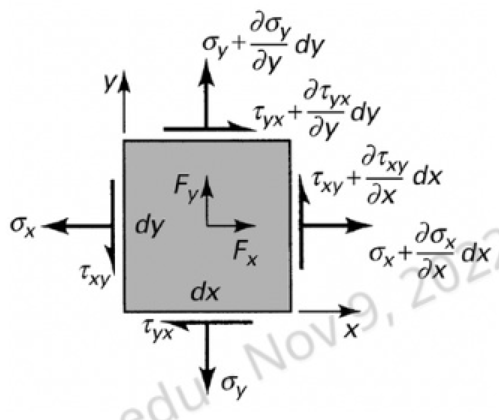

Figure 1: Solid mechanics infinitesimal square. Note that because of the symmetry of the stress tensor, the shear components are equal ($xy \Leftrightarrow yx$). I made this substitution in the derivation implicitly.

Assume the body Ω is in equilibrium so that the sum of the forces in x

$$\begin{aligned}\sum f_x &= F_x dxdydz + (\cancel{\sigma_{xx}} + \frac{\partial \sigma_{xx}}{\partial x} dx)dydz + (\cancel{\tau_{xy}} + \frac{\partial \tau_{xy}}{\partial y} dy)dxdz - \cancel{\sigma_{xx} dydz} - \cancel{\tau_{xy} dxdz} \\ &= F_x dxdydz + \frac{\partial \sigma_{xx}}{\partial x} dxdydz + \frac{\partial \tau_{xy}}{\partial y} dxdydz = 0. \end{aligned} \tag{1}$$

This implies

$$F_x + \frac{\partial \sigma_{xx}}{\partial x} + \frac{\partial \tau_{xy}}{\partial y} = 0. \tag{2}$$

Similarly for y,

$$\begin{aligned}\sum f_y &= F_y dxdydz + (\cancel{\sigma_{yy}} + \frac{\partial \sigma_{yy}}{\partial y} dy)dxdz + (\cancel{\tau_{yx}} + \frac{\partial \tau_{yx}}{\partial x} dx)dydz - \cancel{\sigma_{yy} dxdz} - \cancel{\tau_{yx} dydz} \\ &= F_y dxdydz + \frac{\partial \sigma_{yy}}{\partial y} dxdydz + \frac{\partial \tau_{yx}}{\partial x} dxdydz = 0. \end{aligned} \tag{3}$$

This implies

$$F_y + \frac{\partial \sigma_{yy}}{\partial y} + \frac{\partial \tau_{yx}}{\partial x} = 0. \tag{4}$$

Eqs. 2 and 4 can be rewritten as

$$F_x + \sigma_{xx,x} + \tau_{xy,y} = 0, \quad F_y + \sigma_{yy,y} + \tau_{yx,x} = 0 \tag{5}$$

Note that $\tau_{xy},\ \tau_{yx}$ are components in stress tensor $[\boldsymbol{\sigma}]$, just like $\sigma_{xx},\ \sigma_{yy}$. So, it is permissible to write

$$F_x + \sigma_{xx,x} + \sigma_{xy,y} = 0, \quad F_y + \sigma_{yy,y} + \sigma_{yx,x} = 0. \tag{6}$$

Generalizing,

$$F_i + \sigma_{ij,j} = 0. \tag{7}$$

Eq. 7 is the strong form. It is an equilibrium equation of force per volume (newtons per meters cubed). F is a body force that acts volumetrically, so that $F = f/V$. In this case, f is a true force in newtons. Stress $\sigma = f/A$; the spatial derivative of stress is also a volumetric term, because $\frac{d}{\mathrm{dm}}\mathrm{Nm}^{-2} = -2\mathrm{Nm}^{-3}$, speaking in terms of units.

Surface traction

$$t_i = \sigma_{ij} n_j \tag{8}$$

is the normal component of stress. Here n is a vector normal to the surface of the body.

A Neumann boundary condition in this context is some imposition on $\bar{t}_i$. On the other hand, a Dirichlet boundary condition is an imposition on displacement $u_i(0,0) = \bar{u}_0$.

3.2 Weak form

Recall the strong form Eq. 7 ($\sigma_{ij,j} + F_i = 0$). The principle of virtual work (PVW) is defined as the act of multiplying the governing equation by a virtual displacement ($F \times \delta u = \delta W$) and then integrating over the domain Ω. That is,

$$\int_\Omega (\sigma_{ij,j} + F_i)\delta u_i d\Omega = 0. \tag{9}$$

This implies

$$\int_\Omega \underbrace{\sigma_{ij,j}\delta u_i}_{\mathbf{I.}} d\Omega + \int_\Omega F_i \delta u_i d\Omega = 0. \tag{10}$$

Now, as an aside, consider the chain rule

$$(\sigma_{ij}\delta u_i)_{,j} = \underbrace{\sigma_{ij,j}\delta u_i}_{\mathbf{I.}} + \sigma_{ij}\delta u_{i,j} \Longrightarrow \underbrace{\sigma_{ij,j}\delta u_i}_{\mathbf{I.}} = (\sigma_{ij}\delta u_i)_{,j} - \sigma_{ij}\delta u_{i,j} \tag{11}$$

Substituting Eq. 11 into Eq. 10,

$$\int_\Omega [(\sigma_{ij}\delta u_i)_{,j} - \sigma_{ij}\delta u_{i,j}] d\Omega + \int_\Omega F_i \delta u_i d\Omega = 0 \tag{12}$$

implies

$$\underbrace{\int_\Omega (\sigma_{ij}\delta u_i)_{,j} d\Omega}_{\mathbf{II.}} - \int_\Omega \sigma_{ij}\delta u_{i,j} d\Omega + \int_\Omega F_i \delta u_i d\Omega = 0. \tag{13}$$

The divergence theorem ($\underbrace{\int_\Omega (w_j)_{,j} dV}_{\textbf{II.}} = \underbrace{\int_{\partial\Omega} (w_j) n_j dA}_{\textbf{III.}}$) transforms Eq. 13 into

$$\underbrace{\int_\Gamma (\sigma_{ij}\delta u_i) n_j d\Gamma}_{\textbf{III.}} - \int_\Omega \sigma_{ij}\delta u_{i,j} d\Omega + \int_\Omega F_i \delta u_i d\Omega = 0. \tag{14}$$

Because of Eq. 8 ($\sigma_{ij} n_j = \bar{t}_i$),

$$\int_\Gamma \bar{t}_i \delta u_i d\Gamma - \int_\Omega \sigma_{ij}\delta u_{i,j} d\Omega + \int_\Omega F_i \delta u_i d\Omega = 0. \tag{15}$$

Any tensor $\mathbf{A}$ can be broken into its symmetric and skew parts $\text{sym}\mathbf{A} + \text{skw}\mathbf{A}$. For displacement gradient,

$$u_{i,j} = \epsilon_{ij} + \omega_{ij} = \text{sym}(u_{i,j}) + \text{skw}(u_{i,j}). \tag{16}$$

Note that displacement gradients are effectively strains, because strain $\epsilon = \delta/L$ is deformation with respect to the length of the body. So, $\delta\epsilon$ is a virtual strain. Substituting,

$$\int_\Gamma \bar{t}_i \delta u_i d\Gamma - \int_\Omega \sigma_{ij}\delta(\epsilon_{ij} + \omega_{ij}) d\Omega + \int_\Omega F_i \delta u_i d\Omega = 0 \tag{17}$$

implies

$$\int_\Gamma \bar{t}_i \delta u_i d\Gamma - \int_\Omega \sigma_{ij}\delta\epsilon_{ij} d\Omega + \int_\Omega \sigma_{ij}\delta\omega_{ij} d\Omega + \int_\Omega F_i \delta u_i d\Omega = 0. \tag{18}$$

The elementwise product of any symmetric and skew matrix $A_{ij}B_{ij} \Leftrightarrow \mathbf{A} : \mathbf{B}$ is 0. This is because

$$\mathbf{A} : \mathbf{B} \Leftrightarrow A_{ij}B_{ij} = A_{ji}(-B_{ji}) \Leftrightarrow -(\mathbf{A} : \mathbf{B}) \Longrightarrow 2(\mathbf{A} : \mathbf{B}) = 0 \Longrightarrow \mathbf{A} : \mathbf{B} = 0. \tag{19}$$

ω is skew while σ is symmetric. Therefore, Eq. 18 becomes

$$\int_\Gamma \bar{t}_i \delta u_i d\Gamma - \int_\Omega \sigma_{ij}\delta\epsilon_{ij} d\Omega + 0 + \int_\Omega F_i \delta u_i d\Omega = 0. \tag{20}$$

Rearranged,

$$\underbrace{\int_\Gamma \bar{t}_i \delta u_i d\Gamma + \int_\Omega F_i \delta u_i d\Omega}_{\delta W_{ext}.} = \underbrace{\int_\Omega \sigma_{ij}\delta\epsilon_{ij} d\Omega.}_{\delta W_{int}.} \tag{21}$$

The LHS term is the external virtual work δW_{ext} done by surface tractions and body forces. The RHS term is the internal virtual work δW_{int} done by virtual strain. Therefore,

$$\delta W_{ext} = \delta W_{int}. \tag{22}$$

Eq. 21 is the weak form.

3.3 Develop CST element

3.3.1 Define element

3.3.2 Shape functions

3.3.3 Strain/displacement relationship

3.3.4 Stress/strain relationship

3.3.5 Virtual quantities

3.3.6 Invoke PVW

3.3.7 Global stiffness matrix/boundary conditions

4 2D QUAD4

4.1 Strong form/weak form

For QUAD4, the strong and weak forms are the same as in CST. Those are Eq. 7 and 21, or

$$F_i + \sigma_{ij,j} = 0 \tag{23}$$

and

$$\underbrace{\int_\Omega \sigma_{ij}\delta\epsilon_{ij}d\Omega.}_{\delta W_{int.}} = \underbrace{\int_\Gamma \bar{t}_i \delta u_i d\Gamma + \int_\Omega F_i \delta u_i d\Omega}_{\delta W_{ext.}} \tag{24}$$

The LHS term is the internal virtual work δW_{int} done by virtual strain. The RHS term is the external virtual work δW_{ext} done by surface tractions and body forces. Recall also from Sec. 3.2 that surface traction

$$t_i = \sigma_{ij}n_j \tag{25}$$

is the normal component of stress. In this case n is a vector normal to the surface of the body. A Neumann boundary condition in this context is some imposition on $\bar{t}_i$. On the other hand, a Dirichlet boundary condition is an imposition on displacement $u_i(0,0) = \bar{u}_0$.

4.2 Develop QUAD4 element

4.2.1 Define element

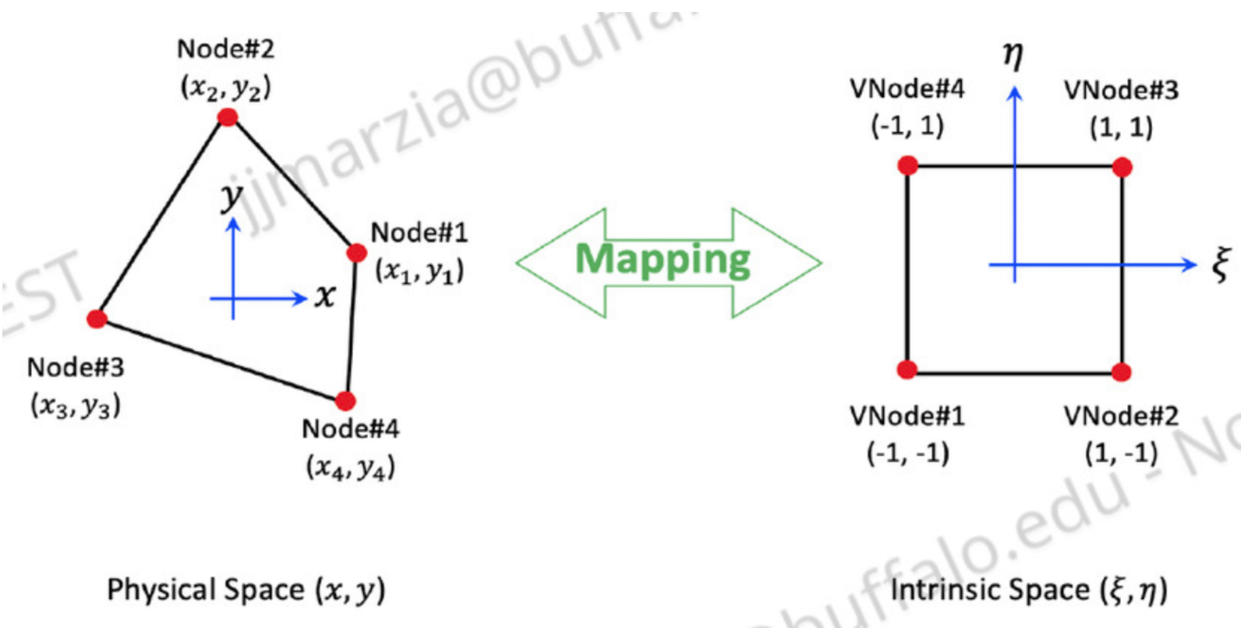

Figure 2: Mapping between intrinsic and physical spaces

Fig. 2 describes the mapping that isoparametric elements use between the intrinsic (reference) space

$$\{(\xi_i, \eta_i)\} = \{(-1,-1),(1,-1),(1,1),(-1,1)\} \tag{26}$$

and the physical (deformed) space

$$\{(x_i, y_i)\} = \{(x_1,y_1),(x_2,y_2),(x_3,y_3),(x_4,y_4)\}. \tag{27}$$

The deformed configuration can be expressed as a function of its reference, such that

$$x = x(\xi,\eta), \quad y = y(\xi,\eta). \tag{28}$$

Of course, the converse

$$\xi = \xi(x,y), \quad \eta = \eta(x,y) \tag{29}$$

is also true.

4.2.2 Shape functions

Determining the shape functions is first of all a matter of determining the functional form of Eq. 28. Fig. 3 shows the relationship between Pascal's triangle and the polynomial terms that comprise $x(\xi,\eta)$ and $y(\xi,\eta)$.

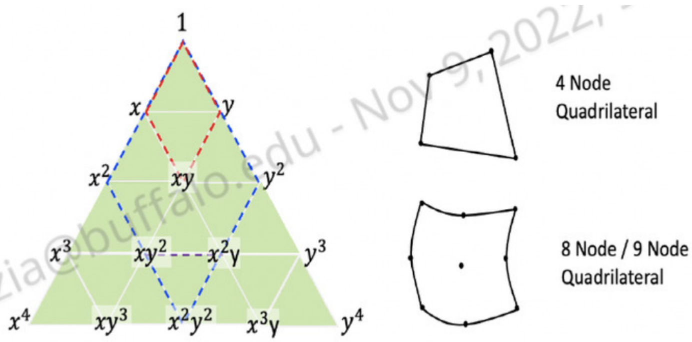

Figure 3: Pascal's triangle. Note that this is completely analogous to the intrinsic space, in that $xy \Leftrightarrow \xi\eta$.

For QUAD4,

$$x(\xi,\eta) = \alpha_1 1 + \alpha_2\xi + \alpha_3\eta + \alpha_4\xi\eta = \begin{bmatrix}1 & \xi & \eta & \xi\eta\end{bmatrix}\begin{bmatrix}\alpha_1\\ \alpha_2\\ \alpha_3\\ \alpha_4\end{bmatrix} = \begin{bmatrix}1 & \xi & \eta & \xi\eta\end{bmatrix}\{\boldsymbol{\alpha}^e\},$$

$$y(\xi,\eta) = \alpha_5 1 + \alpha_6\xi + \alpha_7\eta + \alpha_8\xi\eta = \begin{bmatrix} 1 & \xi & \eta & \xi\eta \end{bmatrix} \begin{bmatrix} \alpha_5 \\ \alpha_6 \\ \alpha_7 \\ \alpha_8 \end{bmatrix} = \begin{bmatrix} 1 & \xi & \eta & \xi\eta \end{bmatrix} \{\boldsymbol{\alpha}^e\}, \tag{30}$$

where α_i are coefficients. Substituting Eq. 26 $\{(\xi_i,\eta_i)\} = \{(-1,-1),(1,-1),(1,1),(-1,1)\}$ into Eq. 30,

$$x_1(\xi_1,\eta_1) = \begin{bmatrix} \underbrace{1}_{1} & \underbrace{-1}_{\xi} & \underbrace{-1}_{\eta} & \underbrace{1}_{\xi\eta} \end{bmatrix} \begin{bmatrix} \alpha_1 \\ \alpha_2 \\ \alpha_3 \\ \alpha_4 \end{bmatrix} = \alpha_1 - \alpha_2 - \alpha_3 + \alpha_4, \tag{31}$$

$$x_2(\xi_2,\eta_2) = \begin{bmatrix} 1 & 1 & -1 & -1 \end{bmatrix} \begin{bmatrix} \alpha_1 \\ \alpha_2 \\ \alpha_3 \\ \alpha_4 \end{bmatrix} = \alpha_1 + \alpha_2 - \alpha_3 - \alpha_4, \tag{32}$$

$$x_3(\xi_3,\eta_3) = \begin{bmatrix} 1 & 1 & 1 & 1 \end{bmatrix} \begin{bmatrix} \alpha_1 \\ \alpha_2 \\ \alpha_3 \\ \alpha_4 \end{bmatrix} = \alpha_1 + \alpha_2 + \alpha_3 + \alpha_4, \tag{33}$$

$$x_4(\xi_4,\eta_4) = \begin{bmatrix} 1 & -1 & 1 & -1 \end{bmatrix} \begin{bmatrix} \alpha_1 \\ \alpha_2 \\ \alpha_3 \\ \alpha_4 \end{bmatrix} = \alpha_1 - \alpha_2 + \alpha_3 - \alpha_4, \tag{34}$$

$$y_1(\xi_1,\eta_1) = \begin{bmatrix} 1 & -1 & -1 & 1 \end{bmatrix} \begin{bmatrix} \alpha_5 \\ \alpha_6 \\ \alpha_7 \\ \alpha_8 \end{bmatrix} = \alpha_5 - \alpha_6 - \alpha_7 + \alpha_8, \tag{35}$$

$$y_2(\xi_2,\eta_2) = \begin{bmatrix} 1 & 1 & -1 & -1 \end{bmatrix} \begin{bmatrix} \alpha_5 \\ \alpha_6 \\ \alpha_7 \\ \alpha_8 \end{bmatrix} = \alpha_5 + \alpha_6 - \alpha_7 - \alpha_8, \tag{36}$$

$$y_3(\xi_3,\eta_3) = \begin{bmatrix} 1 & 1 & 1 & 1 \end{bmatrix} \begin{bmatrix} \alpha_5 \\ \alpha_6 \\ \alpha_7 \\ \alpha_8 \end{bmatrix} = \alpha_5 + \alpha_6 + \alpha_7 + \alpha_8, \tag{37}$$

$$y_4(\xi_4,\eta_4) = \begin{bmatrix} 1 & -1 & 1 & -1 \end{bmatrix} \begin{bmatrix} \alpha_5 \\ \alpha_6 \\ \alpha_7 \\ \alpha_8 \end{bmatrix} = \alpha_5 - \alpha_6 + \alpha_7 - \alpha_8. \tag{38}$$

Altogether,

$$\underbrace{\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix}}_{\mathbf{x}^e} = \underbrace{\begin{bmatrix} 1 & -1 & -1 & 1 \\ 1 & 1 & -1 & -1 \\ 1 & 1 & 1 & 1 \\ 1 & -1 & 1 & -1 \end{bmatrix}}_{\mathbf{A}} \underbrace{\begin{bmatrix} \alpha_1 \\ \alpha_2 \\ \alpha_3 \\ \alpha_4 \end{bmatrix}}_{\boldsymbol{\alpha}^e}, \quad \underbrace{\begin{bmatrix} y_1 \\ y_2 \\ y_3 \\ y_4 \end{bmatrix}}_{\mathbf{y}^e} = \underbrace{\begin{bmatrix} 1 & -1 & -1 & 1 \\ 1 & 1 & -1 & -1 \\ 1 & 1 & 1 & 1 \\ 1 & -1 & 1 & -1 \end{bmatrix}}_{\mathbf{A}} \underbrace{\begin{bmatrix} \alpha_5 \\ \alpha_6 \\ \alpha_7 \\ \alpha_8 \end{bmatrix}}_{\boldsymbol{\alpha}^e}, \tag{39}$$

or

$$\mathbf{x}^e = \mathbf{A}\boldsymbol{\alpha}^e, \quad \mathbf{y}^e = \mathbf{A}\boldsymbol{\alpha}^e. \tag{40}$$

This means

$$\boldsymbol{\alpha}^e = \mathbf{A}^{-1}\mathbf{x}^e, \quad \boldsymbol{\alpha}^e = \mathbf{A}^{-1}\mathbf{y}^e, \tag{41}$$

where the inverse of $\mathbf{A}$

$$\mathbf{A}^{-1} = \frac{1}{4}\begin{bmatrix} 1 & 1 & 1 & 1 \\ -1 & 1 & 1 & -1 \\ -1 & -1 & 1 & 1 \\ 1 & -1 & 1 & -1 \end{bmatrix}. \tag{42}$$

Substituting Eq. 41 into Eq. 30,

$$x(\xi,\eta) = \begin{bmatrix} 1 & \xi & \eta & \xi\eta \end{bmatrix}\{\boldsymbol{\alpha}^e\} = \begin{bmatrix} 1 & \xi & \eta & \xi\eta \end{bmatrix}\{\mathbf{A}^{-1}\mathbf{x}^e\}$$

$$= \underbrace{\frac{1}{4}\begin{bmatrix} 1 & \xi & \eta & \xi\eta \end{bmatrix}\begin{bmatrix} 1 & 1 & 1 & 1 \\ -1 & 1 & 1 & -1 \\ -1 & -1 & 1 & 1 \\ 1 & -1 & 1 & -1 \end{bmatrix}}_{\mathbf{N}} \underbrace{\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix}}_{\mathbf{x}^e} = x(\xi,\eta) = \mathbf{N}(\xi,\eta)\mathbf{x}^e, \tag{43}$$

$$y(\xi,\eta) = \begin{bmatrix} 1 & \xi & \eta & \xi\eta \end{bmatrix}\{\boldsymbol{\alpha}^e\} = \begin{bmatrix} 1 & \xi & \eta & \xi\eta \end{bmatrix}\{\mathbf{A}^{-1}\mathbf{y}^e\}$$

$$= \underbrace{\frac{1}{4}\begin{bmatrix} 1 & \xi & \eta & \xi\eta \end{bmatrix}\begin{bmatrix} 1 & 1 & 1 & 1 \\ -1 & 1 & 1 & -1 \\ -1 & -1 & 1 & 1 \\ 1 & -1 & 1 & -1 \end{bmatrix}}_{\mathbf{N}} \underbrace{\begin{bmatrix} y_1 \\ y_2 \\ y_3 \\ y_4 \end{bmatrix}}_{\mathbf{y}^e} = x(\xi,\eta) = \mathbf{N}(\xi,\eta)\mathbf{y}^e, \tag{44}$$

where $\mathbf{N} \Leftrightarrow N_i(\xi,\eta)$ are the shape functions. $\mathbf{N}$ can be condensed to

$$\frac{1}{4}\begin{bmatrix} 1-\xi-\eta+\xi\eta \\ 1+\xi-\eta-\xi\eta \\ 1+\xi+\eta+\xi\eta \\ 1-\xi+\eta-\xi\eta \end{bmatrix} = \begin{bmatrix} (1/4)(1-\xi)(1-\eta) \\ (1/4)(1+\xi)(1-\eta) \\ (1/4)(1+\xi)(1+\eta) \\ (1/4)(1-\xi)(1+\eta) \end{bmatrix} = \begin{bmatrix} N_1 \\ N_2 \\ N_3 \\ N_4 \end{bmatrix} = \mathbf{N}(\xi,\eta) \Leftrightarrow N_i. \tag{45}$$

Consider

$$N_1(\xi_1,\eta_1) = \frac{1}{4}(1-\xi_1)(1-\eta_1) = \frac{1}{4}(1+1)(1+1) = 1 \tag{46}$$

and

$$N_1(\xi_2, \eta_2) = \frac{1}{4}(1-\xi_2)(1-\eta_2) = \frac{1}{4}(1-1)(1+1) = 0. \tag{47}$$

This is an example of the general rule

$$N_i(\xi_j, \eta_j) \Leftrightarrow N_i(x_j) = \delta_{ij} = \begin{cases} 1, & i = j \\ 0, & i \neq j. \end{cases} \tag{48}$$

4.2.3 Strain/displacement relationship

To obtain two separate equations for horizontal and vertical displacement u and v, let us redefine the system of shape functions as

$$\mathbf{N} = \begin{bmatrix} N_1 & 0 & N_2 & 0 & N_3 & 0 & N_4 & 0 \\ 0 & N_1 & 0 & N_2 & 0 & N_3 & 0 & N_4 \end{bmatrix}. \tag{49}$$

In general, strain

$$\boldsymbol{\epsilon} = \frac{\partial \mathbf{u}}{\partial \mathbf{x}} = \begin{bmatrix} \partial u/\partial x & \partial u/\partial y \\ \partial v/\partial x & \partial v/\partial y \end{bmatrix}. \tag{50}$$

Rewritten in Voigt notation,

$$\boldsymbol{\epsilon} = \begin{bmatrix} \partial u/\partial x \\ \partial v/\partial y \\ \partial u/\partial y + \partial v/\partial x \end{bmatrix}. \tag{51}$$

This can be decomposed as

$$\boldsymbol{\epsilon} = \begin{bmatrix} \partial/\partial x & 0 \\ 0 & \partial/\partial y \\ \partial/\partial y & \partial/\partial x \end{bmatrix} \begin{bmatrix} u \\ v \end{bmatrix} \quad \left(\text{Voigt: } \frac{\partial}{\partial \mathbf{x}} = \begin{bmatrix} \partial/\partial x & 0 \\ 0 & \partial/\partial y \\ \partial/\partial y & \partial/\partial x \end{bmatrix}\right). \tag{52}$$

Recall also that generalized strain

$$\boldsymbol{\epsilon} = \frac{\partial}{\partial \mathbf{x}}(\mathbf{u}) = \frac{\partial}{\partial \mathbf{x}}(\mathbf{N}\mathbf{u}^e) = \underbrace{\frac{\partial}{\partial \mathbf{x}}(\mathbf{N})}_{\mathbf{I}.} \mathbf{u}^e = \underbrace{\mathbf{B}}_{\mathbf{I}.} \mathbf{u}^e. \tag{53}$$

This means

$$\mathbf{B} = \frac{\partial}{\partial \mathbf{x}} \mathbf{N} = \begin{bmatrix} \partial/\partial x & 0 \\ 0 & \partial/\partial y \\ \partial/\partial y & \partial/\partial x \end{bmatrix} \begin{bmatrix} N_1 & 0 & N_2 & 0 & N_3 & 0 & N_4 & 0 \\ 0 & N_1 & 0 & N_2 & 0 & N_3 & 0 & N_4 \end{bmatrix}$$

$$= \begin{bmatrix} \partial N_1/\partial x & 0 & \partial N_2/\partial x & 0 & \partial N_3/\partial x & 0 & \partial N_4/\partial x & 0 \\ 0 & \partial N_1/\partial y & 0 & \partial N_2/\partial y & 0 & \partial N_3/\partial y & 0 & \partial N_4/\partial y \\ \partial N_1/\partial y & \partial N_1/\partial x & \partial N_2/\partial y & \partial N_2/\partial x & \partial N_3/\partial y & \partial N_3/\partial x & \partial N_4/\partial y & \partial N_4/\partial x \end{bmatrix}. \tag{54}$$

The chain rules for $f(g,h),\ g(x,y),\ h(x,y)$ are

$$\frac{\partial f}{\partial x} = \frac{\partial f}{\partial g}\frac{\partial g}{\partial x} + \frac{\partial f}{\partial h}\frac{\partial h}{\partial x}, \qquad \frac{\partial f}{\partial y} = \frac{\partial f}{\partial g}\frac{\partial g}{\partial y} + \frac{\partial f}{\partial h}\frac{\partial h}{\partial y}. \tag{55}$$

If $g = \xi,\ h = \eta,$ and $f(g,h) = N_i(\xi,\eta)$,

$$\frac{\partial N_i}{\partial x} = \frac{\partial N_i}{\partial \xi}\underbrace{\frac{\partial \xi}{\partial x}}_{\hat{\mathbf{J}}_{11}} + \frac{\partial N_i}{\partial \eta}\underbrace{\frac{\partial \eta}{\partial x}}_{\hat{\mathbf{J}}_{21}}, \qquad \frac{\partial N_i}{\partial y} = \frac{\partial N_i}{\partial \xi}\underbrace{\frac{\partial \xi}{\partial y}}_{\hat{\mathbf{J}}_{12}} + \frac{\partial N_i}{\partial \eta}\underbrace{\frac{\partial \eta}{\partial y}}_{\hat{\mathbf{J}}_{22}}. \tag{56}$$

Using Eq. 45,

$$\begin{bmatrix} N_1 \\ N_2 \\ N_3 \\ N_4 \end{bmatrix} = \frac{1}{4}\begin{bmatrix} 1-\xi-\eta+\xi\eta \\ 1+\xi-\eta-\xi\eta \\ 1+\xi+\eta+\xi\eta \\ 1-\xi+\eta-\xi\eta \end{bmatrix}$$

$$\implies \begin{bmatrix} \partial N_1/\partial\xi \\ \partial N_2/\partial\xi \\ \partial N_3/\partial\xi \\ \partial N_4/\partial\xi \end{bmatrix} = \frac{1}{4}\begin{bmatrix} -1+\eta \\ 1-\eta \\ 1+\eta \\ -1-\eta \end{bmatrix} = \begin{bmatrix} -(1/4)(1-\eta) \\ (1/4)(1-\eta) \\ (1/4)(1+\eta) \\ -(1/4)(1+\eta) \end{bmatrix},$$

$$\implies \begin{bmatrix} \partial N_1/\partial\eta \\ \partial N_2/\partial\eta \\ \partial N_3/\partial\eta \\ \partial N_4/\partial\eta \end{bmatrix} = \frac{1}{4}\begin{bmatrix} -1+\xi \\ -1-\xi \\ 1+\xi \\ 1-\xi \end{bmatrix} = \begin{bmatrix} -(1/4)(1-\xi) \\ -(1/4)(1+\xi) \\ (1/4)(1+\xi) \\ (1/4)(1-\xi) \end{bmatrix}. \tag{57}$$

Expanding Eq. 56 and substituting terms contained in Eq. 57 where appropriate,

$$\begin{bmatrix} \partial N_1/\partial x \\ \partial N_2/\partial x \\ \partial N_3/\partial x \\ \partial N_4/\partial x \end{bmatrix} = \begin{bmatrix} (\partial N_1/\partial\xi)\hat{\mathbf{J}}_{11} + (\partial N_1/\partial\eta)\hat{\mathbf{J}}_{21} \\ (\partial N_2/\partial\xi)\hat{\mathbf{J}}_{11} + (\partial N_2/\partial\eta)\hat{\mathbf{J}}_{21} \\ (\partial N_3/\partial\xi)\hat{\mathbf{J}}_{11} + (\partial N_3/\partial\eta)\hat{\mathbf{J}}_{21} \\ (\partial N_4/\partial\xi)\hat{\mathbf{J}}_{11} + (\partial N_4/\partial\eta)\hat{\mathbf{J}}_{21} \end{bmatrix} = \frac{1}{4}\begin{bmatrix} -(1-\eta)\hat{\mathbf{J}}_{11} - (1-\xi)\hat{\mathbf{J}}_{21} \\ (1-\eta)\hat{\mathbf{J}}_{11} - (1+\xi)\hat{\mathbf{J}}_{21} \\ (1+\eta)\hat{\mathbf{J}}_{11} + (1+\xi)\hat{\mathbf{J}}_{21} \\ -(1+\eta)\hat{\mathbf{J}}_{11} + (1-\xi)\hat{\mathbf{J}}_{21} \end{bmatrix} \tag{58}$$

and

$$\begin{bmatrix} \partial N_1/\partial y \\ \partial N_2/\partial y \\ \partial N_3/\partial y \\ \partial N_4/\partial y \end{bmatrix} = \begin{bmatrix} (\partial N_1/\partial\xi)\hat{\mathbf{J}}_{12} + (\partial N_1/\partial\eta)\hat{\mathbf{J}}_{22} \\ (\partial N_2/\partial\xi)\hat{\mathbf{J}}_{12} + (\partial N_2/\partial\eta)\hat{\mathbf{J}}_{22} \\ (\partial N_3/\partial\xi)\hat{\mathbf{J}}_{12} + (\partial N_3/\partial\eta)\hat{\mathbf{J}}_{22} \\ (\partial N_4/\partial\xi)\hat{\mathbf{J}}_{12} + (\partial N_4/\partial\eta)\hat{\mathbf{J}}_{22} \end{bmatrix} = \frac{1}{4}\begin{bmatrix} -(1-\eta)\hat{\mathbf{J}}_{12} - (1-\xi)\hat{\mathbf{J}}_{22} \\ (1-\eta)\hat{\mathbf{J}}_{12} - (1+\xi)\hat{\mathbf{J}}_{22} \\ (1+\eta)\hat{\mathbf{J}}_{12} + (1+\xi)\hat{\mathbf{J}}_{22} \\ -(1+\eta)\hat{\mathbf{J}}_{12} + (1-\xi)\hat{\mathbf{J}}_{22} \end{bmatrix}. \tag{59}$$

4.2.4 Jacobian

Matrix

$$[\hat{\mathbf{J}}] = \begin{bmatrix} \hat{\mathbf{J}}_{11} & \hat{\mathbf{J}}_{12} \\ \hat{\mathbf{J}}_{21} & \hat{\mathbf{J}}_{22} \end{bmatrix} = \begin{bmatrix} \partial\xi/\partial x & \partial\xi/\partial y \\ \partial\eta/\partial x & \partial\eta/\partial y \end{bmatrix} = [\mathbf{J}^{-1}] \tag{60}$$

is the inverse of

$$[\mathbf{J}] = \begin{bmatrix} \mathbf{J}_{11} & \mathbf{J}_{12} \\ \mathbf{J}_{21} & \mathbf{J}_{22} \end{bmatrix} = \begin{bmatrix} \partial x/\partial \xi & \partial y/\partial \xi \\ \partial x/\partial \eta & \partial y/\partial \eta \end{bmatrix} \Longleftrightarrow \mathbf{J}(\xi, \eta), \tag{61}$$

the Jacobian. Using the shape function definition $\mathbf{x} = \mathbf{N}\mathbf{x}^e \Leftrightarrow x = N_i x_i$, Eq. 61 becomes

$$[\mathbf{J}] = \begin{bmatrix} (\partial N_i/\partial \xi) x_i & (\partial N_i/\partial \xi) y_i \\ (\partial N_i/\partial \eta) x_i & (\partial N_i/\partial \eta) y_i \end{bmatrix}$$

$$= \begin{bmatrix} (\partial N_1/\partial \xi)x_1 + (\partial N_2/\partial \xi)x_2 + (\partial N_3/\partial \xi)x_3 + (\partial N_4/\partial \xi)x_4, & (\partial N_1/\partial \xi)y_1 + (\partial N_2/\partial \xi)y_2 + \dots \\ (\partial N_1/\partial \eta)x_1 + (\partial N_2/\partial \eta)x_2 + (\partial N_3/\partial \eta)x_3 + (\partial N_4/\partial \eta)x_4, & (\partial N_1/\partial \eta)y_1 + (\partial N_2/\partial \eta)y_2 + \dots \end{bmatrix}$$

$$= \begin{bmatrix} \partial N_1/\partial \xi & \partial N_2/\partial \xi & \partial N_3/\partial \xi & \partial N_4/\partial \xi \\ \partial N_1/\partial \eta & \partial N_2/\partial \eta & \partial N_3/\partial \eta & \partial N_4/\partial \eta \end{bmatrix} \begin{bmatrix} x_1 & y_1 \\ x_2 & y_2 \\ x_3 & y_3 \\ x_4 & y_4 \end{bmatrix}. \tag{62}$$

Again substituting appropriate terms in Eq. 57,

$$= \frac{1}{4} \begin{bmatrix} -(1-\eta) & (1-\eta) & (1+\eta) & -(1+\eta) \\ -(1-\xi) & -(1+\xi) & (1+\xi) & (1-\xi) \end{bmatrix} \begin{bmatrix} x_1 & y_1 \\ x_2 & y_2 \\ x_3 & y_3 \\ x_4 & y_4 \end{bmatrix} = \begin{bmatrix} \mathbf{J}_{11} & \mathbf{J}_{12} \\ \mathbf{J}_{21} & \mathbf{J}_{22} \end{bmatrix} = [\mathbf{J}]. \tag{63}$$

A physically realistic deformation requires

$$\det \mathbf{J} > 0. \tag{64}$$

If $\mathbf{J}$ is known, its inverse is nothing more than

$$[\mathbf{J}^{-1}] = \begin{bmatrix} \mathbf{J}_{22} & -\mathbf{J}_{12} \\ -\mathbf{J}_{21} & \mathbf{J}_{11} \end{bmatrix} = \begin{bmatrix} \hat{\mathbf{J}}_{11} & \hat{\mathbf{J}}_{12} \\ \hat{\mathbf{J}}_{21} & \hat{\mathbf{J}}_{22} \end{bmatrix} = [\hat{\mathbf{J}}]. \tag{65}$$

Both Eqs. 61 and 63 are valid ways to calculate $\mathbf{J}$. Eq. 61 is convenient if $x = N_i x_i,\ y = N_i y_i$ are already given. Eq. 63 is convenient if only $x_i,\ y_i$ are given and $x,\ y$ would need to be found otherwise.

An interesting special case is the 1D bar element, where $\xi_1 = -1,\ \xi_2 = 1$ and

$$x = \alpha_1 + \alpha_2 \xi = \begin{bmatrix} 1 & \xi \end{bmatrix} \begin{bmatrix} \alpha_1 \\ \alpha_2 \end{bmatrix} \Rightarrow \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \underbrace{\begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix}}_{\mathbf{A}} \begin{bmatrix} \alpha_1 \\ \alpha_2 \end{bmatrix} \tag{66}$$

$$\Rightarrow \underbrace{\frac{1}{2} \begin{bmatrix} 1 & 1 \\ -1 & 1 \end{bmatrix}}_{\mathbf{A}^{-1}} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} \alpha_1 \\ \alpha_2 \end{bmatrix} \Rightarrow x = \begin{bmatrix} 1 & \xi \end{bmatrix} \begin{bmatrix} 1/2 & 1/2 \\ -1/2 & 1/2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

$$= \begin{bmatrix} 1 - \xi/2 & 1 + \xi/2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + N_i x_i = x, \tag{67}$$

Then

$$J = \frac{\partial x}{\partial \xi} = \frac{\partial N_i}{\partial \xi} x_i = -\frac{1}{2} x_1 + \frac{1}{2} x_2 = \frac{L}{2}, \tag{68}$$

where $x_2 - x_1 = L$.

4.2.5 Stress/strain relationship

The stress/strain relationship in QUAD4 is the same as in CST. For linear elasticity, Hooke's law states that stress

$$\boldsymbol{\sigma} = \mathbf{C}\boldsymbol{\epsilon} = \mathbf{C}\mathbf{B}\mathbf{u}^e, \tag{69}$$

given Eq. 53 and using Voigt notation

$$\boldsymbol{\sigma} = \begin{bmatrix} \sigma_{11} \\ \sigma_{22} \\ \sigma_{12} \end{bmatrix}, \quad \boldsymbol{\epsilon} = \begin{bmatrix} \epsilon_{11} \\ \epsilon_{22} \\ \epsilon_{12} \end{bmatrix}. \tag{70}$$

For a plane stress problem,

$$\mathbf{C} = \frac{E}{1-\nu^2} \begin{bmatrix} 1 & \nu & 0 \\ \nu & 1 & 0 \\ 0 & 0 & 1-\nu \end{bmatrix}. \tag{71}$$

For a plane strain problem,

$$\mathbf{C} = \frac{E}{(1+\nu)(1-2\nu)} \begin{bmatrix} 1-\nu & \nu & 0 \\ \nu & 1-\nu & 0 \\ 0 & 0 & 1-2\nu \end{bmatrix}. \tag{72}$$

Here, E is Young's modulus and ν is Poisson's ratio.

4.2.6 Virtual quantities

Just like for CST, virtual quantities

$$\delta\mathbf{u} = \mathbf{N}\delta\mathbf{u}^e, \qquad \delta\boldsymbol{\epsilon} = \mathbf{B}\delta\mathbf{u}^e. \tag{73}$$

4.2.7 Invoke PVW

Recall the weak form for the whole domain that is Eq. 24,

$$\underbrace{\int_\Omega \sigma_{ij}\delta\epsilon_{ij}d\Omega.}_{\delta W_{int}.} = \underbrace{\int_\Gamma \bar{t}_i\delta u_i d\Gamma + \int_\Omega F_i\delta u_i d\Omega.}_{\delta W_{ext}.}$$

In matrix notation, noting that a symmetric $\boldsymbol{\epsilon}$ permits $\delta\boldsymbol{\epsilon}\boldsymbol{\sigma} = \delta\boldsymbol{\epsilon}^T\boldsymbol{\sigma}$ and using Eqs. 69 ($\boldsymbol{\sigma} = \mathbf{C}\mathbf{B}\mathbf{u}^e$) and 73 ($\delta\boldsymbol{\epsilon} = \mathbf{B}\delta\mathbf{u}^e$), the LHS at the elemental level ($\Omega \Rightarrow \Omega^e$) is

$$\int_{\Omega^e} (\delta\boldsymbol{\epsilon}^T)(\boldsymbol{\sigma})d\Omega^e = \int_{\Omega^e} ([\delta\mathbf{u}^e]^T\mathbf{B}^T)(\mathbf{C}\mathbf{B}\mathbf{u}^e)d\Omega^e = [\delta\mathbf{u}^e]^T\mathbf{K}^e\delta\mathbf{u}^e \tag{74}$$

where elemental stiffness matrix

$$\mathbf{K}^e = \int_{\Omega^e} \mathbf{B}^T\mathbf{C}\mathbf{B}d\Omega^e. \tag{75}$$

We assume the body has some thickness t, which is simply multiplied as a scalar. Then for a plane stress problem, the conversion from the physical space $x,\ y \in \Omega^e$ to the intrinsic space $\xi,\ \eta \in [-1, 1]$ is

$$\mathbf{K}^e = t \int_{\Omega^e} \mathbf{B}(x, y)^T \mathbf{C} \mathbf{B}(x, y) dx dy \tag{76}$$

$$= t \int_{-1}^{1} \int_{-1}^{1} \mathbf{B}(\xi, \eta)^T \mathbf{C} \mathbf{B}(\xi, \eta) [\det \mathbf{J}(\xi, \eta)] d\xi d\eta. \tag{77}$$

We rely on Gauss quadrature to approximate the solution to this integral. The more Gauss points, the finer the distribution and the closer the approximation. Fig. 4 is a visualization of Gauss quadrature.

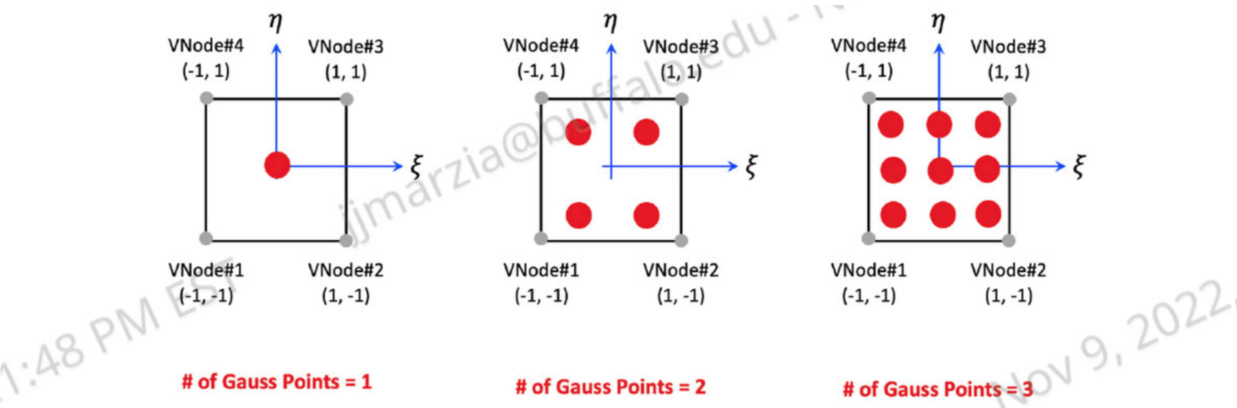

Figure 4: Gauss points $GP = \{1, 2, 3\}$

Then Eq. 77 is approximately

$$\mathbf{K}^e = t \sum_{i=1}^{\mathrm{GP}} \sum_{j=1}^{\mathrm{GP}} w_i w_j \mathbf{B}(\xi_i, \eta_j)^T \mathbf{C} \mathbf{B}(\xi_i, \eta_j) [\det \mathbf{J}(\xi_i, \eta_j)], \tag{78}$$

where weights w are explained in further detail in Sec. 4.3. If $GP = 1$,

$$\mathbf{K}^e = t w_1 w_1 \mathbf{B}(\xi_1, \eta_1)^T \mathbf{C} \mathbf{B}(\xi_1, \eta_1) [\det \mathbf{J}(\xi_1, \eta_1)]. \tag{79}$$

If $GP = 2$,

$$\begin{aligned} \mathbf{K}^e = {} & t w_1 w_1 \mathbf{B}(\xi_1, \eta_1)^T \mathbf{C} \mathbf{B}(\xi_1, \eta_1) [\det \mathbf{J}(\xi_1, \eta_1)] \\ & + t w_1 w_2 \mathbf{B}(\xi_1, \eta_2)^T \mathbf{C} \mathbf{B}(\xi_1, \eta_2) [\det \mathbf{J}(\xi_1, \eta_2)] \\ & + t w_2 w_1 \mathbf{B}(\xi_2, \eta_1)^T \mathbf{C} \mathbf{B}(\xi_2, \eta_1) [\det \mathbf{J}(\xi_2, \eta_1)] \\ & + t w_2 w_2 \mathbf{B}(\xi_2, \eta_2)^T \mathbf{C} \mathbf{B}(\xi_2, \eta_2) [\det \mathbf{J}(\xi_2, \eta_2)]. \end{aligned} \tag{80}$$

If $GP = 3$,

$$\mathbf{K}^e = t \int_{\Omega^e} \mathbf{B}(x, y)^T \mathbf{C} \mathbf{B}(x, y) dx dy$$

$$
\begin{aligned}
\mathbf{K}^e = & \, tw_1w_1\mathbf{B}(\xi_1,\eta_1)^T\mathbf{C}\mathbf{B}(\xi_1,\eta_1)[\det\mathbf{J}(\xi_1,\eta_1)] \\
& +tw_1w_2\mathbf{B}(\xi_1,\eta_2)^T\mathbf{C}\mathbf{B}(\xi_1,\eta_2)[\det\mathbf{J}(\xi_1,\eta_2)] \\
& +tw_1w_3\mathbf{B}(\xi_1,\eta_3)^T\mathbf{C}\mathbf{B}(\xi_1,\eta_3)[\det\mathbf{J}(\xi_1,\eta_3)] \\
& +tw_2w_1\mathbf{B}(\xi_2,\eta_1)^T\mathbf{C}\mathbf{B}(\xi_2,\eta_1)[\det\mathbf{J}(\xi_2,\eta_1)] \\
& +tw_2w_2\mathbf{B}(\xi_2,\eta_2)^T\mathbf{C}\mathbf{B}(\xi_2,\eta_2)[\det\mathbf{J}(\xi_2,\eta_2)] \\
& +tw_2w_3\mathbf{B}(\xi_2,\eta_3)^T\mathbf{C}\mathbf{B}(\xi_2,\eta_3)[\det\mathbf{J}(\xi_2,\eta_3)] \\
& +tw_3w_1\mathbf{B}(\xi_3,\eta_1)^T\mathbf{C}\mathbf{B}(\xi_3,\eta_1)[\det\mathbf{J}(\xi_3,\eta_1)] \\
& +tw_3w_2\mathbf{B}(\xi_3,\eta_2)^T\mathbf{C}\mathbf{B}(\xi_3,\eta_2)[\det\mathbf{J}(\xi_3,\eta_2)] \\
& +tw_3w_3\mathbf{B}(\xi_3,\eta_3)^T\mathbf{C}\mathbf{B}(\xi_3,\eta_3)[\det\mathbf{J}(\xi_3,\eta_3)].
\end{aligned} \tag{81}
$$

4.2.8 Forcing term

Recall once again the weak form for the whole domain that is Eq. 24,

$$
\underbrace{\int_\Omega \sigma_{ij}\delta\epsilon_{ij}d\Omega.}_{\delta W_{int.}} = \underbrace{\int_\Omega F_i\delta u_i d\Omega + \int_\Gamma \bar{t}_i\delta u_i d\Gamma.}_{\delta W_{ext.}}
$$

The left hand side is addressed in Sec. 4.2.7. As for the right hand side, its representation in matrix notation is

$$
\int_\Omega F_i\delta u_i d\Omega + \int_\Gamma \bar{t}_i\delta u_i d\Gamma \Longleftrightarrow \int_{\Omega^e}\delta\mathbf{u}^T\bar{\mathbf{g}}d\Omega^e + \int_{\Gamma^e}\delta\mathbf{u}^T\bar{\mathbf{t}}d\Gamma^e, \tag{82}
$$

where body force $F_i \Leftrightarrow \bar{\mathbf{g}}$ so as to not confuse body force $\bar{\mathbf{g}}$ with overall forcing term

$$
\mathbf{F}^e = \bar{\mathbf{b}}^e + \bar{\mathbf{T}}^e = \int_{\Omega^e}\mathbf{N}^T\bar{\mathbf{g}}d\Omega^e + \int_{\Gamma^e}\mathbf{N}^T\bar{\mathbf{t}}d\Gamma^e, \tag{83}
$$

in which

$$
\bar{\mathbf{b}}^e = \sum_{i=1}^{\mathrm{GP}}\sum_{j=1}^{\mathrm{GP}} w_iw_j\mathbf{N}^T(\xi_i,\eta_j)\left(\sum_{k=1}^{SF}N_k(\xi_i,\eta_j)\bar{\mathbf{g}}_k\right)[\det\mathbf{J}(\xi_i,\eta_j)] \tag{84}
$$

and

$$
\bar{\mathbf{T}}^e = \sum_{i=1}^{\mathrm{GP}} w_i\mathbf{N}^T(\xi_i,\eta_i)\left(\sum_{k=1}^{SF}N_k(\xi_i,\eta_i)\bar{\mathbf{t}}_k\right)[\det\mathbf{J}(\xi_i,\eta_i)] \tag{85}
$$

For elemental body force $\bar{\mathbf{b}}^e$ and external force $\bar{\mathbf{T}}^e$, index k goes from 1 to SF, which is the total number of shape functions (N's) in the problem. Note also that in the same way as

$$
x = \sum_{i=i}^{SF}N_ix_i, \quad y = \sum_{i=i}^{SF}N_iy_i, \quad u = \sum_{i=i}^{SF}N_iu_i, \quad v = \sum_{i=i}^{SF}N_iv_i, \tag{86}
$$

the forcing term can be constructed using contributions from nodes. That is,

$$
\bar{f} = \sum_{k=i}^{SF}N_k\bar{f}_k. \tag{87}
$$

4.3 Numerical integration/Gauss quadrature

A quadrature approximates a definite integral. It is usually a weighted sum of function values at specific points within the domain, written as

$$\int_{-1}^{1} f(x)dx \approx \sum_{i=1}^{n} w_i f(\xi_i). \tag{88}$$

provided the domain $\Omega^e = [-1, 1]$. In terms of a more general Ω^e, and in higher dimensions,

$$\int_{\Omega^e} f(x, y, z)d\Omega^e = \int_{-1}^{1}\int_{-1}^{1}\int_{-1}^{1} f(\xi, \eta, \zeta)[\det \mathbf{J}]d\xi d\eta d\zeta \approx \sum_{i=1}^{GP}\sum_{j=1}^{GP}\sum_{k=1}^{GP} w_i w_j w_k f(\xi_i, \eta_j, \zeta_k)[\det \mathbf{J}]. \tag{89}$$

An n- (GP-) point quadrature rule yields a result for polynomials f with degree $2n-1$ or less. Weights

$$w_i = \frac{2(1-\xi_i^2)}{\left[n\mathcal{P}_{n-1}(\xi_i)\right]^2}, \tag{90}$$

Where $\mathcal{P}_n(\xi)$ is the nth Legendre polynomial, given by

$$\mathcal{P}_0(\xi) = 1, \quad \mathcal{P}_1(\xi) = \xi,$$
$$\mathcal{P}_n(\xi) = \frac{2n-1}{n}\xi\mathcal{P}_{n-1}(\xi) - \frac{n-1}{n}\mathcal{P}_{n-2}(\xi). \tag{91}$$

Therefore,

$$\begin{aligned} \mathcal{P}_2(\xi) &= \frac{3}{2}\xi^2 - \frac{1}{2}, \\ \mathcal{P}_3(\xi) &= \frac{5}{3}\xi\left(\frac{3}{2}\xi^2 - \frac{1}{2}\right) - \frac{2}{3}\xi = \frac{5}{2}\xi^3 - \frac{3}{2}\xi, \\ &\dots . \end{aligned} \tag{92}$$

Setting $\mathcal{P}_n(\xi_i) = 0$ reveals solutions for ξ_i, followed by w_i (plugging ξ_i into Eq. 90 and separetely calculating $\mathcal{P}_{n-1}(\xi_i)$). The first few solutions are

$$0 = \mathcal{P}_1 = \xi \Longrightarrow \xi = 0 \Longrightarrow w_1 = 2;$$

$$0 = \mathcal{P}_2 = \frac{3}{2}\xi^2 - \frac{1}{2} \Longrightarrow \xi_1,\ \xi_2 = \pm\frac{1}{\sqrt{3}} \Longrightarrow w_1,\ w_2 = 1;$$

$$0 = \mathcal{P}_3 = \frac{5}{2}\xi^3 - \frac{3}{2}\xi \Longrightarrow \xi_1,\ \xi_2 = \sqrt{\frac{3}{5}},\ \xi_3 = 0 \Longrightarrow w_1,\ w_2 = \frac{5}{9},\ w_3 = \frac{8}{9}.$$

Fig. 5 provides some low order weights which can be applied to the quadrature rule.

Number of points, n	Points, \xi_i		Weights, w_i	
1	0		2	
2	$\pm\frac{1}{\sqrt{3}}$	±0.57735...	1	
3	0		$\frac{8}{9}$	0.888889...
	$\pm\sqrt{\frac{3}{5}}$	±0.774597...	$\frac{5}{9}$	0.555556...
4	$\pm\sqrt{\frac{3}{7}-\frac{2}{7}\sqrt{\frac{6}{5}}}$	±0.339981...	$\frac{18+\sqrt{30}}{36}$	0.652145...
	$\pm\sqrt{\frac{3}{7}+\frac{2}{7}\sqrt{\frac{6}{5}}}$	±0.861136...	$\frac{18-\sqrt{30}}{36}$	0.347855...
5	0		$\frac{128}{225}$	0.568889...
	$\pm\frac{1}{3}\sqrt{5-2\sqrt{\frac{10}{7}}}$	±0.538469...	$\frac{322+13\sqrt{70}}{900}$	0.478629...
	$\pm\frac{1}{3}\sqrt{5+2\sqrt{\frac{10}{7}}}$	±0.90618...	$\frac{322-13\sqrt{70}}{900}$	0.236927...

Figure 5: Low order weights w_i over interval $[-1, 1]$ given number of Gauss points n.

4.3.1 Example A

Suppose we wished to evaluate

$$I_A = \int_a^b f(x)dx = \int_3^7 \frac{1}{1.1+x}dx. \tag{93}$$

To solve such an equation, let

$$\xi = \frac{2x-b-a}{b-a} \Longrightarrow x = \frac{b\xi - a\xi + b + a}{2} = \frac{b-a}{2}\xi + \frac{b+a}{2}. \tag{94}$$

This causes

$$d\xi = \frac{2}{b-a}dx \Longrightarrow dx = \frac{b-a}{2}d\xi. \tag{95}$$

New bounds are

$$a' = \frac{2a-b-a}{b-a} = \frac{a-b}{b-a} = -\frac{b-a}{b-a} = -1, \quad b' = \frac{2b-b-a}{b-a} = \frac{b-a}{b-a} = 1. \tag{96}$$

For this problem in particular,

$$x = \frac{7\xi - 3\xi + 7 + 3}{2} = 2\xi + 5, \qquad dx = \frac{7-3}{2}d\xi = 2d\xi. \tag{97}$$

Note it is no coincidence that $2 = \partial x/\partial\xi = \det \mathbf{J} = J$ in one dimension. Substituting Eqs. 97 into Eq. 93,

$$I_A = \int_a^b f(x)dx = \int_{-1}^1 \underbrace{\frac{2}{1.1+2\xi+5}}_{f(\xi)} d\xi. \tag{98}$$

Using two Gauss points in the 1D governing equation Eq. 88 $\left(\int_{-1}^1 f(x)dx \approx \sum_{i=1}^{GP} w_i f(\xi_i)\right)$,

$$I_A \approx w_1 f(\xi_1) + w_2 f(\xi_2) = w_1 \frac{2}{1.1+2\xi_1+5} + w_2 \frac{2}{1.1+2\xi_2+5}. \tag{99}$$

According to Fig. 5,

$$\xi_i = \pm\frac{1}{\sqrt{3}} \Rightarrow \xi_1 = \frac{1}{\sqrt{3}},\ \xi_2 = -\frac{1}{\sqrt{3}}; \qquad w_i = 1 \Rightarrow w_1 = 1, \quad w_2 = 1. \tag{100}$$

Substituting,

$$I_A \approx \frac{2}{1.1+2/\sqrt{3}+5} + \frac{2}{1.1-2/\sqrt{3}+5} = 0.680107776642. \tag{101}$$

To compute the answer using MATLAB,

```
funInt = @(x) (1./(1.1+x));        result = integral(funInt, 3,7);        disp(result)
```

4.3.2 Example B

Suppose we wished to solve

$$I_B = \int_0^{\pi} \int_0^3 (x^2 - x) \sin y dx dy. \tag{102}$$

Let $[0, \pi] = [c, d]$ and $[0, 3] = [a, b]$. Now, recall the function mapping

$$\begin{aligned} \xi &= \frac{2x - b - a}{b - a} \Longrightarrow x = \frac{b\xi - a\xi + b + a}{2} = \frac{b-a}{2}\xi + \frac{b+a}{2}, \\ \eta &= \frac{2y - d - c}{d - c} \Longrightarrow y = \frac{d\xi - c\xi + d + c}{2} = \frac{d-c}{2}\xi + \frac{d+c}{2}, \end{aligned} \tag{103}$$

which is the double-integral analog of Eq. 94. In particular,

$$x = \frac{3}{2}\xi + \frac{3}{2}, \quad y = \frac{\pi}{2}\eta + \frac{\pi}{2}. \tag{104}$$

Then, of course, bounds

$$a' = \frac{2a - b - a}{b - a} = -1,\ b' = \frac{2b - b - a}{b - a} = 1,\ c' = \frac{2c - d - c}{d - c} = -1,\ d' = \frac{2d - d - c}{d - c} = 1. \tag{105}$$

Also,

$$\det \mathbf{J} = \det \begin{bmatrix} \partial x/\partial\xi & \partial x/\partial\eta \\ \partial y/\partial\xi & \partial y/\partial\eta \end{bmatrix} = \det \begin{bmatrix} 3/2 & 0 \\ 0 & \pi/2 \end{bmatrix} = \frac{3\pi}{4}. \tag{106}$$

Substituting according to the 2D form of Eq. 89, which is

$$I_B = \int_{\Omega^e} f(x, y, z) d\Omega^e = \int_{-1}^{1} \int_{-1}^{1} f(\xi, \eta)[\det \mathbf{J}] d\xi d\eta \approx \sum_{i=1}^{GP} \sum_{j=1}^{GP} w_i w_j f(\xi_i, \eta_j)[\det \mathbf{J}], \tag{107}$$

we receive

$$I_B = \int_{-1}^{1} \int_{-1}^{1} \underbrace{\left[\left(\frac{3}{2}\xi + \frac{3}{2}\right)^2 - \left(\frac{3}{2}\xi + \frac{3}{2}\right)\right]}_{x^2 - x} \sin \underbrace{\left(\frac{\pi}{2}\eta + \frac{\pi}{2}\right)}_{y} \underbrace{\left(\frac{3\pi}{4}\right)}_{\det \mathbf{J}} \underbrace{\left(\frac{3}{2}d\xi\right)}_{dx} \underbrace{\left(\frac{\pi}{2}d\eta\right)}_{dy}. \tag{108}$$

If $GP = 1 \Rightarrow \xi_i, \eta_i = 0,\ w_1 = 2,$

$$I_B \approx w_1 w_1 [(\frac{3}{2}\xi_1 + \frac{3}{2})^2 - (\frac{3}{2}\xi_1 + \frac{3}{2})] \sin(\frac{\pi}{2}\eta_1 + \frac{\pi}{2})(\frac{3\pi}{4}) \tag{109}$$

$$= 4(\frac{9}{4} - \frac{3}{2}) \sin(\frac{\pi}{2}) \frac{3\pi}{4} = \frac{9\pi}{16} = 1.76714586764. \tag{110}$$

If $GP = 2 \Rightarrow \xi_1,\ \eta_1 = 1/\sqrt{3};\ x_2,\ \eta_2 = -1/\sqrt{3};\ w_1 = 1;\ w_2 = 1,$

$$I_B \approx \frac{3\pi}{4}\Big(w_1 w_1 f(\xi_1,\eta_1) + w_1 w_2 f(\xi_1,\eta_2) + w_2 w_1 f(\xi_2,\eta_1) + w_2 w_2 f(\xi_2,\eta_2)\Big) = 8.7122. \quad (111)$$

If $GP = 3 \Rightarrow \xi_1,\ \eta_1 = 0;\ \xi_2,\ \eta_2 = \sqrt{3/5};\ \xi_3,\ \eta_3 = -\sqrt{3/5};\ w_1 = 8/9,\ w_2 = 5/9,\ w_3 = 5/9,$

$$\begin{aligned} I_B \approx \frac{3\pi}{4}\Big(& w_1 w_1 f(\xi_1,\eta_1) + w_1 w_2 f(\xi_1,\eta_2) + w_1 w_3 f(\xi_1,\eta_3) \\ & + w_2 w_1 f(\xi_2,\eta_1) + w_2 w_2 f(\xi_2,\eta_2) + w_2 w_3 f(\xi_2,\eta_3) \\ & + w_3 w_1 f(\xi_3,\eta_1) + w_3 w_2 f(\xi_3,\eta_2) + w_3 w_3 f(\xi_3,\eta_3)\Big) = 9.0063. \end{aligned} \quad (112)$$

To compute the answer using MATLAB,

```
funInt = @(x,y) (x.^2-x).*sin(y);     result = integral2(funInt, 0,3 0,pi);     disp(result)
```

5 3D HEX8

5.1 Strong form

Consider the 3D material body subject to the force distribution drawn in Fig. 6. The

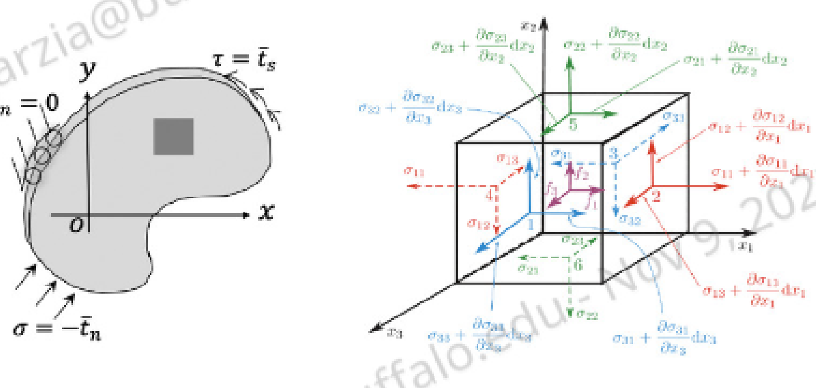

Figure 6: Solid mechanics infinitesimal cube.

equilibrium equation for the electrostatic continuum is the same as in CST and QUAD4 (Eq. 7, Eq. 23), but generalized to 3 dimensions. It is

$$\sigma_{ij,j} + F_i = 0. \tag{113}$$

With the addition of dynamics,

$$\sigma_{ij,j} + \rho^0 f_i = \rho^0 \ddot{u}_i. \tag{114}$$

ρ^0 is mass density. Like in QUAD4, this is still in units Nm^{-3}, because $\rho^0 \ddot{u}_i = m\ddot{u}_i/V^0$. Recall also from Sec. 3.2 and from QUAD4 that surface traction

$$t_i = \sigma_{ij} n_j \tag{115}$$

is the normal component of stress. In this case n is a vector normal to the surface of the body. A Neumann boundary condition in this context is some imposition on $\bar{t}_i$. On the other hand, a Dirichlet boundary condition is an imposition on displacement $u_i(0,0,0) = \bar{u}_0$.

5.2 Develop HEX8 element

5.2.1 Define element

Fig. 7 illustrates the mapping between the physical space x, y, z and intrinsic space ξ, η, ζ.

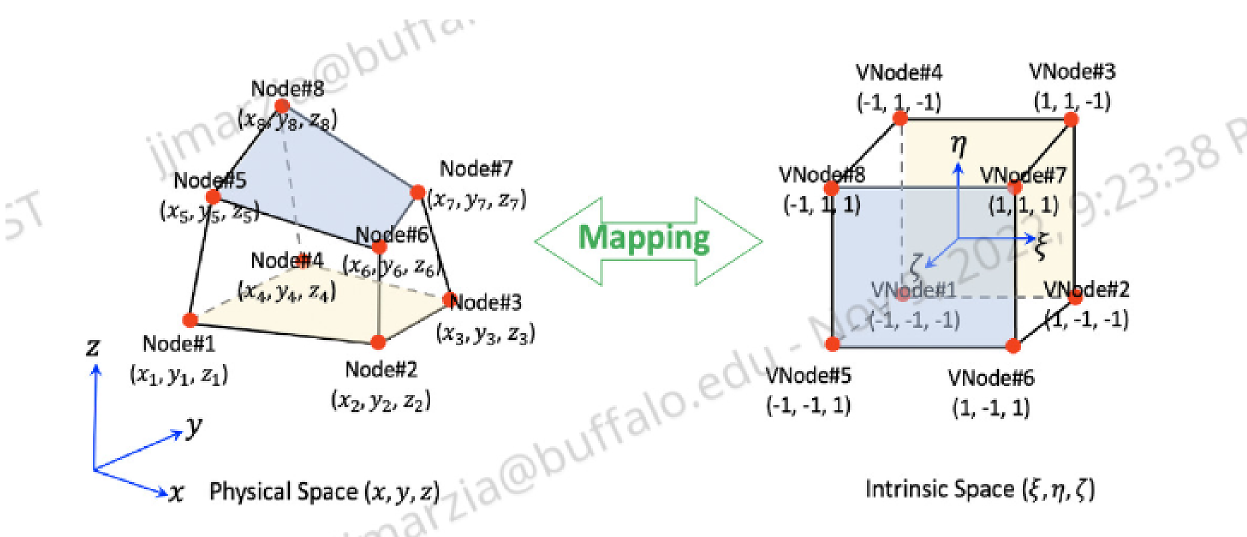

Figure 7: Mapping between physical and intrinsic space.

As in Fig. 7,

$$\begin{gathered}(\xi_1, \eta_1, \zeta_1) = (-1, -1, -1), \quad (\xi_2, \eta_2, \zeta_2) = (1, -1, -1), \\ (\xi_3, \eta_3, \zeta_3) = (1, 1, -1), \quad (\xi_4, \eta_4, \zeta_4) = (-1, 1, -1), \\ (\xi_5, \eta_5, \zeta_5) = (-1, -1, 1), \quad (\xi_6, \eta_6, \zeta_6) = (1, -1, 1), \\ (\xi_7, \eta_7, \zeta_7) = (1, 1, 1), \quad (\xi_8, \eta_8, \zeta_8) = (-1, 1, 1).\end{gathered} \tag{116}$$

5.2.2 Shape functions

Shape functions

$$\begin{gathered}N_1 = \frac{1}{8}(1-\xi)(1-\eta)(1-\zeta), \quad N_2 = \frac{1}{8}(1+\xi)(1-\eta)(1-\zeta), \\ N_3 = \frac{1}{8}(1+\xi)(1+\eta)(1-\zeta), \quad N_4 = \frac{1}{8}(1-\xi)(1+\eta)(1-\zeta), \\ N_5 = \frac{1}{8}(1-\xi)(1-\eta)(1+\zeta), \quad N_6 = \frac{1}{8}(1+\xi)(1-\eta)(1+\zeta), \\ N_7 = \frac{1}{8}(1+\xi)(1+\eta)(1+\zeta), \quad N_8 = \frac{1}{8}(1-\xi)(1+\eta)(1+\zeta).\end{gathered} \tag{117}$$

Notice

$$N_1(\xi_1, \eta_1, \zeta_1) = \frac{1}{8}(1+1)(1+1)(1+1) = 1, \tag{118}$$

but

$$N_1(\xi_4, \eta_4, \zeta_4) = \frac{1}{8}(1+1)(1-1)(1+1) = 0. \tag{119}$$

In general, like in QUAD4,

$$N_i(\xi_j, \eta_j, \zeta_j) \Leftrightarrow N_i(x_j) = \delta_{ij}. \tag{120}$$

To obtain three separate equations for displacement X, Y, Z, let us construct $\mathbf{N}$ as

$$\begin{bmatrix} N_1 & 0 & 0 & N_2 & 0 & 0 & \dots & N_8 & 0 & 0 \\ 0 & N_1 & 0 & 0 & N_2 & 0 & \dots & 0 & N_8 & 0 \\ 0 & 0 & N_1 & 0 & 0 & N_2 & \dots & 0 & 0 & N_8 \end{bmatrix} = [\mathbf{N}] \tag{121}$$

so that

$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} N_1 & 0 & 0 & N_2 & 0 & 0 & \dots & N_8 & 0 & 0 \\ 0 & N_1 & 0 & 0 & N_2 & 0 & \dots & 0 & N_8 & 0 \\ 0 & 0 & N_1 & 0 & 0 & N_2 & \dots & 0 & 0 & N_8 \end{bmatrix} \begin{bmatrix} X_1 \\ Y_1 \\ Z_1 \\ X_2 \\ Y_2 \\ Z_2 \\ \dots \\ X_8 \\ Y_8 \\ Z_8 \end{bmatrix} \tag{122}$$

Note that $\mathbf{N}$ is 3×24. Thus,

$$x_i = N_{i\alpha} X_\alpha, \quad \alpha = \{1, 2, \dots, 24\}, \quad i = \{1, 2, 3\}. \tag{123}$$

Also, by definition of the shape function, demonstrated in Eq. 86,

$$A = N_\alpha A_\alpha \tag{124}$$

for any variable A. This means that any variable can be approximated by its corresponding values at each node.

5.2.3 Strain/displacement relationship

Let $A = A(\xi, \eta, \zeta)$. Then,

$$\begin{aligned} A_{,x} &= \frac{\partial A}{\partial x} = \frac{\partial}{\partial x}(N_\alpha) A_\alpha = [\frac{\partial N_\alpha}{\partial \xi}\frac{\partial \xi}{\partial x} + \frac{\partial N_\alpha}{\partial \eta}\frac{\partial \eta}{\partial x} + \frac{\partial N_\alpha}{\partial \zeta}\frac{\partial \zeta}{\partial x}] A_\alpha = [N_{\alpha,\xi}\xi_{,x} + N_{\alpha,\eta}\eta_{,x} + N_{\alpha,\zeta}\zeta_{,x}] A_\alpha; \\ A_{,y} &= \frac{\partial A}{\partial y} = \frac{\partial}{\partial x}(N_\alpha) A_\alpha = [\frac{\partial N_\alpha}{\partial \xi}\frac{\partial \xi}{\partial y} + \frac{\partial N_\alpha}{\partial \eta}\frac{\partial \eta}{\partial y} + \frac{\partial N_\alpha}{\partial \zeta}\frac{\partial \zeta}{\partial y}] A_\alpha = [N_{\alpha,\xi}\xi_{,y} + N_{\alpha,\eta}\eta_{,y} + N_{\alpha,\zeta}\zeta_{,y}] A_\alpha; \\ A_{,z} &= \frac{\partial A}{\partial z} = \frac{\partial}{\partial x}(N_\alpha) A_\alpha = [\frac{\partial N_\alpha}{\partial \xi}\frac{\partial \xi}{\partial z} + \frac{\partial N_\alpha}{\partial \eta}\frac{\partial \eta}{\partial z} + \frac{\partial N_\alpha}{\partial \zeta}\frac{\partial \zeta}{\partial z}] A_\alpha = [N_{\alpha,\xi}\xi_{,z} + N_{\alpha,\eta}\eta_{,z} + N_{\alpha,\zeta}\zeta_{,z}] A_\alpha. \end{aligned} \tag{125}$$

Now, consider displacement

$$u_i = N_{i\alpha} U_\alpha, \tag{126}$$

where $U \Leftrightarrow \mathbf{u}^e$ is the displacement at the nodes. Then, displacement gradient

$$u_{i,j} = N_{i\alpha,j}U_\alpha = [N_{i\alpha,\xi}\xi_{,j} + N_{i\alpha,\eta}\eta_{,j} + N_{i\alpha,\zeta}\zeta_{,j}]U_\alpha. \tag{127}$$

By definition,

$$N_{i\alpha,j} := B_{ij\alpha} \Leftrightarrow \frac{\partial}{\partial x_j}\mathbf{N} = \mathbf{B}. \tag{128}$$

Therefore,

$$u_{i,j} = B_{ij\alpha}U_\alpha. \tag{129}$$

i, j are free indices while α is a dummy index, so it gets summed over. Then the strain is the symmetric component of the displacement gradient

$$\mathrm{sym}(u_{i,j}) = \frac{1}{2}(u_{i,j} + u_{j,i}) = \sum_\alpha \frac{1}{2}(B_{ij\alpha} + B_{ji\alpha}) = \epsilon_{ij}. \tag{130}$$

5.2.4 Jacobian

5.3 Galerkin weak form

From Eq. 114,

$$\sigma_{ij,j} + \rho^0 f_i = \rho^0 \ddot{u}_i \Longrightarrow \rho^0\ddot{u}_i - \sigma_{ij,j} - \rho^0 f_i = 0. \tag{131}$$

Invoking PVW,

$$\int_\Omega (\rho^0\ddot{u}_i - \sigma_{ij,j} - \rho^0 f_i)\delta u_i d\Omega = 0 \tag{132}$$

implies

$$\underbrace{\int_\Omega \rho^0\ddot{u}_i\delta u_i d\Omega}_{\mathbf{I}} - \underbrace{\int_\Omega \sigma_{ij,j}\delta u_i d\Omega}_{\mathbf{II}} - \underbrace{\int_\Omega \rho^0 f_i\delta u_i d\Omega}_{\mathbf{III}} = 0. \tag{133}$$

There are three terms. Let us simplify one at a time, starting with **I**. First of all, notice that

$$u_i = N_{i\alpha}U_\alpha \Rightarrow \delta u_i = N_{i\alpha}\delta U_\alpha. \tag{134}$$

Substituting into **I**,

$$\int_\Omega \rho^0\ddot{u}_i\delta u_i d\Omega = \int_\Omega \rho^0 N_{i\beta}\ddot{U}_\beta N_{i\alpha}\delta U_\alpha d\Omega = \delta U_\alpha \ddot{U}_\beta \int_\Omega \rho^0 N_{i\beta}N_{i\alpha}d\Omega := \delta U_\alpha(\ddot{U}_\beta M_{\alpha\beta}). \tag{135}$$

As for **II**,

$$\int_\Omega -\sigma_{ij,j}\delta u_i d\Omega = \int_\Omega [(-\sigma_{ij}\delta u_i)_{,j} - (-\sigma_{ij}\delta u_{i,j})]d\Omega \tag{136}$$

$$= \int_\Gamma -\sigma_{ij}\delta u_i n_j d\Gamma + \int_\Omega \sigma_{ij}\delta u_{i,j}d\Omega = \int_\Gamma -\bar{t}_i(\delta u_i)d\Gamma + \int_\Omega \sigma_{ij}(\delta\epsilon_{ij})d\Omega \tag{137}$$

$$= \int_\Gamma -\bar{t}_i(N_{i\alpha}\delta U_\alpha)d\Gamma + \int_\Omega \sigma_{ij}(B_{ij\alpha}\delta U_\alpha)d\Omega \tag{138}$$

$$= \delta U_\alpha \bigg(\int_\Gamma -\bar{t}_i N_{i\alpha} d\Gamma + \int_\Omega \sigma_{ij} B_{ij\alpha} d\Omega \bigg) := \delta U_\alpha (-F_\alpha^{\text{st}} + F_\alpha^{\text{int}}). \tag{139}$$

Lastly, for **III**,

$$\int_\Omega -\rho^0 f_i (N_{i\alpha}) \delta U_\alpha d\Omega = -\delta U_\alpha \int_\Omega \rho^0 f_i N_{i\alpha} d\Omega := \delta U_\alpha (-F_\alpha^{\text{bf}}). \tag{140}$$

Altogether,

$$\{\mathbf{I}\} - \{\mathbf{II}\} - \{\mathbf{III}\} = \delta U_\alpha \bigg(\ddot{U}_\beta M_{\alpha\beta} - F_\alpha^{\text{st}} + F_\alpha^{\text{int}} - F_\alpha^{\text{bf}} \bigg) = 0, \tag{141}$$

implies the Galerkin weak form

$$\ddot{U}_\beta M_{\alpha\beta} - F_\alpha^{\text{st}} + F_\alpha^{\text{int}} - F_\alpha^{\text{bf}} = 0, \tag{142}$$

where

$$M_{\alpha\beta} = \int_\Omega \rho^0 N_{i\beta} N_{i\alpha} d\Omega, \tag{143}$$

$$F_\alpha^{\text{st}} = \int_\Gamma \bar{t}_i N_{i\alpha} d\Gamma, \tag{144}$$

$$F_\alpha^{\text{int}} = \int_\Omega \sigma_{ij} B_{ij\alpha} d\Omega, \tag{145}$$

$$F_\alpha^{\text{bf}} = \int_\Omega \rho^0 f_i N_{i\alpha} d\Omega. \tag{146}$$

5.3.1 Linear elastic constitutive

For a linear elastic material, stress

$$\sigma_{ij} = C_{ijkl} \epsilon_{kl}. \tag{147}$$

Therefore,

$$F_\alpha^{\text{int}} = \int_\Omega \sigma_{ij} B_{ij\alpha} d\Omega = \int_\Omega C_{ijkl} (\epsilon_{kl}) B_{ij\alpha} \delta U_\alpha d\Omega \tag{148}$$

$$= \int_\Omega C_{ijkl} (B_{kl\beta} U_\beta) B_{ij\alpha} d\Omega = U_\beta \int_\Omega C_{ijkl} B_{kl\beta} B_{ij\alpha} d\Omega := U_\beta K_{\alpha\beta}. \tag{149}$$

Of course, this means

$$K_{\alpha\beta} = \int_\Omega C_{ijkl} B_{kl\beta} B_{ij\alpha} d\Omega. \tag{150}$$

The weak form Eq. 142 then becomes

$$0 = \ddot{U}_\beta M_{\alpha\beta} - F_\alpha^{\text{st}} + F_\alpha^{\text{int}} - F_\alpha^{\text{bf}} = \ddot{U}_\beta M_{\alpha\beta} - F_\alpha^{\text{st}} + U_\beta K_{\alpha\beta} - F_\alpha^{\text{bf}}, \tag{151}$$

which implies

$$\ddot{U}_\beta M_{\alpha\beta} + U_\beta K_{\alpha\beta} = F_\alpha^{\text{st}} + F_\alpha^{\text{bf}}. \tag{152}$$

5.3.2 Viscoelastic constitutive

For a viscoelastic material, stress is generalized as

$$\sigma_{ij} = C_{ijkl}\epsilon_{kl} + D_{ijkl}\dot{\epsilon}_{kl}. \tag{153}$$

D is Rayleigh damping. Strain rate $\dot{\epsilon}$ is a damping term, which is a good way to think about viscosity. This means

$$F_\alpha^{\text{int}} = \int_\Omega \sigma_{ij}B_{ij\alpha}d\Omega = \int_\Omega (C_{ijkl}\epsilon_{kl} + D_{ijkl}\dot{\epsilon}_{kl})B_{ij\alpha}d\Omega \tag{154}$$

$$= \int_\Omega C_{ijkl}\epsilon_{kl}B_{ij\alpha}d\Omega + \int_\Omega D_{ijkl}\dot{\epsilon}_{kl}B_{ij\alpha}B_{ij\alpha}d\Omega \tag{155}$$

$$U_\beta \int_\Omega C_{ijkl}B_{kl\beta}B_{ij\alpha}d\Omega + \dot{U}_\beta \int_\Omega D_{ijkl}B_{kl\beta}B_{ij\alpha}d\Omega := U_\beta K_{\alpha\beta} + \dot{U}_\beta C_{\alpha\beta}. \tag{156}$$

Clearly,

$$K_{\alpha\beta} = \int_\Omega C_{ijkl}B_{kl\beta}B_{ij\alpha}d\Omega, \tag{157}$$

$$C_{\alpha\beta} = \int_\Omega D_{ijkl}B_{kl\beta}B_{ij\alpha}d\Omega. \tag{158}$$

Then the weak form Eq. 142 becomes

$$0 = \ddot{U}_\beta M_{\alpha\beta} - F_\alpha^{\text{st}} + F_\alpha^{\text{int}} - F_\alpha^{\text{bf}} = \ddot{U}_\beta M_{\alpha\beta} - F_\alpha^{\text{st}} + U_\beta K_{\alpha\beta} + \dot{U}_\beta C_{\alpha\beta} - F_\alpha^{\text{bf}}, \tag{159}$$

which implies

$$\ddot{U}_\beta M_{\alpha\beta} + U_\beta K_{\alpha\beta} + \dot{U}_\beta C_{\alpha\beta} = F_\alpha^{\text{st}} + F_\alpha^{\text{bf}}. \tag{160}$$

5.3.3 Thermoviscoelastic constitutive

For a thermoviscoelastic material, stress

$$\sigma_{ij} = C_{ijkl}\epsilon_{kl} + D_{ijkl}\dot{\epsilon}_{kl} - \beta_{ij}T + b_{ijk}T_{,k}. \tag{161}$$

Added to this term are the set of thermal expansion coefficients β_{ij}, temperature T, temperature gradient $T_{,k}$, and tensor b_{ijk}. Notice by virtue of the shape function that

$$T = N_\beta T_\beta. \tag{162}$$

This means

$$F_\alpha^{\text{int}} = \int_\Omega \sigma_{ij}B_{ij\alpha}d\Omega = \int_\Omega (C_{ijkl}\epsilon_{kl} + D_{ijkl}\dot{\epsilon}_{kl} - \beta_{ij}T + b_{ijk}T_{,k})B_{ij\alpha}d\Omega \tag{163}$$

$$= U_\beta \int_\Omega C_{ijkl}B_{kl\beta}B_{ij\alpha}d\Omega + \dot{U}_\beta \int_\Omega D_{ijkl}B_{kl\beta}B_{ij\alpha}d\Omega$$

$$-T_\beta \int_\Omega \beta_{ij} N_\beta B_{ij\alpha} d\Omega + T_\beta \int_\Omega b_{ijk} N_{\beta,k} B_{ij\alpha} d\Omega := U_\beta K_{\alpha\beta} + \dot{U}_\beta C_{\alpha\beta} + T_\beta(-P_{\alpha\beta} + G_{\alpha\beta}). \quad (164)$$

Here,

$$K_{\alpha\beta} = \int_\Omega C_{ijkl} B_{kl\beta} B_{ij\alpha} d\Omega, \quad (165)$$

$$C_{\alpha\beta} = \int_\Omega D_{ijkl} B_{kl\beta} B_{ij\alpha} d\Omega, \quad (166)$$

$$P_{\alpha\beta} = \int_\Omega \beta_{ij} N_\beta B_{ij\alpha} d\Omega, \quad (167)$$

$$G_{\alpha\beta} = \int_\Omega b_{ijk} N_{\beta,k} B_{ij\alpha} d\Omega = \int_\Omega b_{ijk} B_{k\beta} B_{ij\alpha} d\Omega. \quad (168)$$

The term $N_{\beta,k}$ in Eq. 168 changes to $B_{k\beta}$ by virtue of Eq. 128. Then the weak form Eq. 142 becomes

$$0 = \ddot{U}_\beta M_{\alpha\beta} - F_\alpha^{\text{st}} + F_\alpha^{\text{int}} - F_\alpha^{\text{bf}} = \ddot{U}_\beta M_{\alpha\beta} - F_\alpha^{\text{st}} + U_\beta K_{\alpha\beta} + \dot{U}_\beta C_{\alpha\beta} + T_\beta(-P_{\alpha\beta} + G_{\alpha\beta}) - F_\alpha^{\text{bf}}, \quad (169)$$

which implies the updated weak form

$$\ddot{U}_\beta M_{\alpha\beta} + U_\beta K_{\alpha\beta} + \dot{U}_\beta C_{\alpha\beta} + T_\beta(-P_{\alpha\beta} + G_{\alpha\beta}) = F_\alpha^{\text{st}} + F_\alpha^{\text{bf}}. \quad (170)$$

Now, Eq. 170 is only one of two governing equations for this system. That is because temperature T must also obey the energy conservation law

$$\rho^0 \gamma \dot{T} + T^0 \beta_{ij} \dot{u}_{i,j} = -q_{k,k} + \rho^0 h \Longrightarrow \rho^0 \gamma \dot{T} + T^0 \beta_{ij} \dot{u}_{i,j} + q_{k,k} - \rho^0 h = 0, \quad (171)$$

where ρ^0 is mass density, T is temperature, T^0 is the reference temperature, γ is the heat conductivity, β_{ij} are the set of damping coefficients, q is the heat flux, and h is the heat source. We also define the deformation rate tensor as the symmetric component of the time derivative of the displacement gradient. It is also the strain rate. That is,

$$d_{ij} = \frac{1}{2}(\dot{u}_{ij} + \dot{u}_{j,i}) = \dot{\epsilon}_{ij}. \quad (172)$$

This causes

$$\beta_{ij} d_{ij} = \beta_{ij} \dot{u}_{i,j} = \beta_{ij} \dot{\epsilon}_{ij} = \beta_{ij} B_{ij\alpha} \dot{U}_\alpha. \quad (173)$$

Now, invoking PVW on Eq. 171,

$$0 = \int_\Omega \rho^0 \gamma \dot{T} \delta T d\Omega + \int_\Omega T^0 \beta_{ij} \dot{u}_{i,j} \delta T d\Omega + \int_\Omega q_{k,k} \delta T d\Omega - \int_\Omega \rho^0 h \delta T d\Omega \quad (174)$$

$$= \int_\Omega \rho^0 \gamma \dot{T} (\delta T_\alpha N_\alpha) d\Omega + \int_\Omega T^0 \beta_{ij} \dot{u}_{i,j} (\delta T_\alpha N_\alpha) d\Omega + \int_\Omega q_{k,k} \delta T d\Omega - \int_\Omega \rho^0 h (\delta T_\alpha N_\alpha) d\Omega \quad (175)$$

$$= \underbrace{\int_\Omega \rho^0 \gamma (\dot{T}_\beta N_\beta)(\delta T_\alpha N_\alpha) d\Omega}_{\mathbf{I}} + \underbrace{\int_\Omega T^0 (\beta_{ij} B_{ij\beta} \dot{U}_\beta)(\delta T_\alpha N_\alpha) d\Omega}_{\mathbf{II}}$$

$$+\underbrace{\int_\Omega q_{k,k}\delta T d\Omega}_{\textbf{III}} - \underbrace{\int_\Omega \rho^0 h(\delta T_\alpha N_\alpha) d\Omega}_{\textbf{IV}} \tag{176}$$

$$:= \underbrace{\delta T_\alpha \dot{T}_\beta \Gamma_{\alpha\beta}}_{\textbf{I}} + \underbrace{\delta T_\alpha \dot{U}_\beta T^0 P_{\beta\alpha}}_{\textbf{II}} + \underbrace{\int_\Omega q_{k,k}\delta T d\Omega}_{\textbf{III}} - \underbrace{\delta T_\alpha \bar{Q}^s_\alpha}_{\textbf{IV}}. \tag{177}$$

Notice **III** in Eq. 177 went unexamined. This is because we can reexpress heat flux

$$q_k = -H_{kl}T_{,l} - T^0 b_{ijl}\dot{\epsilon}_{ij}. \tag{178}$$

Now addressing **III** in Eq. 177,

$$\begin{aligned}
\int_\Omega q_{k,k}\delta T d\Omega &= \int_\Omega (q_k \delta T)_{,k} d\Omega - \int_\Omega q_k \delta T_{,k} d\Omega \\
&= \int_\Gamma q_k(\delta T) n_k d\Gamma - \int_\Omega (-H_{kl}T_{,l} - T^0 b_{ijl}\dot{\epsilon}_{ij})\delta T_{,k} d\Omega \\
&= \int_\Gamma \bar{q}(\delta T_\alpha N_\alpha) d\Gamma + \int_\Omega H_{kl}(T_{,l})(\delta T_{,k}) d\Omega + \int_\Omega T^0 b_{ijl}\dot{\epsilon}_{ij}(\delta T_{,k}) d\Omega \\
&= \delta T_\alpha \int_\Gamma \bar{q} N_\alpha d\Gamma + \int_\Omega H_{kl}(T_\beta N_{\beta,l})(\delta T_\alpha N_{\alpha,k}) d\Omega + \int_\Omega T^0 b_{ijl}\dot{\epsilon}_{ij}(\delta T_\alpha N_{\alpha,k}) d\Omega \\
&= \delta T_\alpha \int_\Gamma \bar{q} N_\alpha d\Gamma + \delta T_\alpha \int_\Omega H_{kl}(T_\beta N_{\beta,l}) N_{\alpha,k} d\Omega + \delta T_\alpha \int_\Omega T^0 b_{ijl}\dot{\epsilon}_{ij} N_{\alpha,k} d\Omega \qquad (179) \\
&:= \delta T_\alpha[\bar{Q}^f_\alpha + T_\beta \hat{H}_{\alpha\beta} + T^0 \dot{U}_\beta G_{\beta\alpha}] = \textbf{III}. \qquad (180)
\end{aligned}$$

Substituting Eq. 180 into Eq. 177,

$$0 = \underbrace{\delta T_\alpha \dot{T}_\beta \Gamma_{\alpha\beta}}_{\textbf{I}} + \underbrace{\delta T_\alpha \dot{U}_\beta T^0 P_{\beta\alpha}}_{\textbf{II}} + \underbrace{\delta T_\alpha[\bar{Q}^f_\alpha + T_\beta \hat{H}_{\alpha\beta} + T^0 \dot{U}_\beta G_{\beta\alpha}]}_{\textbf{III}} - \underbrace{\delta T_\alpha \bar{Q}^s_\alpha}_{\textbf{IV}}. \tag{181}$$

Dropping δT_α and rearranging, the second part of the weak form is

$$\dot{T}_\beta \Gamma_{\alpha\beta} + T^0 \dot{U}_\beta (P_{\beta\alpha} + G_{\beta\alpha}) + T_\beta \hat{H}_{\alpha\beta} = \bar{Q}^s_\alpha - \bar{Q}^f_\alpha, \tag{182}$$

where

$$\Gamma_{\alpha\beta} = \int_\Omega \rho^0 \gamma N_\beta N_\alpha d\Omega, \tag{183}$$

$$P_{\beta\alpha} = \int_\Omega \beta_{ij} B_{ij\beta} N_\alpha d\Omega, \tag{184}$$

$$\bar{Q}^s_\alpha = \int_\Omega \rho^0 h N_\alpha d\Omega, \tag{185}$$

$$\hat{H}_{\alpha\beta} = \int_\Omega H_{kl} N_{\beta,l} N_{\alpha,k} d\Omega, \tag{186}$$

$$G_{\beta\alpha} = \int_\Omega b_{ijl}\dot{\epsilon}_{ij} N_{\alpha,k} d\Omega = \int_\Omega b_{ijl} B_{ij\beta} N_{\alpha,k} d\Omega, \tag{187}$$

$$\bar{Q}^f_\alpha = \int_\Gamma \bar{q} N_\alpha d\Gamma. \tag{188}$$