# MAE 524 - Elasticity

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## Contents

# 1 Ch1 Mathematical preliminaries

## 1.1 Vectors

Vector

$$\mathbf{x} = x_1\mathbf{e}_1 + x_2\mathbf{e}_2 + x_3\mathbf{e}_3. \tag{1}$$

Basis vectors are unit vectors so that

$$|\mathbf{e}_1| = |\mathbf{e}_2| = |\mathbf{e}_3| = 1. \tag{2}$$

The basis vectors are mutually perpendicular so that

$$\mathbf{e}_1 \cdot \mathbf{e}_2 = \mathbf{e}_1 \cdot \mathbf{e}_3 = \mathbf{e}_2 \cdot \mathbf{e}_3 = 0. \tag{3}$$

In general

$$\mathbf{e}_i \cdot \mathbf{e}_j = \delta_{ij} \tag{4}$$

where Kronecker delta

$$\delta_{ij} = \begin{cases} 1, & i = j, \\ 0, & i \neq j. \end{cases} \tag{5}$$

Therefore by definition these three vectors form an orthonormal basis.

We can rewrite $\mathbf{x}$ as

$$\mathbf{x} = x_i\mathbf{e}_i \tag{6}$$

where summation convention acts like

$$\delta_{ii} = \begin{cases} \delta_{11} + \delta_{22}, & \text{2D}, \\ \delta_{11} + \delta_{22} + \delta_{33}, & \text{3D}. \end{cases} \tag{7}$$

The Kronecker delta is the index notation form of identity matrix $\mathbf{I}$ and so $\delta_{ii} = \mathrm{tr}\mathbf{I}$ which is defined as the sum of the diagonal entries.

The basis vectors form a right handed orthogonal triad and so

$$\mathbf{e}_1 \times \mathbf{e}_2 = \mathbf{e}_3, \quad \mathbf{e}_2 \times \mathbf{e}_3 = \mathbf{e}_1, \quad \mathbf{e}_3 \times \mathbf{e}_1 = \mathbf{e}_2 \tag{8}$$

but

$$\mathbf{e}_1 \times \mathbf{e}_3 = -\mathbf{e}_2, \quad \mathbf{e}_3 \times \mathbf{e}_2 = -\mathbf{e}_1, \quad \mathbf{e}_2 \times \mathbf{e}_1 = -\mathbf{e}_3. \tag{9}$$

Generalizing in index notation,

$$\mathbf{e}_i \times \mathbf{e}_j = \epsilon_{ijk}\mathbf{e}_k \tag{10}$$

where

$$\epsilon_{ijk} = \begin{cases} 1, & ijk \Leftrightarrow 1 \to 2 \to 3 \to \\ -1, & ijk \Leftrightarrow 3 \to 2 \to 1 \to \\ 0, & ijk \Leftrightarrow \text{incohesive loop}. \end{cases} \tag{11}$$

For example

$$\epsilon_{123} = \epsilon_{231} = \epsilon_{312} = 1, \tag{12}$$

$$\epsilon_{321} = \epsilon_{213} = \epsilon_{132} = -1, \tag{13}$$

$$\epsilon_{112} = \epsilon_{233} = \ldots = 0. \tag{14}$$

If there are two vectors

$$\mathbf{a} = a_i \mathbf{e}_i, \quad \mathbf{b} = b_i \mathbf{e}_i, \tag{15}$$

then their dot product is

$$\mathbf{a} \cdot \mathbf{b} = a_i \mathbf{e}_i \cdot b_j \mathbf{e}_j = a_i b_j \mathbf{e}_i \cdot \mathbf{e}_j = a_i b_j \delta_{ij} = a_i b_j \delta_{ji} = a_i b_i = a_1 b_1 + a_2 b_2 + a_3 b_3. \tag{16}$$

The difference between two vectors

$$\mathbf{c} = \mathbf{a} - \mathbf{b} \tag{17}$$

has a magnitude which can be solved for using

$$(a - b\cos\theta)^2 + (b \sin\theta)^2 = c^2 \Longrightarrow a^2 - 2ab\cos\theta + b^2\cos^2\theta + b^2\sin^2\theta = c^2$$

$$\Longleftrightarrow a^2 + b^2 - 2ab\cos\theta = c^2 \tag{18}$$

where $\theta$ is the angle that separates $\mathbf{a}$ and $\mathbf{b}$. The length or magnitude of any $\mathbf{a}$ is

$$a = |\mathbf{a}| = \sqrt{\mathbf{a} \cdot \mathbf{a}} = \sqrt{a_i a_i} = \sqrt{a_1^2 + a_2^2 + a_3^2}. \tag{19}$$

Substituting into Eq. 18,

$$a_1^2 + a_2^2 + a_3^2 + b_1^2 + b_2^2 + b_3^2 - 2ab\cos\theta = c_1^2 + c_2^2 + c_3^2 \tag{20}$$

$$\Longleftrightarrow a_1^2 + a_2^2 + a_3^2 + b_1^2 + b_2^2 + b_3^2 - 2ab\cos\theta = (a_1 - b_1)^2 + (a_2 - b_2)^2 + (a_3 - b_3)^2 \tag{21}$$

$$\Longleftrightarrow a_1^2 + a_2^2 + a_3^2 + b_1^2 + b_2^2 + b_3^2 - 2ab\cos\theta = a_1^2 - 2a_1b_1 + b_1^2 + a_2^2 - 2a_2b_2 + b_2^2 + a_3^2 - 2a_3b_3 + b_3^2 \tag{22}$$

$$\Longleftrightarrow -2ab\cos\theta = -2a_1b_1 - 2a_2b_2 - 2a_3b_3 \tag{23}$$

$$\Longleftrightarrow ab\cos\theta = a_1b_1 + a_2b_2 + a_3b_3 \Leftrightarrow \mathbf{a} \cdot \mathbf{b}. \tag{24}$$

Therefore, dot product

$$\mathbf{a} \cdot \mathbf{b} = ab\cos\theta. \tag{25}$$

Cross product

$$\mathbf{a} \times \mathbf{b} = a_i \mathbf{e}_i \times \mathbf{b}_j \mathbf{e}_j = a_i b_j \mathbf{e}_i \times \mathbf{e}_j = a_i b_j \epsilon_{ijk} \mathbf{e}_k. \tag{26}$$

Expanding,

$$\mathbf{a} \times \mathbf{b} = \cancel{a_1 b_1 \epsilon_{111} \mathbf{e}_1} + \cancel{a_1 b_1 \epsilon_{112} \mathbf{e}_2} + \cancel{a_1 b_1 \epsilon_{113} \mathbf{e}_3} \tag{27}$$

$$+\cancel{a_1 b_2 \epsilon_{121} \mathbf{e}_1} + \cancel{a_1 b_2 \epsilon_{122} \mathbf{e}_2} + a_1 b_2 \epsilon_{123} \mathbf{e}_3 \tag{28}$$

$$+\cancel{a_1 b_3 \epsilon_{131} \mathbf{e}_1} + a_1 b_3 \epsilon_{132} \mathbf{e}_2 + \cancel{a_1 b_3 \epsilon_{133} \mathbf{e}_3} \tag{29}$$

$$+\cancel{a_2 b_1 \epsilon_{211} \mathbf{e}_1} + \cancel{a_2 b_1 \epsilon_{212} \mathbf{e}_2} + a_2 b_1 \epsilon_{213} \mathbf{e}_3 \tag{30}$$

$$+\cancel{a_2 b_2 \epsilon_{221} \mathbf{e}_1} + \cancel{a_2 b_2 \epsilon_{222} \mathbf{e}_2} + \cancel{a_2 b_2 \epsilon_{223} \mathbf{e}_3} \tag{31}$$

$$+a_2b_3\epsilon_{231}\mathbf{e}_1 + \cancel{a_2b_3\epsilon_{232}}\mathbf{e}_2 + \cancel{a_2b_3\epsilon_{233}}\mathbf{e}_3 \tag{32}$$

$$+\cancel{a_3b_1\epsilon_{311}}\mathbf{e}_1 + a_3b_1\epsilon_{312}\mathbf{e}_2 + \cancel{a_3b_1\epsilon_{313}}\mathbf{e}_3 \tag{33}$$

$$+a_3b_2\epsilon_{321}\mathbf{e}_1 + \cancel{a_3b_2\epsilon_{322}}\mathbf{e}_2 + \cancel{a_3b_2\epsilon_{323}}\mathbf{e}_3 \tag{34}$$

$$+\cancel{a_3b_3\epsilon_{331}}\mathbf{e}_1 + \cancel{a_3b_3\epsilon_{332}}\mathbf{e}_2 + \cancel{a_3b_3\epsilon_{333}}\mathbf{e}_3 \tag{35}$$

$$= \mathbf{e}_1(a_2b_3 - a_3b_2) + \mathbf{e}_2(a_3b_1 - a_1b_3) + \mathbf{e}_3(a_1b_2 - a_2b_1) \Leftrightarrow \det \begin{bmatrix} \mathbf{e}_1 & \mathbf{e}_2 & \mathbf{e}_3 \\ a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \end{bmatrix}. \tag{36}$$

It is also provable that

$$\mathbf{a} \times \mathbf{b} = ab\sin\theta\hat{\mathbf{n}} \Leftrightarrow |\mathbf{a}||\mathbf{b}|\sin\theta\mathbf{e}_n \tag{37}$$

where $\mathbf{e}_n$ points in the direction normal to the plane formed by $\mathbf{a}$ and $\mathbf{b}$ and can be identified using the right hand rule. Volume of three new vectors $\mathbf{a}$, $\mathbf{b}$, $\mathbf{c}$

$$\mathbf{a} \cdot (\mathbf{b} \times \mathbf{c}) = (\mathbf{a}) \cdot (|\mathbf{b}||\mathbf{c}|\sin\theta\mathbf{e}_n) = |\mathbf{a}||\mathbf{b}||\mathbf{c}|\sin\theta\cos\alpha \tag{38}$$

where $\theta$ is the angle between $\mathbf{b}$ and $\mathbf{c}$ and $\alpha$ is the angle between $\mathbf{a}$ and vector $\mathbf{b} \times \mathbf{c}$. In index notation

$$\mathbf{a} \cdot (\mathbf{b} \times \mathbf{c}) = \mathbf{a} \cdot (b_ic_j\epsilon_{ijk}\mathbf{e}_k) = a_m\mathbf{e}_m \cdot b_ic_j\epsilon_{ijk}\mathbf{e}_k = a_mb_ic_j\epsilon_{ijk}\mathbf{e}_m \cdot \mathbf{e}_k \tag{39}$$

$$\Leftrightarrow a_mb_ic_j\epsilon_{ijk}\delta_{mk} = a_m\delta_{mk}b_ic_j\epsilon_{ijk} = a_kb_ic_j\epsilon_{ijk}. \tag{40}$$

Note indices are arbitrary in that

$$a_kb_ic_j\epsilon_{ijk} \Leftrightarrow a_ib_jc_k\underbrace{\epsilon_{jki}}_{\textbf{I.}} = a_ib_jc_k\underbrace{\epsilon_{ijk}}_{\textbf{I.}} = \mathbf{a} \cdot (\mathbf{b} \times \mathbf{c}). \tag{41}$$

Because of the arbitrariness of the indices it can be shown that

$$\mathbf{a} \cdot (\mathbf{b} \times \mathbf{c}) = \mathbf{b} \cdot (\mathbf{c} \times \mathbf{a}) = \mathbf{c} \cdot (\mathbf{a} \times \mathbf{b}) \tag{42}$$

so long as the permutation between a,b,c remains intact so that $\epsilon$ does not change sign (as evidenced by $\mathbf{a} \times \mathbf{b} = -\mathbf{b} \times \mathbf{a}$). Note that a vector is direction times magnitude. So, a vector divided by its magnitude is just its direction ($\mathbf{a}/|\mathbf{a}|$). With that said the definition of the projection $\mathbf{p}$ of $\mathbf{b}$ onto $\mathbf{a}$ is

$$\mathbf{p} = |\mathbf{b}|\cos\theta\frac{\mathbf{a}}{|\mathbf{a}|} \tag{43}$$

where $\theta$ is the angle between $\mathbf{a}$ and $\mathbf{b}$. In other words this is the horizontal component of $\mathbf{b}$ in the direction of $\mathbf{a}$. Note

$$|\mathbf{b}|\cos\theta\frac{\mathbf{a}}{|\mathbf{a}|} = \frac{|\mathbf{a}||\mathbf{b}|\cos\theta}{|\mathbf{a}|}\frac{\mathbf{a}}{|\mathbf{a}|} = \frac{\mathbf{a} \cdot \mathbf{b}}{|\mathbf{a}|}\frac{\mathbf{a}}{|\mathbf{a}|} \tag{44}$$

because of the definition of dot product.

## 1.2 Change of basis

A set of basis vectors is an arbitrary way to judge the location of a point. Sometimes it might be mathematically more simple to change the set of basis vectors as we desire, which because of its arbitrariness we are totally allowed to do. Consider vector

$$\mathbf{v} = v_1\mathbf{e}_1 + v_2\mathbf{e}_2 + v_3\mathbf{e}_3 = v_i\mathbf{e}_i \tag{45}$$

where $v_i$ are the components of $\mathbf{v}$ and $\mathbf{e}_i$ are the basis vectors. Let us define a new orthonormal basis

$$\mathbf{v} = \sum_{i=1}^{3} v_i\mathbf{e}_i = \sum_{i=1}^{3} \bar{v}_i\bar{\mathbf{e}}_i \tag{46}$$

where the barred quantities are also components of $\mathbf{v}$ and basis vectors. Then

$$v_i\mathbf{e}_i = \bar{v}_i\bar{\mathbf{e}}_i \Longleftrightarrow v_i \underbrace{\mathbf{e}_i \cdot \mathbf{e}_j}_{\textbf{I.}} = \bar{v}_i \underbrace{\bar{\mathbf{e}}_i \cdot \mathbf{e}_j}_{\textbf{II.}} \Longleftrightarrow \overbrace{v_i \underbrace{\delta_{ij}}_{\textbf{I.}}}^{\textbf{III.}} = \bar{v}_i \underbrace{R_{ij}}_{\textbf{II.}} \Longleftrightarrow \overbrace{v_j}^{\textbf{III.}} = \bar{v}_i R_{ij}. \tag{47}$$

where $R_{ij} = \bar{\mathbf{e}}_i \cdot \mathbf{e}_j$ is the dot product between the old and new coordinate systems. Note that this will not necessarily be identity $\mathbf{I}$ because it is not necessarily true that $\bar{\mathbf{e}}_1 \cdot \mathbf{e}_1 = \cos 0 = 1$, etc. Further,

$$v_j = \bar{v}_i R_{ij} = R^T_{ji}\bar{v}_i \Longleftrightarrow \mathbf{v} = \mathbf{R}^T\bar{\mathbf{v}}. \tag{48}$$

Instead of how we started with Eq. 47 which was to multiply both sides by $\mathbf{e}_j$, we could have also multiplied by $\bar{\mathbf{e}}_j$. What follows is

$$v_i\mathbf{e}_i = \bar{v}_i\bar{\mathbf{e}}_i \Longleftrightarrow v_i \underbrace{\mathbf{e}_i \cdot \bar{\mathbf{e}}_j}_{\textbf{I.}} = \bar{v}_i \underbrace{\bar{\mathbf{e}}_i \cdot \bar{\mathbf{e}}_j}_{\textbf{II.}} \Longleftrightarrow v_i \underbrace{R_{ji}}_{\textbf{I.}} = \overbrace{\bar{v}_i \underbrace{\delta_{ij}}_{\textbf{II.}}}^{\textbf{III.}} \Longleftrightarrow R_{ji}v_i = \overbrace{\bar{v}_j}^{\textbf{III}}. \tag{49}$$

$$\Longleftrightarrow \bar{\mathbf{v}} = \mathbf{R}\mathbf{v} \Longleftrightarrow \mathbf{v} = \mathbf{R}^{-1}\bar{\mathbf{v}}. \tag{50}$$

Combining Eq. 48 and Eq. 50,

$$\mathbf{R}^{-1} = \mathbf{R}^T \Leftrightarrow \mathbf{R}\mathbf{R}^T = \mathbf{R}\mathbf{R}^{-1} = \mathbf{I} \Longleftrightarrow R_{ik}R_{jk} = \delta_{ij} \tag{51}$$

where the indexing $R_{ik}R_{jk}$ defies the conventional rule of matrix multiplication that the dummy index $k$ neighbors itself (as in $R_{ik}R_{kj}$) because of the transpose operation on $\mathbf{R}^T$.

A matrix $\mathbf{R}$ that satisfies $\mathbf{R}^T = \mathbf{R}^{-1}$ is said to be orthogonal. Rotation matrices are always orthogonal. Consider

$$\mathbf{R} = \begin{bmatrix} \cos\theta & \sin\theta & 0 \\ -\sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{bmatrix}. \tag{52}$$

This rotation matrix, when applied to a vector such that

$$\begin{Bmatrix} \bar{v}_1 \\ \bar{v}_2 \\ \bar{v}_3 \end{Bmatrix} = \begin{bmatrix} \cos\theta & \sin\theta & 0 \\ -\sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{Bmatrix} v_1 \\ v_2 \\ v_3 \end{Bmatrix}, \tag{53}$$

is describing the component by component transformation

$$\bar{v}_1 = v_1 \cos\theta + v_2 \sin\theta, \quad \bar{v}_2 = -v_2 \sin\theta + v_2 \cos\theta, \quad \bar{v}_3 = v_3 \tag{54}$$

where $\theta$ is the angle of rotation. This particular transformation is describing a $\theta$ degrees counterclockwise rotation between orthonormal bases in the dimensions $\mathbf{e}_1$ and $\mathbf{e}_2$.

The rules of determinants for matrices are

$$\det \mathbf{S}^T = \det \mathbf{S}, \quad \det \mathbf{ST} = \det \mathbf{S} \det \mathbf{T}, \quad \det \mathbf{S}^{-1} = (\det \mathbf{S})^{-1}. \tag{55}$$

Accepting this,

$$1 = \det \mathbf{I} = \det \mathbf{RR}^T = \det \mathbf{R} \det \mathbf{R}^T = \det \mathbf{R} \det \mathbf{R} = (\det \mathbf{R})^2. \tag{56}$$

Therefore,

$$\det \mathbf{R} = \pm 1 \tag{57}$$

if $\mathbf{R}$ is orthogonal. The signage determines the functionality of $\mathbf{R}$. Particularly

$$\det \mathbf{R} = \begin{cases} 1, & \text{rotation, e.g.} \begin{bmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix} \text{(90 degree clockwise rotation in } xy\text{)} \\ -1, & \text{rotation and reflection, e.g.} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{bmatrix} \text{(reflection in } z\text{).} \end{cases} \tag{58}$$

Note that under an orthogonal coordinate transformation the magnitude of the vector $\mathbf{v} \longrightarrow \bar{\mathbf{v}}$ does not change. Its square length

$$\bar{v}_i \bar{v}_i = R_{ij} v_j R_{ik} v_k = \delta_{jk} v_j v_k = v_j v_j. \tag{59}$$

A vector undergoing a transformation by $\mathbf{R}$ is considered a tensor with rank 1. A tensor of rank 0 is a scalar, and a tensor of rank 2 is an $m \times n$ matrix.

If the relationships

$$\bar{\mathbf{v}} = \mathbf{Rv} \rightarrow \mathbf{R}^T \bar{\mathbf{v}} = \mathbf{v}, \quad \bar{\mathbf{u}} = \mathbf{Ru} \rightarrow \mathbf{R}^T \bar{\mathbf{u}} = \mathbf{u}, \quad \mathbf{v} = \mathbf{Mu} \Longleftrightarrow \bar{\mathbf{v}} = \overbrace{\bar{\mathbf{M}}}^{\mathbf{I.}} \bar{\mathbf{u}} \tag{60}$$

hold, then

$$(\mathbf{v}) = \mathbf{M}(\mathbf{u}) \longrightarrow (\mathbf{R}^T \bar{\mathbf{v}}) = \mathbf{M}(\mathbf{R}^T \bar{\mathbf{u}}) \longrightarrow \mathbf{RR}^T \bar{\mathbf{v}} = \mathbf{RMR}^T \bar{\mathbf{u}} \longrightarrow \bar{\mathbf{v}} = \overbrace{\mathbf{RMR}^T}^{\mathbf{I.}} \bar{\mathbf{u}} \tag{61}$$

implies

$$\underbrace{\mathbf{RMR}^T = \bar{\mathbf{M}}}_{\mathbf{I.}} \Longleftrightarrow R_{ik} M_{kl} R_{jl} = \bar{M}_{ij}. \tag{62}$$

In general this is how to transform a second order tensor. In general for a tensor of any rank,

$$\bar{A}_{ij...k} = R_{ip}R_{jq}...R_{kr}A_{pq...r}. \tag{63}$$

The trace of a matrix

$$\text{tr}\bar{\mathbf{M}} \Longleftrightarrow \bar{M}_{ii} = R_{ij}M_{jk}R_{ik} = \underbrace{R_{ij}R_{ik}}_{\mathbf{I}.} M_{jk} = \underbrace{\delta_{jk}}_{\mathbf{I}.} M_{jk} = M_{kk} \Longleftrightarrow \text{tr}\mathbf{M}. \tag{64}$$

So the trace of a matrix under orthogonal transformation is invariant.

An isotropic tensor is one that does not change because of a coordinate transformation. For example the Kronecker delta is isotropic in that

$$\bar{\delta}_{ij} = R_{ik}R_{jl}\delta_{kl} = R_{il}R_{jl} = \delta_{ij}. \tag{65}$$

In matrix notation this is more obvious as

$$\bar{\mathbf{I}} = \mathbf{R}\mathbf{I}\mathbf{R}^T = \mathbf{R}\mathbf{R}^T = \mathbf{I}. \tag{66}$$

## 1.3 Symmetry and skew symmetry

A matrix $\mathbf{S}$ is symmetric if

$$\mathbf{S} = \mathbf{S}^T \longleftrightarrow S_{ij} = S_{ji} \longrightarrow [\mathbf{S}] = \begin{bmatrix} S_{11} & S_{12} & S_{13} \\ S_{12} & S_{22} & S_{23} \\ S_{13} & S_{23} & S_{33} \end{bmatrix}. \tag{67}$$

A matrix $\mathbf{A}$ is skew symmetric if

$$\mathbf{A} = -\mathbf{A}^T \longleftrightarrow A_{ij} = -A_{ji} \longrightarrow [\mathbf{A}] = \begin{bmatrix} 0 & A_{12} & A_{13} \\ -A_{12} & 0 & A_{23} \\ -A_{13} & -A_{23} & 0 \end{bmatrix}. \tag{68}$$

Any matrix $\mathbf{M}$ has symmetric and skewsymmetric components

$$\mathbf{M} = \mathbf{S} + \mathbf{A} \quad \text{where} \quad \mathbf{S} = \frac{\mathbf{M} + \mathbf{M}^T}{2} = \mathbf{S}^T, \quad \mathbf{A} = \frac{\mathbf{M} - \mathbf{M}^T}{2} = -\mathbf{A}^T. \tag{69}$$

For example

$$\mathbf{M} = \begin{bmatrix} 3 & 5 & 7 \\ 1 & 2 & 8 \\ 9 & 6 & 4 \end{bmatrix} = \begin{bmatrix} 3 & 3 & 8 \\ 3 & 2 & 7 \\ 8 & 7 & 4 \end{bmatrix} + \begin{bmatrix} 0 & 2 & -1 \\ -2 & 0 & 1 \\ 1 & -1 & 0 \end{bmatrix} = \mathbf{S} + \mathbf{A}. \tag{70}$$

Note that for skew symmetric $\mathbf{A}$,

$$\mathbf{x} \cdot \mathbf{A}\mathbf{x} = \mathbf{x}^T\mathbf{A}\mathbf{x} = \mathbf{x}^T\mathbf{A}^T\mathbf{x}^{T^T} = \mathbf{x}^T\mathbf{A}^T\mathbf{x} = -\mathbf{x}^T\mathbf{A}\mathbf{x} \tag{71}$$

which implies

$$\mathbf{x}^T\mathbf{A}\mathbf{x} = -\mathbf{x}^T\mathbf{A}\mathbf{x} \longrightarrow \mathbf{x} \cdot \mathbf{A}\mathbf{x} = 0 \forall \mathbf{x}. \tag{72}$$

Consider then a matrix $\mathbf{M}$ subjected to the matrix product

$$\mathbf{x} \cdot \mathbf{M}\mathbf{x} = \mathbf{x} \cdot (\mathbf{S} + \mathbf{A})\mathbf{x} = \mathbf{x} \cdot \mathbf{S}\mathbf{x} + \mathbf{x} \cdot \mathbf{A}\mathbf{x} = \mathbf{x} \cdot \mathbf{S}\mathbf{x}. \tag{73}$$

This is called the quadratic form of $\mathbf{M}$. Expanded, the quadratic form is

$$\mathbf{x}^T\mathbf{M}\mathbf{x} \Longleftrightarrow \mathbf{x} \cdot \mathbf{M}\mathbf{x} \Longleftrightarrow x_i M_{ij} x_j \tag{74}$$

$$= x_1(M_{11}x_1 + M_{12}x_2 + M_{13}x_3) + x_2(M_{21}x_1 + M_{22}x_2 + M_{23}x_3) + x_3(M_{31}x_1 + M_{32}x_2 + M_{33}x_3)$$

$$= x_1^2 M_{11} + x_2^2 M_{22} + x_3^2 M_{33} + x_1x_2(M_{12} + M_{21}) + x_1x_3(M_{13} + M_{31}) + x_2x_3(M_{23} + M_{32}). \tag{75}$$

$\mathbf{M}$ is positive definite if its quadratic form $\mathbf{x} \cdot \mathbf{M}\mathbf{x} > 0 \forall \mathbf{x}$ and positive semidefinite if $\mathbf{x} \cdot \mathbf{M}\mathbf{x} \geq 0 \forall \mathbf{x}$.

## 1.4 Derivatives and divergence

Consider the scalar function $\phi(x_j)$. The chain rule states

$$\frac{\partial \phi}{\partial \bar{x}_i} = \frac{\partial \phi}{\partial x_j} \frac{\partial x_j}{\partial \bar{x}_i}. \tag{76}$$

If $x_j = R_{kj}\bar{x}_k$, then

$$\frac{\partial x_j}{\partial \bar{x}_i} = \frac{\partial}{\partial \bar{x}_i}(R_{kj}\bar{x}_k) = R_{kj}\frac{\partial \bar{x}_k}{\partial \bar{x}_i} = R_{kj}\delta_{ki} = \delta_{ik}R_{kj} = R_{ij}. \tag{77}$$

Substituting this into Eq. 76,

$$\frac{\partial \phi}{\partial \bar{x}_i} = R_{ij}\frac{\partial \phi}{\partial x_j}. \tag{78}$$

The result of this is a tensor of rank 1. The del or nabla operator

$$\nabla = \mathbf{e}_1\frac{\partial}{\partial x_1} + \mathbf{e}_2\frac{\partial}{\partial x_2} + \mathbf{e}_3\frac{\partial}{\partial x_3} = \mathbf{e}_i\frac{\partial}{\partial x_i}. \tag{79}$$

Gradient

$$\nabla \phi = \frac{\partial \phi}{\partial x_1}\mathbf{e}_1 + \frac{\partial \phi}{\partial x_2}\mathbf{e}_2 + \frac{\partial \phi}{\partial x_3}\mathbf{e}_3 = \frac{\partial \phi}{\partial x_i}\mathbf{e}_i = \phi_{,i}\mathbf{e}_i. \tag{80}$$

This turns the scalar $\phi$ into a vector. Gradient increases rank.

A directional derivative is the amount that a function's gradient aligns with direction $\mathbf{e}_s$. It is a scalar. It is

$$\frac{\partial \phi}{\partial s} = \mathbf{e}_s \cdot \nabla \phi \Leftrightarrow |\mathbf{e}_s||\nabla \phi| \cos\theta = |\nabla \phi| \cos\theta \tag{81}$$

where $\theta$ is the angle between vector $\mathbf{e}_s$ and vector $\nabla \phi$.

Now consider vector function $\mathbf{f}(x_j) = f_j\mathbf{e}_j$. Divergence of $\mathbf{f}$

$$\nabla \cdot \mathbf{f} = \left(\mathbf{e}_i\frac{\partial}{\partial x_i}\right) \cdot \left(f_j\mathbf{e}_j\right) = (\mathbf{e}_i \cdot \mathbf{e}_j)\frac{\partial f_j}{\partial x_i} = \delta_{ij}\frac{\partial f_j}{\partial x_i} = \frac{\partial f_i}{\partial x_i} = f_{i,i}. \tag{82}$$

This operation turns $\mathbf{f}$ from a vector into a scalar. Divergence decreases rank.

The Laplacian maintains rank. The Laplacian is the divergence of the gradient. Scalar $\phi$ has Laplacian

$$\nabla^2\phi := \nabla \cdot \nabla\phi = \nabla \cdot (\frac{\partial \phi}{\partial x_i}\mathbf{e}_i) = (\frac{\partial}{\partial x_j}\mathbf{e}_j) \cdot (\frac{\partial \phi}{\partial x_i}\mathbf{e}_i) = (\mathbf{e}_j \cdot \mathbf{e}_i)\frac{\partial^2 \phi}{\partial x_j \partial x_i} \tag{83}$$

$$= \delta_{ji}\frac{\partial^2 \phi}{\partial x_j \partial x_i} = \frac{\partial^2 \phi}{\partial^2 x_i} = \frac{\partial^2 \phi}{\partial x_1^2} + \frac{\partial^2 \phi}{\partial x_2^2} + \frac{\partial^2 \phi}{\partial x_3^2} = \phi_{,ii}. \tag{84}$$

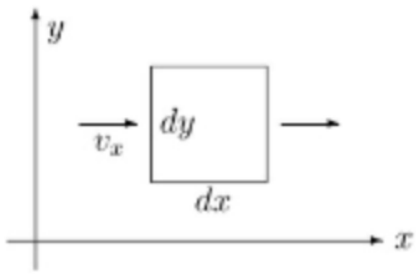
## 1.5 Divergence theorem

Consider a fluid with density $\rho = \rho(x, y, z)$ and velocity

$$\mathbf{v} = v_x\mathbf{e}_x + v_y\mathbf{e}_y + v_z\mathbf{e}_z \tag{85}$$

where $v_x = v_x(x, y, z)$, $v_y = v_y(x, y, z)$, $v_z = v_z(x, y, z)$. The meaning of this is that $v_i$ and $\rho$ can change in magnitude based on the specific point $(x, y, z)$, but this has no bearing towards the directionality of the components of the velocity vector $(\circ)\mathbf{e}_i$ which always point in the $x_i$ direction.

Imagine this fluid is flowing through a small cube.




On the left side, the fluid enters at a rate

$$\text{rate in} = (\rho)(v_x)(dydz) = \frac{\text{mass}}{\cancel{xyz \text{ vol}}}\frac{\cancel{x \text{ dist}}}{\text{time}}\cancel{yz \text{ area}} = \frac{\text{mass}}{\text{time}}. \tag{86}$$

On the right side, the fluid exits at a rate

$$\text{rate out} = (\rho v_x + \frac{\partial \rho v_x}{\partial x}dx)dydz \tag{87}$$

$$\rho v_x dydz + \frac{\partial \rho}{\partial x}v_x dxdydz + \rho\frac{\partial v_x}{\partial x}dxdydz \tag{88}$$

$$= \frac{\text{mass}}{\text{time}} + \frac{\text{mass}}{xyz \text{ vol} \times x \text{ dist}}\frac{x \text{ dist}}{\text{time}}xyz \text{ vol} + \frac{\text{mass}}{xyz \text{ vol}}\frac{x \text{ dist}}{\text{time} \times x \text{ dist}}xyz \text{ vol} = \frac{\text{mass}}{\text{time}}. \tag{89}$$

Then the total gain of mass per time is

$$\text{rate in} - \text{rate out} = \rho v_x dydz - \rho v_x dydz - \frac{\partial \rho v_x}{\partial x}dxdydz = -\frac{\partial}{\partial x}(\rho v_x)dxdydz. \tag{90}$$

where the total loss is the negative of the total gain. Considering all directions, total loss is

$$\frac{\partial}{\partial x}(\rho v_x)dxdydz + \frac{\partial}{\partial y}(\rho v_y)dxdydz + \frac{\partial}{\partial z}(\rho v_z)dxdydz \tag{91}$$

$$= \frac{\partial}{\partial x_i}(\rho v_i)dxdydz = \nabla \cdot (\rho v_i)dxdydz. \tag{92}$$

If bounded by volume $V$ then this becomes

$$\text{total loss per time} = \int_V \nabla \cdot (\rho \mathbf{v})dV. \tag{93}$$

The divergence theorem is the relationship between the amount of fluid exiting with respect to the volume of the body and the amount of fluid crossing the outer surface across the perimeter. Physically they are the same thing. The relationship for this problem is

$$\int_V \nabla \cdot (\rho \mathbf{v})dV = \oint_S \rho \mathbf{v} \cdot \mathbf{n}dS. \tag{94}$$

In general for a vector $\mathbf{f}$, the divergence theorem

$$\int_V \nabla \cdot \mathbf{f}dV = \oint_S \mathbf{f} \cdot \mathbf{n}dS \Longleftrightarrow \int_V f_{i,i}dV = \oint_S f_i n_i dS. \tag{95}$$

Similar rules are the gradient theorem for scalar $f$

$$\int_V \nabla f dV = \oint_S f\mathbf{n}dS \tag{96}$$

and the curl theorem for vector $\mathbf{f}$

$$\int_V \nabla \times \mathbf{f}dV = \oint_S \mathbf{n} \times \mathbf{f}dS. \tag{97}$$

The divergence theorem can be approximated about a point $P$ as

$$(\nabla \cdot \mathbf{f})_P \approx \frac{1}{\Delta V} \oint_S \mathbf{f} \cdot \mathbf{n}dS \tag{98}$$

where $\Delta V$ is a small volume element surrounding point $P$. This means that the divergence of $\mathbf{f}$ can be thought of as the outward flow of $\mathbf{f}$ normal to the surface per unit volume. It can be said about the divergence theorem that the sum of the sources and sinks ( $V$) is equal to the net flow in and out of the surface ($S$).

The utility of the integral theorems can also be demonstrated by considering the applied pressure $p(x, y, z)$ on a body. Pressure is force/area, so force is pressure $\times$ area or (pressure/dist) $\times$ volume, and this force will act normal to the surface of the body. Then the gradient theorem dictates

$$\mathbf{F} = -\oint_S p\mathbf{n}DS = -\int_V \nabla p dV. \tag{99}$$

If pressure is constant then force is zero because a uniform pressure load across the entire body will result in no net force in any particular direction.

## 1.6 Eigenvalue problems

If for square matrix $\mathbf{A}$ there is some pair $\mathbf{x}, \lambda$ such that

$$\mathbf{A}\mathbf{x} = \lambda\mathbf{x} \tag{100}$$

then $\mathbf{x}, \lambda$ are an eigenvector and eigenvalue pair of system matrix $\mathbf{A}$. Consider

$$\lambda\mathbf{x} = \lambda\mathbf{I}\mathbf{x} = \mathbf{A}\mathbf{x} \longrightarrow (\mathbf{A} - \lambda\mathbf{I})\mathbf{x} = \mathbf{0}. \tag{101}$$

For nontrivial $\mathbf{x} \neq \mathbf{0}$, it must be that $(\mathbf{A} - \lambda\mathbf{I})$ is singular, meaning that by definition

$$\det(\mathbf{A} - \lambda\mathbf{I}) = \mathbf{0}. \tag{102}$$

This is called the characteristic equation or characteristic polynomial for the eigenvalue problem. If $\mathrm{A} = n \times n$ then the polynomial will be of degree $n$, will have $n$ roots, and thus will have $n$ eigenvalues. For example suppose

$$[\mathbf{A}] = \begin{bmatrix} 2 & -1 \\ -1 & 1 \end{bmatrix} \longrightarrow \det \mathbf{A} - \lambda\mathbf{I} = (2-\lambda)(1-\lambda) - (-1)(-1) = 1 - 3\lambda + \lambda^2 = 0. \tag{103}$$

Then

$$\lambda = \frac{3 \pm \sqrt{5}}{2} = \{0.382,\ 2.618\} \tag{104}$$

$$\longrightarrow \mathbf{0} = \begin{Bmatrix} 0 \\ 0 \end{Bmatrix} = (\mathbf{A} - \lambda\mathbf{x}) = \begin{bmatrix} 1.618 & -1 \\ -1 & 0.618 \end{bmatrix} \begin{Bmatrix} x_1^{(1)} \\ x_2^{(1)} \end{Bmatrix} \longrightarrow \mathbf{x}^{(1)} = \begin{Bmatrix} 0.618 \\ 1 \end{Bmatrix}, \tag{105}$$

$$\longrightarrow \mathbf{0} = \begin{Bmatrix} 0 \\ 0 \end{Bmatrix} = (\mathbf{A} - \lambda\mathbf{x}) = \begin{bmatrix} -0.618 & -1 \\ -1 & -1.618 \end{bmatrix} \begin{Bmatrix} x_1^{(2)} \\ x_2^{(2)} \end{Bmatrix} \longrightarrow \mathbf{x}^{(2)} = \begin{Bmatrix} -1.618 \\ 1 \end{Bmatrix}. \tag{106}$$

## 1.7 Even and odd functions

A function $f$ is even if $f(-x) = f(x)$, meaning the $y$ values on the left are the same as the $y$ values on the right. It is odd if $f(-x) = -f(x)$, meaning the $y$ values on the left are the opposite of the $y$ values on the right. The following properties hold.

- $f$ is even and smooth $\longrightarrow f'(0) = 0$.
- $f$ is odd $\longrightarrow f(0) = 0$.
- $f, g$ are even $\longrightarrow fg$ is even.
- $f, g$ are odd $\longrightarrow fg$ is even.
- $f$ is even, $g$ is odd $\longrightarrow fg$ is odd.
- $f$ is even $\longrightarrow f'$ is odd.
- $f$ is odd $\longrightarrow f'$ is even.

- $f$ is even $\longrightarrow \int_{-a}^{a} f(x)dx = 2\int_{0}^{a} f(x)dx$.
- $f$ is odd $\longrightarrow \int_{-a}^{a} f(x)dx = 0$.

Any $f$ can be broken into even and odd parts $f_e, f_o$ so that

$$f(x) = \underbrace{f_e(x)}_{\textbf{I.}} + \underbrace{f_o(x)}_{\textbf{II.}} = \underbrace{\frac{1}{2}[f(x) + f(-x)]}_{\textbf{I.}} + \underbrace{\frac{1}{2}[f(x) - f(-x)]}_{\textbf{II.}}. \tag{107}$$

# 2 Strain

## 2.1 Admissible deformation

Consider an object undergoing deformation so that $(x_1, x_2, x_3) \longrightarrow (\xi_1, \xi_2, \xi_3)$ is the transformation between the coordinates of a point $P$ in its original state to the coordinates of the same point in a deformed state. We can express the deformed coordinates as a function of each of the original coordinates, so that $\xi_i = \xi_i(x_j)$. Inversely, we can say $x_i = x_i(\xi_j)$. The derivative operators are related by

$$\frac{\partial}{\partial x_1} = \frac{\partial}{\partial \xi_1}\frac{\partial \xi_1}{\partial x_1} + \frac{\partial}{\partial \xi_2}\frac{\partial \xi_2}{\partial x_1} + \frac{\partial}{\partial \xi_3}\frac{\partial \xi_3}{\partial x_1}, \tag{108}$$

$$\frac{\partial}{\partial x_2} = \frac{\partial}{\partial \xi_1}\frac{\partial \xi_1}{\partial x_2} + \frac{\partial}{\partial \xi_2}\frac{\partial \xi_2}{\partial x_2} + \frac{\partial}{\partial \xi_3}\frac{\partial \xi_3}{\partial x_2}, \tag{109}$$

$$\frac{\partial}{\partial x_3} = \frac{\partial}{\partial \xi_1}\frac{\partial \xi_1}{\partial x_3} + \frac{\partial}{\partial \xi_2}\frac{\partial \xi_2}{\partial x_3} + \frac{\partial}{\partial \xi_3}\frac{\partial \xi_3}{\partial x_3}. \tag{110}$$

As a system of equations,

$$\begin{Bmatrix} \partial/\partial x_1 \\ \partial/\partial x_2 \\ \partial/\partial x_3 \end{Bmatrix} = \underbrace{\begin{bmatrix} \partial\xi_1/\partial x_1 & \partial\xi_2/\partial x_1 & \partial\xi_3/\partial x_1 \\ \partial\xi_1/\partial x_2 & \partial\xi_2/\partial x_2 & \partial\xi_3/\partial x_2 \\ \partial\xi_1/\partial x_3 & \partial\xi_2/\partial x_3 & \partial\xi_3/\partial x_3 \end{bmatrix}}_{[\mathbf{J}]} \begin{Bmatrix} \partial/\partial \xi_1 \\ \partial/\partial \xi_2 \\ \partial/\partial \xi_3 \end{Bmatrix} = [\mathbf{J}] \begin{Bmatrix} \partial/\partial \xi_1 \\ \partial/\partial \xi_2 \\ \partial/\partial \xi_3 \end{Bmatrix}. \tag{111}$$

Jacobian matrix $\mathbf{J}$ is the $3 \times 3$ transformation tensor. A physically real transformation requires $\det \mathbf{J} > 0$. For the inverse Jacobian $\mathbf{J}^{-1}$ to exist, $\det \mathbf{J} \neq 0$. If the body is not deformed at all then $\det \mathbf{J} = 1$. This is the case where there is no distinction between $x$ and $\xi$ and so $\mathbf{J}$ becomes $\mathbf{I}$.

Displacement vector

$$\begin{Bmatrix} u_1 \\ u_2 \\ u_3 \end{Bmatrix} = \begin{Bmatrix} \xi_1 - x_1 \\ \xi_2 - x_2 \\ \xi_3 - x_3 \end{Bmatrix} \tag{112}$$

implies

$$\begin{Bmatrix} \xi_1 \\ \xi_2 \\ \xi_3 \end{Bmatrix} = \begin{Bmatrix} x_1 + u_1 \\ x_2 + u_2 \\ x_3 + u_3 \end{Bmatrix} \longrightarrow [\mathbf{J}] = \begin{bmatrix} 1 + \partial u_1/\partial x_1 & \partial u_2/\partial x_1 & \partial u_3/\partial x_1 \\ \partial u_1/\partial x_2 & 1 + \partial u_2/\partial x_2 & \partial u_3/\partial x_2 \\ \partial u_1/\partial x_3 & \partial u_2/\partial x_3 & 1 + \partial u_3/\partial x_3 \end{bmatrix}. \tag{113}$$

For example if

$$\begin{Bmatrix} u_1 \\ u_2 \\ u_3 \end{Bmatrix} = \begin{Bmatrix} x_1 - 2x_2 \\ 3x_1 + 2x_2 \\ 5x_3 \end{Bmatrix} \tag{114}$$

then

$$\begin{Bmatrix} \xi_1 \\ \xi_2 \\ \xi_3 \end{Bmatrix} = \begin{Bmatrix} 2x_1 - 2x_2 \\ 3x_1 + 3x_2 \\ 6x_3 \end{Bmatrix} \longrightarrow [\mathbf{J}] = \begin{bmatrix} 2 & 3 & 0 \\ -2 & 3 & 0 \\ 0 & 0 & 6 \end{bmatrix} \tag{115}$$

and $\det \mathbf{J} = 2(18) + -3(-12) = 72 > 0$, making it admissible.

$\mathbf{J}$ can be thought of as the ratio between the volume of the deformed configuration and the undeformed configuration (new/old). Vectors

$$\mathbf{dx}_1 = \begin{Bmatrix} \partial\xi_1/\partial x_1 \\ \partial\xi_2/\partial x_1 \\ \partial\xi_3/\partial x_1 \end{Bmatrix} dx_1, \quad \mathbf{dx}_2 = \begin{Bmatrix} \partial\xi_1/\partial x_2 \\ \partial\xi_2/\partial x_2 \\ \partial\xi_3/\partial x_2 \end{Bmatrix} dx_2, \quad \mathbf{dx}_3 = \begin{Bmatrix} \partial\xi_1/\partial x_3 \\ \partial\xi_2/\partial x_3 \\ \partial\xi_3/\partial x_3 \end{Bmatrix} dx_3 \tag{116}$$

are tangent to the coordinate curves of $x_1,\ x_2$, and $x_3$ respectively, where for example the $x_1$ coordinate curve is obtained by fixing $x_2, x_3$ and changing $x_1$. In the Cartesian coordinate system the coordinate curves are just the axes. For example consider Fig. 1. Going from $(x_1, x_2, x_3) \longrightarrow (x_1 + \Delta x_1, x_2, x_3)$ causes a change in both $\xi_1, \xi_2$ so that

$$\Delta x_1 \cos\theta = \Delta\xi_1 \longrightarrow \frac{\Delta x_1}{\Delta\xi_1} = \cos\theta, \tag{117}$$

$$\Delta x_1 \sin\theta = \Delta\xi_2 \longrightarrow \frac{\Delta x_1}{\Delta\xi_2} = \sin\theta. \tag{118}$$

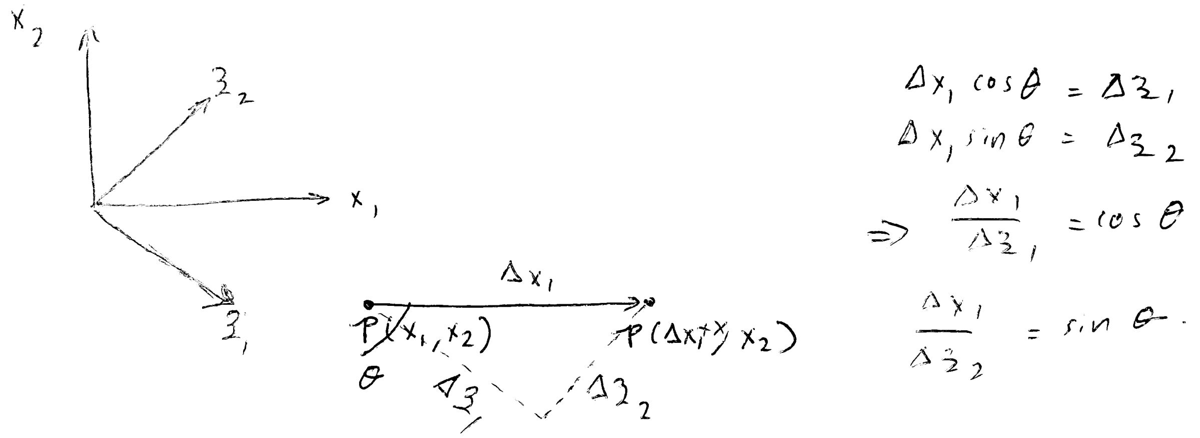


Figure 1: 2D coordinate transformation

Because of the triple scalar product identity

$$\mathbf{a} \cdot \mathbf{b} \times \mathbf{c} = \det \begin{bmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{bmatrix}, \tag{119}$$

we find

$$\mathbf{dx}_1 \cdot \mathbf{dx}_2 \times \mathbf{dx}_3 = \det \begin{bmatrix} \partial\xi_1/\partial x_1 & \partial\xi_2/\partial x_1 & \partial\xi_3/\partial x_1 \\ \partial\xi_1/\partial x_2 & \partial\xi_2/\partial x_2 & \partial\xi_3/\partial x_2 \\ \partial\xi_1/\partial x_3 & \partial\xi_2/\partial x_3 & \partial\xi_3/\partial x_3 \end{bmatrix} dx_1 dx_2 dx_3 = \det \mathbf{J} dx_1 dx_2 dx_3 = dV. \tag{120}$$

This justifies the claim that $\det \mathbf{J}$ must be positive because it leads to a volume element that is positive and one cannot have negative volume.

Since the determinant is a norm or magnitude of the matrix, $\det \mathbf{J}$ can be thought of as a ratio between new and old volume, in the sense that it is the magnitude of the change in new coordinates with respect to the old coordinates. Therefore,

$$\det \mathbf{J} = \frac{V + \Delta V}{V} = 1 + \frac{\Delta V}{V}, \tag{121}$$

where the change in volume with respect to the original volume $\Delta V/V$ is called the volumetric strain. Recalling the representation of $\mathbf{J}$ that is Eq. 113,

$$\det \mathbf{J} = \det \begin{bmatrix} 1 + \partial u_1/\partial x_1 & \partial u_2/\partial x_1 & \partial u_3/\partial x_1 \\ \partial u_1/\partial x_2 & 1 + \partial u_2/\partial x_2 & \partial u_3/\partial x_2 \\ \partial u_1/\partial x_3 & \partial u_2/\partial x_3 & 1 + \partial u_3/\partial x_3 \end{bmatrix} \Leftrightarrow \det \begin{bmatrix} 1 + u_{1,1} & u_{2,1} & u_{3,1} \\ u_{1,2} & 1 + u_{2,2} & u_{3,2} \\ u_{1,3} & u_{2,3} & 1 + u_{3,3} \end{bmatrix}$$

$$= (1 + u_{1,1})[(1 + u_{2,2})(1 + u_{3,3}) - u_{2,3}u_{3,2}] \tag{122}$$

$$-u_{2,1}[u_{1,2}(1 + u_{3,3}) - u_{1,3}u_{3,2}] + u_{3,1}(u_{1,2}u_{2,3} - u_{1,3}(1 + u_{2,2})) \tag{123}$$

$$= 1 + u_{1,1} + u_{2,2} + u_{3,3} + \underbrace{\cdots}_{\text{combinations of product terms}} . \tag{124}$$

If the displacement is small then the product terms are negligible because small times small is extremely small. So in this case we can simplify to say

$$\det \mathbf{J} = 1 + \frac{\Delta V}{V} = 1 + u_{i,i}, \tag{125}$$

of course meaning

$$\frac{\Delta V}{V} = u_{i,i}. \tag{126}$$

## 2.2 Affine transformations

Let $x_i$ be the original coordinates and $x'_i(x_j)$ be the new coordinates. Here we are only concerned with the deformation behavior itself and not how it happens (temperature, force, etc.). A special type of deformation is an affine transformation, which is when the function describing the relationship between the deformed coordinates and the original coordinates is linear. That is,

$$x'_i = \underbrace{x_i}_{\text{original coordinate vector}} + \underbrace{\alpha_{i0}}_{\text{translation vector}} + \underbrace{\alpha_{ij}x_j}_{\text{rotation and stretch}} \Leftrightarrow \left\{ \begin{matrix} x'_1 = x_1 + \alpha_{10} + \alpha_{1j}x_j \\ x'_2 = x_2 + \alpha_{20} + \alpha_{2j}x_j \\ x'_3 = x_3 + \alpha_{30} + \alpha_{3j}x_j \end{matrix} \right\} \tag{127}$$

which implies

$$x'_i = \delta_{ij}x_j + \alpha_{i0} + \alpha_{ij}x_j \longrightarrow x'_i = \alpha_{i0} + (\delta_{ij} + \alpha_{ij})x_j \tag{128}$$

or

$$\mathbf{x}' = \boldsymbol{\alpha}_0 + (\mathbf{I} + \boldsymbol{\alpha})\mathbf{x}. \tag{129}$$

So the term $\boldsymbol{\alpha}$ would have to be $\mathbf{0}$, not $\mathbf{I}$ if it was assumed that there was no rotation and no stretch. As another example the matrix

$$\mathbf{I} + \boldsymbol{\alpha} = \begin{bmatrix} c & 0 \\ 0 & c \end{bmatrix} \longrightarrow \boldsymbol{\alpha} = (c-1)\mathbf{I} \tag{130}$$

represents uniform stretch by the factor $c$, NOT by the factor $c - 1$. The matrix

$$\mathbf{I} + \boldsymbol{\alpha} + \mathbf{I} = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} \longrightarrow \boldsymbol{\alpha} = \begin{bmatrix} -1 & -1 \\ 1 & -1 \end{bmatrix} \tag{131}$$

represents a 90 degree CCW rotation in that

$$\begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} \begin{Bmatrix} x_1 \\ x_2 \end{Bmatrix} = \begin{Bmatrix} x_1' \\ x_2' \end{Bmatrix} \tag{132}$$

implies

$$\begin{Bmatrix} x_1' \\ x_2' \end{Bmatrix} = \begin{Bmatrix} -x_2 \\ x_1 \end{Bmatrix} \tag{133}$$

which can be thought of visually, turning the axes $x_1, x_2$ 90 degrees counterclockwise. Like earlier, note that this does NOT imply the transformation $\boldsymbol{\alpha}\mathbf{x} = \mathbf{x}'$ but rather $(\boldsymbol{\alpha} + \mathbf{I})\mathbf{x} = \mathbf{x}'$. In the same way we can solve for $\mathbf{x}'$ in terms of $\mathbf{x}$ using Eq. 128, we also can solve for $\mathbf{x}$ in terms of $\mathbf{x}'$. So there must exist some $\boldsymbol{\beta}_0, \boldsymbol{\beta}$ such that

$$x_i = \beta_{i0} + (\delta_{ij} + \beta_{ij})x_j', \tag{134}$$

and this is also an affine/linear coordinate transformation.

Affine transformations have two interesting properties. First it transforms planes into other planes. The general equation for a plane is

$$Ax + By + Cz = D$$

and if we plug in the affine transformations into this equation then we receive another linear equation for a plane. The second interesting property of affine transformations is that straight lines transform into other straight lines. This is a consequence of (1) since lines are just intersections of planes. If planes turn into planes, then the straight lines on that plane turn into other straight lines.

As a consequence of (2), a vector

$$\mathbf{A} = A_i \mathbf{e}_i \Longrightarrow^{\text{affine}} A_i' \mathbf{e}_i' = \mathbf{A}' \tag{135}$$

turns into another vector under an affine transformation. Let $\mathbf{A}$ be a vector within a body that goes from one point $x_{i0}$ to another point $x_i$. Note this is NOT the displacement vector that maps the undeformed coordinates to the deformed coordinates. This is simply a vector that travels across the body in its undeformed state from one point to another point. So

$$A_i = x_i - x_{i0}. \tag{136}$$

Then suppose the body undergoes an affine transformation. Then

$$A'_i = x'_i - x'_{i0} = (x_i + \cancel{\alpha_{i0}} + \alpha_{ij}x_j) - (x_{i0} + \cancel{\alpha_{i0}} + \alpha_{ij}x_{j0}) \tag{137}$$

$$= (x_i - x_{i0}) + \alpha_{ij}(x_j - x_{j0}) = A_i + \alpha_{ij}A_j. \tag{138}$$

Let

$$\delta A_i = \alpha_{ij}A_j = A'_i - A_i, \tag{139}$$

where $\delta A_i = \{\delta A_1, \delta A_2, \delta A_3\}$ are the components of the change between the vector before and after deformation. It is also defined as the components of the rotation/stretch vector. Both of these definitions follow from reading Eq. 139. Also, the length of $\mathbf{A}$ is

$$\sqrt{\mathbf{A}\cdot\mathbf{A}} = \sqrt{A^2\cos 0} = A \Longrightarrow A^2 = \mathbf{A}\cdot\mathbf{A} = A_iA_i, \tag{140}$$

and the length of $\delta A_i$ is

$$\delta A = \sqrt{\delta A_i \delta A_i}. \tag{141}$$

Then

$$2A\delta A = 2\sqrt{A_iA_i}\sqrt{\delta A_i\delta A_i} = 2\sqrt{A_i\delta A_i}\sqrt{A_i\delta A_i} = 2A_i\delta A_i \Longrightarrow A\delta A = A_i\delta A_i. \tag{142}$$

Substituting in Eq. 139,

$$A\delta A = A_i \underbrace{\delta A_i}_{\mathbf{I.}} = A_i \underbrace{\alpha_{ij}A_j}_{\mathbf{I.}} = \alpha_{ij}A_iA_j. \tag{143}$$

If there is rotation but no stretch, then the change in the length of vector $\mathbf{A}$ does not change. Therefore $\delta A = 0$ and

$$\alpha_{ij}A_iA_j = 0 \quad \forall A_i. \tag{144}$$

Expanding,

$$0 = \alpha_{11}A_1A_1 + \alpha_{12}A_1A_2 + \alpha_{13}A_1A_3 \tag{145}$$

$$+\alpha_{21}A_2A_1 + \alpha_{22}A_2A_2 + \alpha_{23}A_2A_3 \tag{146}$$

$$+\alpha_{31}A_3A_1 + \alpha_{32}A_3A_2 + \alpha_{33}A_3A_3 \tag{147}$$

$$= 0 = \alpha_{11}A_1^2 + \alpha_{22}A_2^2 + \alpha_{33}A_3^2 + A_1A_2(\alpha_{12}+\alpha_{21}) + A_1A_3(\alpha_{13}+\alpha_{31}) + A_2A_3(\alpha_{23}+\alpha_{32}). \tag{148}$$

If this is true for any $A_1, A_2, A_3 = A_i$, then

$$\alpha_{ii} = 0, \quad \alpha_{ij} = -\alpha_{ji}. \tag{149}$$

This means $\alpha_{ij}$ is skew symmetric. Please note that this is in the specific case where there is rotation but no stretch. This is not to say that $\alpha_{ij}$ in general is skew. Speaking more generally, the tensor $\boldsymbol{\alpha}$ like any tensor can be broken up into symmetric and skew parts

$$\alpha_{ij} = \frac{1}{2}(\alpha_{ij} + \alpha_{ji}) + \frac{1}{2}(\alpha_{ij} - \alpha_{ji}) = \epsilon_{ij} + \omega_{ij} \tag{150}$$

where $\boldsymbol{\epsilon}$ is solely dedicated to deformation and $\boldsymbol{\omega}$ is solely dedicated to rotation. $\boldsymbol{\epsilon} \Leftrightarrow \epsilon_{ij}$ is called the strain tensor.

## 2.3 Geometrical interpretations of strain components

Recalling Eq. 143,

$$A\delta A = \alpha_{ij}A_iA_j = (\epsilon_{ij} + \omega_{ij})A_iA_j = \epsilon_{ij}A_iA_j + \omega_{ij}A_iA_j. \tag{151}$$

Consider the last term, recalling that $\omega_{ij}$ is skew. This means

$$\omega_{ij}A_iA_j = -\omega_{ji}A_iA_j \Leftrightarrow -\omega_{ij}A_jA_i \Longrightarrow 2\omega_{ij}A_iA_j = 0 \Longrightarrow \omega_{ij}A_iA_j = 0. \tag{152}$$

Therefore,

$$A\delta A = \epsilon_{ij}A_iA_j \Longrightarrow \frac{\delta A}{A} = \frac{\epsilon_{ij}A_iA_j}{A^2}. \tag{153}$$

This represents the amount that the vector **A** has changed divided by its length. It is rating of relative length change. For example suppose **A** only had a component in the direction $x_1$. This means the length $A = A_1$ and

$$\frac{\delta A}{A_1} = \frac{\epsilon_{11}A_1^2}{A_1^2} \Longrightarrow \frac{\delta A}{A} = \epsilon_{11}. \tag{154}$$

So, the physical interpretation of the diagonal strain components $\epsilon_{ii}$ is that they are a measure of the change in length per unit length in the direction $x_i$. As for off diagonal components, consider two vectors that exist in the body

$$\mathbf{A} = A_2\mathbf{e}_2, \quad \mathbf{B} = B_3\mathbf{e}_3. \tag{155}$$

Here **A** only has a component in the direction $x_2$ and likewise **B** in $x_3$. Because of Eq. 139 ($\delta A_i = \alpha_{ij}A_j$),

$$\delta A_3 = \alpha_{32}A_2, \quad \delta B_2 = \alpha_{23}B_3. \tag{156}$$

The correct interpretation of this equation set is this. Initially $B_2$ is zero but a deformation in the body changes $B_2$ from zero to something that is not zero by the amount $\delta B_2$. This amount is equal to the initial component $B_3$ transformed under the tensor $\alpha_{23}$. The same is true of **A**. Both **B** and **A** change orientation, and this means they have a change in angle in relationship to one another, and the quantity of this change is

$$\alpha_{23} + \alpha_{32} = 2 * \frac{1}{2}(\alpha_{23} + \alpha_{32}) = 2\epsilon_{23} = \text{change in angle between } \mathbf{A} \text{ and } \mathbf{B}. \tag{157}$$

Note that $2\epsilon_{23} = \gamma_{23}$, where $\boldsymbol{\gamma}$ is the engineering strain tensor. So to recap, the diagonal strain components represent the change in length of a vector in a body with respect to its original length, and off-diagonal components represent the shear-induced change in angle between two vectors pointing in the two directions that correspond to the particular component of interest.

## 2.4 Strain as a tensor

Let us prove that strain $\boldsymbol{\epsilon}$ is a tensor. Recall Eq. 153, which is

$$A\delta A = \epsilon_{ij}A_iA_j = A_i \underbrace{\epsilon_{ij}A_j}_{\text{some vector } v_i} = \mathbf{A}\cdot\boldsymbol{\epsilon}\mathbf{A} = \mathbf{A}^T\boldsymbol{\epsilon}\mathbf{A}. \tag{158}$$

The change in length $\delta A$ times the length $A$ is invariant under the transformation of coordinates. Therefore

$$A\delta A = \bar{A}\delta\bar{A} \iff \mathbf{A}^T\boldsymbol{\epsilon}\mathbf{A} = \bar{\mathbf{A}}^T\bar{\boldsymbol{\epsilon}}\bar{\mathbf{A}}. \tag{159}$$

Let us define $\bar{\mathbf{A}}$ as the transformation of $\mathbf{A}$ due to $\mathbf{R}$. Then $\bar{\mathbf{A}} = \mathbf{R}\mathbf{A} \implies \mathbf{A} = \mathbf{R}^T\bar{\mathbf{A}}$ and

$$\bar{\mathbf{A}}^T\bar{\boldsymbol{\epsilon}}\bar{\mathbf{A}} = \mathbf{A}^T\boldsymbol{\epsilon}\mathbf{A} = (\mathbf{R}^T\bar{\mathbf{A}})^T\boldsymbol{\epsilon}(\mathbf{R}^T\bar{\mathbf{A}}) = \bar{\mathbf{A}}^T\mathbf{R}\boldsymbol{\epsilon}\mathbf{R}^T\bar{\mathbf{A}}. \tag{160}$$

Therefore

$$\bar{\boldsymbol{\epsilon}} = \mathbf{R}\boldsymbol{\epsilon}\mathbf{R}^T \implies \bar{\epsilon}_{ij} = R_{ik}\epsilon_{kl}R_{jl} = R_{ik}R_{jl}\epsilon_{kl} \tag{161}$$

which satisfies the definition of a transformed second order tensor.

## 2.5 General infinitesimal deformation

A major consequence of the affine transformation is Eq. 139, which is

$$\delta A_i = \alpha_{ij}A_j = A'_i - A_i. \tag{162}$$

Here $A_j$ is a vector within some body and $\alpha_{ij}$ is a tensor that represents the rotation and deformation of that body. The result is the vector $\delta A_i$ which represents the change between the original vector $A$ and the new vector $A'_i$.

Similarly to the concept of $\mathbf{A}$, which is a vector in the undeformed body, let us consider two points in the undeformed configuration $x_{i0}$ and $x_i$. After deformation, the corresponding displacements are

$$u_{i0} = \overbrace{x'_{i0} - x_{i0}}^{\mathbf{I}}, \qquad u_i = \underbrace{x'_i - x_i}_{\mathbf{II}}. \tag{163}$$

If vector $\mathbf{A}$ represents the distance between $x_i$ and $x_{i0}$, so that

$$A_i = x_i - x_{i0}, \tag{164}$$

$$\delta A_i = A'_i - A_i = (x'_i - x'_{i0}) - (x_i - x_{i0}) = \underbrace{x'_i - x_i}_{\mathbf{II}} \overbrace{-x'_{i0} + x_{i0}}^{-\mathbf{I}} = u_i - u_{i0}. \tag{165}$$

Now we will represent displacement as a Taylor series. In general for some function $f$,

$$f(x) = \sum_{n=0}^{\infty}\frac{f^{(n)}(x_0)}{n!}(x - x_0)^n. \tag{166}$$

Neglecting higher order terms,

$$f(x) = f(x_0) + \underbrace{f'(x_0)(x - x_0)} + \dots . \tag{167}$$

For two variables,

$$f(x, y) = f(x_0, y_0) + \underbrace{\frac{\partial f(x_0, y_0)}{\partial x}(x - x_0) + \frac{\partial f(x_0, y_0)}{\partial y}(y - y_0)} + \dots . \tag{168}$$

For many variables

$$f(x_i) = f(x_{i0}) + \frac{\partial f(x_{j0})}{\partial x_j}(x_j - x_{j0}) + \dots . \tag{169}$$

Substituting in displacement,

$$u_i = u_i(x_i) = u_{i0} + \frac{\partial u_{i0}}{\partial x_j}(x_j - x_{j0}) + \dots . \tag{170}$$

Recalling the assumption Eq. 164 ($A_i = x_i - x_{i0}$),

$$u_i = u_{i0} + u^{?}_{i,j} A_j + \dots . \tag{171}$$

or

$$u_i - u_{i0} = u_{i,j} A_j. \tag{172}$$

Because of Eq. 165 ($\delta A_i = u_i - u_{i0}$),

$$\delta A_i = u_{i,j} A_j. \tag{173}$$

Substituting in Eq. 139 ($\delta A_i = \alpha_{ij} A_j$),

$$\alpha_{ij} = u_{i,j}. \tag{174}$$

The decomposition of $\alpha_{ij}$ in Eq. 150 implies

$$u_{i,j} = \frac{1}{2}(u_{i,j} + u_{j,i}) + \frac{1}{2}(u_{i,j} - u_{j,i}) = \epsilon_{ij} + \omega_{ij}, \tag{175}$$

and the relationship between strain and displacement ($(u_{i,j} + u_{j,i})/2 = \epsilon_{ij}$) is called the strain/displacement equation.

For clarification purposes we now write $u, v, w$ in place of $u_x, u_y, u_z$. Then for instance

$$\epsilon_{xx} = \frac{\partial u}{\partial x}, \quad \epsilon_{yz} = \frac{1}{2}\left(\frac{\partial v}{\partial z} + \frac{\partial w}{\partial y}\right) = \frac{1}{2}\gamma_{yz}, \quad \text{etc.} \tag{176}$$

The diagonal components are called the normal strains and the off diagonal components are called the shear strains.

## 2.6 Compatibility equations

Given displacement field $\mathbf{u}$, strain components

$$\epsilon_{ij} = \frac{1}{2}(u_{i,j} + u_{j,i}). \tag{177}$$

However it is not necessarily true that given a number of strain components we can calculate displacement. To answer this let us make the modification

$$2e_{rip}\epsilon_{ij} = e_{rip}(u_{i,j} + u_{j,i}) \tag{178}$$

where $e$ here is the Levi Civita symbol. Differentiating with respect to $x_p$,

$$2e_{rip}\epsilon_{ij,p} = e_{rip}u_{i,jp} + e_{rip}u_{j,ip}. \tag{179}$$

The last term vanishes because

$$e_{rip}u_{j,ip} = -e_{rpi}u_{j,ip} \Leftrightarrow \underbrace{-e_{rip}u_{j,pi}}_{\text{arbitrary indices}} = \underbrace{-e_{rip}u_{j,ip}}_{\text{arbitrary derivative order}} \Longrightarrow 2e_{rip}u_{j,ip} = 0 \Longrightarrow e_{rip}u_{j,ip} = 0. \tag{180}$$

Therefore

$$2e_{rip}\epsilon_{ij,p} = e_{rip}u_{i,jp} \tag{181}$$

which implies

$$2e_{rip}e_{sjq}\epsilon_{ij,p} = e_{rip}e_{sjq}u_{i,jp} \Longrightarrow 2e_{rip}e_{sjq}\epsilon_{ij,pq} = e_{rip}e_{sjq}u_{i,jpq}. \tag{182}$$

However this RHS term also vanishes for the same reason as Eq. 180, which is that an arbitrary switch of indices changes the sign of the Levi Civita constant but not the derivative terms, meaning the whole term must be equal to be its own negative, meaning the term must be zero. Therefore

$$e_{rip}e_{sjq}\epsilon_{ij,pq} = 0 \tag{183}$$

is a true set of equations called the compatibility equations. If you are given a number of strain components you must be able to solve for this set of equations. Otherwise it is impossible to infer a displacement solution from what strains you are given.

Indices $r, s$ occur once, and so these are free indices. Each equation is unique to one free index, meaning one $r$ and one $s$. The other indices $i, j, p, q$ occur multiple times and so are dummy indices. Each equation has every version of the dummy index among 1,2,3.

Because of the many combinations of $e_{rip}, e_{sjq}$ that are null, and also because of the symmetry properties of $\boldsymbol{\epsilon}$, there are nine total equations based on different $r, s$ but only six of them are unique. The set of index pairs $r, s$ that correspond to each unique equation is $r = s = 1,\ r = 1, s = 2,\ r = 1, s = 3,\ r = 2, s = 2,\ r = 2, s = 3,\ r = 3, s = 3$. The set of equations that correspond to this set is

$$2\epsilon_{32,23} = \epsilon_{22,33} + \epsilon_{33,22} \tag{184}$$

$$2\epsilon_{21,12} = \epsilon_{11,22} + \epsilon_{22,11} \tag{185}$$

$$2\epsilon_{31,13} = \epsilon_{11,33} + \epsilon_{33,11} \tag{186}$$

$$\epsilon_{11,23} = -\epsilon_{23,11} + \epsilon_{13,12} + \epsilon_{12,13} \tag{187}$$

$$\epsilon_{22,13} = -\epsilon_{13,22} + \epsilon_{23,21} + \epsilon_{21,23} \tag{188}$$

$$\epsilon_{33,12} = -\epsilon_{12,33} + \epsilon_{32,31} + \epsilon_{31,32} \tag{189}$$

or in general

$$\epsilon_{ij,kl} = -\epsilon_{kl,ij} + \epsilon_{jl,ik} + \epsilon_{ik,jl} \Longrightarrow \epsilon_{ij,kl} + \epsilon_{kl,ij} - \epsilon_{jl,ik} - \epsilon_{ik,jl} = 0. \tag{190}$$

Eq. 190 is another way to write the set of compatibility equations. It is necessary and sufficient for all of these to be true in order for there to exist a displacement solution given the strains.

What follow from the sufficiency of the compatibility equations are two things. First, zero strains imply no deformation, and this is called rigid body motion, meaning there is only translation and rotation. Second, a set of strains together with a particular set of translation and rotation parameters yields a unique displacement solution.

## 2.7 Integrating the strain displacement equations

We have proven that if a set of given strains satisfies the compatibility equations, then from that set we can infer a displacement solution. For example consider the 2D case

$$\epsilon_{xx} = A = \frac{\partial u}{\partial x},\ \epsilon_{yy} = 0 = \frac{\partial v}{\partial y},\ 2\epsilon_{xy} = 0 = \frac{1}{2}\left(\frac{\partial u}{\partial y} + \frac{\partial v}{\partial x}\right). \tag{191}$$

Then taking antiderivatives of the diagonals leads to

$$u(x) = Ax + f(y), \quad v(y) = g(x). \tag{192}$$

Substituting into the off diagonal,

$$0 = \frac{1}{2}\left(\frac{\partial u}{\partial y} + \frac{\partial v}{\partial x}\right) = \frac{1}{2}\left(\frac{\partial(Ax + f(y))}{\partial y} + \frac{\partial(g(x))}{\partial x}\right) = f'(y) + g'(x), \tag{193}$$

meaning

$$f'(y) = -g'(x). \tag{194}$$

If a function of $y$ is a function of $x \forall x, y$ then the function cannot depend on either $x$ or $y$, meaning it is a constant. So

$$f'(y) = -g'(x) = B \tag{195}$$

which implies

$$f(y) = By + C, \quad g(x) = -Bx + D, \tag{196}$$

which implies

$$u(x) = Ax + By + C, \quad v(y) = Bx + D. \tag{197}$$

As a system of equations,

$$\begin{Bmatrix} u \\ v \end{Bmatrix} = \begin{Bmatrix} A & B \\ -B & 0 \end{Bmatrix} \begin{Bmatrix} x \\ y \end{Bmatrix} + \begin{Bmatrix} C \\ D \end{Bmatrix}. \tag{198}$$

Separating the stretch component associated with $\epsilon_{xx}$, which is $A$, away from rigid body components $B, C, D$,

$$\begin{Bmatrix} u \\ v \end{Bmatrix} = \underbrace{\begin{bmatrix} A & 0 \\ 0 & 0 \end{bmatrix}}_{\text{deformation}} \begin{Bmatrix} x \\ y \end{Bmatrix} + \underbrace{\begin{bmatrix} 0 & B \\ -B & 0 \end{bmatrix}}_{\text{rotation (skew)}} \begin{Bmatrix} x \\ y \end{Bmatrix} + \underbrace{\begin{Bmatrix} C \\ D \end{Bmatrix}}_{\text{translation}}. \tag{199}$$

## 2.8 Principal axes of strain

Given strain tensor

$$[\boldsymbol{\epsilon}] = \begin{bmatrix} \epsilon_{11} & \epsilon_{12} & \epsilon_{13} \\ \epsilon_{21} & \epsilon_{22} & \epsilon_{23} \\ \epsilon_{31} & \epsilon_{32} & \epsilon_{33} \end{bmatrix}, \tag{200}$$

we wish to know if there exists some coordinate rotation $\mathbf{R}$ such that the strain tensor is diagonalized, i.e.

$$\boldsymbol{\epsilon}' \Leftrightarrow \begin{bmatrix} \epsilon'_{11} & 0 & 0 \\ 0 & \epsilon'_{22} & 0 \\ 0 & 0 & \epsilon'_{33} \end{bmatrix} = \mathbf{R}\boldsymbol{\epsilon}\mathbf{R}^T \Longrightarrow \mathbf{R}^T\boldsymbol{\epsilon}' = \mathbf{R}\mathbf{R}^T\boldsymbol{\epsilon}\mathbf{R}^T = \boldsymbol{\epsilon}\mathbf{R}^T. \tag{201}$$

Suppose

$$\mathbf{R} = \begin{Bmatrix} \mathbf{v}_1^T \\ \mathbf{v}_2^T \\ \mathbf{v}_3^T \end{Bmatrix} \Longleftrightarrow \mathbf{R}^T = \begin{Bmatrix} \mathbf{v}_1 & \mathbf{v}_2 & \mathbf{v}_3 \end{Bmatrix}, \tag{202}$$

where $\mathbf{v}_i$ are the $i$ columns of $\mathbf{R}^T$. Then because of Eq. 201 ($\boldsymbol{\epsilon}\mathbf{R}^T = \mathbf{R}^T\boldsymbol{\epsilon}'$),

$$\boldsymbol{\epsilon}\begin{Bmatrix} \mathbf{v}_1 & \mathbf{v}_2 & \mathbf{v}_3 \end{Bmatrix} = \begin{Bmatrix} \mathbf{v}_1 & \mathbf{v}_2 & \mathbf{v}_3 \end{Bmatrix} \begin{bmatrix} \epsilon'_{11} & 0 & 0 \\ 0 & \epsilon'_{22} & 0 \\ 0 & 0 & \epsilon'_{33} \end{bmatrix} = \begin{Bmatrix} \epsilon'_{11}\mathbf{v}_1 & \epsilon'_{22}\mathbf{v}_2 & \epsilon'_{33}\mathbf{v}_3 \end{Bmatrix}. \tag{203}$$

Therefore

$$\boldsymbol{\epsilon}\mathbf{v}_i = \epsilon'_{ii}\mathbf{v}_i \Longleftrightarrow \boldsymbol{\epsilon}\mathbf{v} = \epsilon'\mathbf{v}. \tag{204}$$

This is called an eigenproblem, where $\epsilon'$ are the eigenvalues and $\mathbf{v}$ are the eigenvectors. The goal in solving Eq. 204 is to find nonzero $\mathbf{v}$ which, when transformed by $\boldsymbol{\epsilon}$ (i.e. $\boldsymbol{\epsilon}\mathbf{v}$), produce vectors parallel to $\mathbf{v}$ that are scaled by magnitude $\epsilon'$ (i.e. $\epsilon'\mathbf{v}$). Eq. 204 implies

$$(\boldsymbol{\epsilon} - \epsilon'\mathbf{I})\mathbf{v} = \mathbf{0}. \tag{205}$$

The solution to this is either the trivial solution $\mathbf{v} = \mathbf{0}$, which is uninteresting, or non-trivial solutions where $\boldsymbol{\epsilon} - \epsilon'\mathbf{I}$ is singular, meaning

$$0 = \det \begin{bmatrix} \epsilon_{11} - \epsilon' & \epsilon_{12} & \epsilon_{13} \\ \epsilon_{21} & \epsilon_{22} - \epsilon' & \epsilon_{23} \\ \epsilon_{31} & \epsilon_{32} & \epsilon_{33} - \epsilon' \end{bmatrix} \tag{206}$$

$$= (\epsilon_{11} - \epsilon')[(\epsilon_{22} - \epsilon')(\epsilon_{33} - \epsilon') - \epsilon_{32}\epsilon_{23}]$$
$$-\epsilon_{12}[\epsilon_{21}(\epsilon_{33} - \epsilon') - \epsilon_{31}\epsilon_{23}] + \epsilon_{13}[\epsilon_{21}\epsilon_{32} - \epsilon_{31}(\epsilon_{22} - \epsilon')] \tag{207}$$
$$= -\epsilon'^3 + \theta_1\epsilon'^2 - \theta_2\epsilon' + \theta_3 = 0 \tag{208}$$

where

$$\begin{Bmatrix} \theta_1 \\ \theta_2 \\ \theta_3 \end{Bmatrix} = \begin{Bmatrix} \mathrm{tr}\boldsymbol{\epsilon} \\ (\epsilon_{ii}\epsilon_{jj} - \epsilon_{ij}\epsilon_{ji})/2 \\ \det \boldsymbol{\epsilon} \end{Bmatrix}. \tag{209}$$

The roots $\epsilon' = \{\epsilon'_{11}, \epsilon'_{22}, \epsilon'_{33}\}$ to Eq. 208 are the principal eigenstrains, and the resulting strain tensor is

$$\boldsymbol{\epsilon}' = \begin{bmatrix} \epsilon'_{11} & 0 & 0 \\ 0 & \epsilon'_{22} & 0 \\ 0 & 0 & \epsilon'_{33} \end{bmatrix}, \tag{210}$$

and the corresponding $\mathbf{v}$ are the principal coordinates.

The principal strains $\epsilon'$ are coordinate independent. Therefore so must be the coefficients $\theta$ of the characteristic equation of the eigenproblem Eq. 208, which are called the principal invariants. If $\epsilon'$ are known, they can be solved as

$$\begin{Bmatrix} \theta_1 \\ \theta_2 \\ \theta_3 \end{Bmatrix} = \begin{Bmatrix} \epsilon'_{11} + \epsilon'_{22} + \epsilon'_{33} \\ \epsilon'_{22}\epsilon'_{33} + \epsilon'_{11}\epsilon'_{33} + \epsilon'_{11}\epsilon'_{22} \\ \epsilon'_{11}\epsilon'_{22}\epsilon'_{33} \end{Bmatrix}. \tag{211}$$

This process is true of all second order tensors such as $\boldsymbol{\epsilon}$.

## 2.9 Properties of the real symmetric eigenvalue problem

Note that in the eigenproblem Eq. 204 ($\boldsymbol{\epsilon}\mathbf{v} = \epsilon'\mathbf{v}$), strain $\boldsymbol{\epsilon}$ is symmetric because it is strain, defined as the symmetric part of the displacement gradient. Suppose the components of system matrix $\mathbf{M}$ in

$$\mathbf{M}\mathbf{x} = \lambda\mathbf{x} \tag{212}$$

are real and symmetric. Here $\lambda$ are the eigenvalues and $\mathbf{x}$ are the eigenvectors. Real symmetric eigenproblems have two properties of interest. The first property is that it must yield real eigenvalues. To prove this recall the general definition of a complex conjugate

$$z = a + bi \Longrightarrow z^* = a - bi. \tag{213}$$

Taking the complex conjugate of the eigenproblem as a whole,

$$\mathbf{M}\mathbf{x}^* = \lambda^*\mathbf{x}^*. \tag{214}$$

Note that $\mathbf{M}$ is real, so $\mathbf{M} = \mathbf{M}^*$ necessarily. Respectively from Eq. 212 and Eq. 214 we can deduce

$$\mathbf{x}^{*T}\mathbf{M}\mathbf{x} = \mathbf{x}^{*T}\lambda\mathbf{x}, \quad \mathbf{x}^T\mathbf{M}\mathbf{x}^* = \mathbf{x}^T\lambda^*\mathbf{x}^*. \tag{215}$$

The two Eqs. 215 are actually equal because

$$(\mathbf{x}^{*T}\mathbf{M}\mathbf{x})^T = \mathbf{x}^T\mathbf{M}\mathbf{x}^*. \tag{216}$$

These equations are producing scalars, and all scalars are equal to their own transpose. Therefore

$$\mathbf{x}^{*T}\lambda\mathbf{x} \Leftrightarrow \begin{Bmatrix} x_1^* & x_2^* & x_3^* \end{Bmatrix} \lambda \begin{Bmatrix} x_1 \\ x_2 \\ x_3 \end{Bmatrix} = \lambda x_1 x_1^* + \lambda x_2 x_2^* + \lambda x_3 x_3^*$$

$$= \lambda^* x_1 x_1^* + \lambda^* x_2 x_2^* + \lambda^* x_3 x_3^* = \begin{Bmatrix} x_1 & x_2 & x_3 \end{Bmatrix} \lambda^* \begin{Bmatrix} x_1^* \\ x_2^* \\ x_3^* \end{Bmatrix} = \mathbf{x}^T \lambda^* \mathbf{x}^*. \tag{217}$$

Rearranged,

$$(\lambda - \lambda^*)(x_1 x_1^* + x_2 x_2^* + x_3 x_3^*) = 0. \tag{218}$$

If this is true of all $x_i, x_i^*$, then

$$\lambda = \lambda^*, \tag{219}$$

meaning there is no complex part to the eigenvalues and so they are real.

That was the first property of interest of a real symmetric eigenproblem. The second property is that if the eigenvalues are distinct, then the eigenvectors are orthogonal. Consider two of the possible three eigenvalue/eigenvector pairs that serve as solutions to the same system matrix $\mathbf{M}$ in

$$\mathbf{M}\mathbf{x}_1 = \lambda_1 \mathbf{x}_1, \qquad \mathbf{M}\mathbf{x}_2 = \lambda_2 \mathbf{x}_2. \tag{220}$$

These imply

$$\mathbf{x}_2^T \mathbf{M}\mathbf{x}_1 = \mathbf{x}_2^T \lambda_1 \mathbf{x}_1, \qquad \mathbf{x}_1^T \mathbf{M}\mathbf{x}_2 = \mathbf{x}_1^T \lambda_2 \mathbf{x}_2, \tag{221}$$

or

$$\mathbf{x}_2^T \mathbf{M}\mathbf{x}_1 = \lambda_2 \mathbf{x}_2 \cdot \mathbf{x}_1, \qquad \mathbf{x}_1^T \mathbf{M}\mathbf{x}_2 = \lambda_2 \mathbf{x}_1 \cdot \mathbf{x}_2. \tag{222}$$

Subtracting,

$$\begin{aligned}(\lambda_1 - \lambda_2)(\mathbf{x}_1 \cdot \mathbf{x}_2) &= \mathbf{x}_2^T \mathbf{M}\mathbf{x}_1 - \mathbf{x}_1^T \mathbf{M}\mathbf{x}_2 \\ &= \mathbf{x}_2^T \mathbf{M}\mathbf{x}_1 - (\mathbf{x}_2^T \mathbf{M}\mathbf{x}_1)^T = 0\end{aligned} \tag{223}$$

where we know the whole expression is zero because the transpose of a scalar is itself. Therefore

$$(\lambda_1 - \lambda_2)(\mathbf{x}_1 \cdot \mathbf{x}_2) = 0, \tag{224}$$

and if it is assumed that the eigenvalues are distinct so that $\lambda_1 \neq \lambda_2$, then it must be that $\mathbf{x}_1 \cdot \mathbf{x}_2 = 0$, which is the definition of the two eigenvectors being orthogonal to one another.

## 2.10 Geometrical interpretation of the first invariant

Recall the principal invariants Eq. 209. The first of them is

$$\theta_1 = \mathrm{tr}\boldsymbol{\epsilon} \Leftrightarrow \epsilon_{11} + \epsilon_{22} + \epsilon_{33} = \epsilon_{ii}. \tag{225}$$

Remember that a diagonal strain component indicates the stretch in the direction of the coordinate the component represents. Note also that shear components do not induce volume change. So if the original volume of a cube is

$$V = l_1 l_2 l_3 \tag{226}$$

where $l$ are the side lengths, then the volume change is

$$\Delta V = l_1 \epsilon_{11} l_2 \epsilon_{22} l_3 \epsilon_{33}, \tag{227}$$

and the new volume is

$$\begin{aligned} V + \Delta V &= l_1(1+\epsilon_{11}) l_2(1+\epsilon_{22}) l_3(1+\epsilon_{33}) \\ &= l_1 l_2 l_3 (1+\epsilon_{11})(1+\epsilon_{22})(1+\epsilon_{33}) \\ &= l_1 l_2 l_3 (1+\epsilon_{22}+\epsilon_{11}+\epsilon_{11}\epsilon_{22})(1+\epsilon_{33}) \\ &= l_1 l_2 l_3 (1+\epsilon_{33}+\epsilon_{22}+\epsilon_{22}\epsilon_{33}+\epsilon_{11}+\epsilon_{11}\epsilon_{33}+\epsilon_{11}\epsilon_{22}+\epsilon_{11}\epsilon_{22}\epsilon_{33}) \\ &\approx l_1 l_2 l_3 (1+\epsilon_{11}+\epsilon_{22}+\epsilon_{33}) \end{aligned}$$

where such an approximation is made because products of small strain components are very small and so are considered negligible. Then

$$\begin{aligned} \Delta V = V + \Delta V - V &= l_1 l_2 l_3 (1+\epsilon_{11}+\epsilon_{22}+\epsilon_{33}) - l_1 l_2 l_3 \\ &= l_1 l_2 l_3 (\epsilon_{11}+\epsilon_{22}+\epsilon_{33}) = V\epsilon_{ii} = \Delta V. \end{aligned} \tag{228}$$

Therefore

$$\epsilon_{ii} = \frac{\Delta V}{V} = \theta_1 \tag{229}$$

which is the first invariant. So the first invariant can be interpreted as the volumetric strain, or the change in volume with respect to the original volume.

## 2.11 Finite deformation

In the past we have neglected products of strain components because of the assumption that they were so small that they could be considered negligible. Now though we wish to consider a more general derivation for larger deformations. Displacement

$$u_i = x_i' - x_i \tag{230}$$

If $dl$ is a small distance between $x_i$ and a neighboring point, then

$$dl^2 = dx_i dx_i, \quad dl'^2 = dx_i' dx_i'. \tag{231}$$

From Eq. 230,

$$dx_i' = dx_i + \underbrace{du_i} = dx_i + \underbrace{\frac{\partial u_i}{\partial x_j}} dx_j = dx_i + \underbrace{u_{i,j} dx_j} = dx_i'. \tag{232}$$

Substituting this into Eq. 231,

$$
\begin{aligned}
dl'^2 &= (dx_i + u_{i,j}dx_j)(dx_i + u_{i,k}dx_k) \\
&= dx_i dx_i + dx_i u_{i,k} dx_k + dx_i u_{i,j} dx_j + u_{i,j} u_{i,k} dx_j dx_k \\
&= dl^2 + u_{i,j} dx_i dx_j + \underbrace{u_{i,k} dx_i dx_k} + \underbrace{u_{i,j} u_{i,k} dx_j dx_k} \\
&= dl^2 + u_{i,j} dx_i dx_j + \underbrace{u_{j,i} dx_j dx_i}_{k\to i,\ i\to j} + \underbrace{u_{k,i} u_{k,j} dx_i dx_j}_{i\to k,\ k\to j,\ j\to i}.
\end{aligned} \tag{233}
$$

Therefore

$$
\begin{aligned}
dl'^2 - dl^2 &= u_{i,j} dx_i dx_j + u_{j,i} dx_j dx_i + u_{k,i} u_{k,j} dx_i dx_j \\
&= (u_{i,j} + u_{j,i} + u_{k,i} u_{k,j}) dx_i dx_j.
\end{aligned} \tag{234}
$$

Think of the physical meaning of LHS. It is measuring the squared difference in length between two neighboring points before and after a deformation. This is a rating of the deformation itself. It is measuring to what extent points in the body are separating or stretching. Units wise,

$$
dl'^2 - dl^2 \sim \text{meters}^2, \qquad \epsilon \sim \text{dimensionless}, \tag{235}
$$

so we multiply strain by a representative small box of area. Particularly

$$
dl'^2 - dl^2 = 2\epsilon_{ij} dx_i dx_j \sim \text{meters}^2. \tag{236}
$$

Strain in this case is

$$
\epsilon_{ij} = \frac{1}{2}\frac{dl'^2 - dl^2}{dx_i dx_j} \tag{237}
$$

which is one half the squared change in distance between two points with respect to the area of a square with sides defined by a small unit distance $dx$. Conceptually it is a rating of the extent that points on a body have separated after a deformation process with respect to the original configuration. Substituting Eq. 236 into Eq. 234,

$$
\epsilon_{ij} = \frac{1}{2}\Big(u_{i,j} + u_{j,i} + u_{k,i} u_{k,j}\Big) \tag{238}
$$

where $i, j$ are free and $k$ is dummy. Remember, dummy means you sum over all indices. Free is independent. So for example

$$
\begin{aligned}
\epsilon_{ij}\big|_{i=x,j=x} = \epsilon_{xx} &= \frac{1}{2}\Big(\underbrace{u_{x,x}}_{u_{i,j}} + \underbrace{u_{x,x}}_{u_{j,i}} + \underbrace{(u_{x,x}u_{x,x} + u_{y,x}u_{y,x} + u_{z,x}u_{z,x})}_{u_{k,i}u_{k,j}}\Big) \\
&= \frac{\partial u}{\partial x} + \frac{1}{2}\left(\left(\frac{\partial u}{\partial x}\right)^2 + \left(\frac{\partial v}{\partial x}\right)^2 + \left(\frac{\partial w}{\partial x}\right)^2\right)
\end{aligned} \tag{239}
$$

and

$$
\begin{aligned}
\epsilon_{zy} &= \frac{1}{2}\Big(u_{z,y} + u_{y,z} + u_{x,z}u_{x,y} + u_{y,z}u_{y,y} + u_{z,z}u_{z,y}\Big) \\
&= \frac{1}{2}\left(\frac{\partial w}{\partial y} + \frac{\partial v}{\partial z} + \frac{\partial u}{\partial z}\frac{\partial u}{\partial y} + \frac{\partial v}{\partial z}\frac{\partial v}{\partial y} + \frac{\partial w}{\partial z}\frac{\partial w}{\partial y}\right) = \epsilon_{yz} \quad \text{(symmetric)}.
\end{aligned} \tag{240}
$$

Recall that for the small strain assumption the product terms can be neglected.

# 3 Stress

In terms of force, a body can be acted upon by

- body forces, which act at every point on the body, such as gravity, and
- surface forces, which act only on surface points, such as traction or hydrostatic pressure.

If a body's density is $\rho$ and its volume is $V$, then its mass is $\rho V$ or

$$m = \int_V \rho dV. \tag{241}$$

Therefore gravitational force

$$\mathbf{F}_g = m\mathbf{g} = \int_V \rho \mathbf{g} dV. \tag{242}$$

In general the net body force is $\mathbf{f}$ is

$$\int_V \rho \mathbf{f} dV \tag{243}$$

and torque/moment is

$$\mathbf{r} \times \mathbf{F} = \int_V \mathbf{r} \times \rho \mathbf{f} dV = ||r||||\rho f|| \sin\theta \hat{\mathbf{n}} \tag{244}$$

if $\mathbf{r} = x_i \mathbf{e}_i$ is the position of a point on the body with respect to the origin and $\theta$ is the angle between $\mathbf{r}$ and $\mathbf{F}$ (really $\mathbf{f}$).

Stress is force per area. If a stress vector is $\mathbf{t}$ and the area of a surface is $S$, then surface force is $\mathbf{t}S$ or

$$\oint_S \mathbf{t} dS. \tag{245}$$

Summing the surface forces with the body forces, the net force is

$$\int_V \rho \mathbf{f} dV + \oint_S \mathbf{t} dS \tag{246}$$

and the net torque/moment is

$$\int_V \mathbf{r} \times \rho \mathbf{f} dV + \oint_S \mathbf{r} \times \mathbf{t} dS. \tag{247}$$

Surface force $\mathbf{t} = \mathbf{t}(\mathbf{x}, \mathbf{n})$ depends on the location of a point on the surface $\mathbf{x}$ and on the vector which is normal to the surface $\mathbf{n}$. Let $\sigma_{ij}$ (NO SUM, not a tensor), be the $j$th component of $\mathbf{t}$ if the surface was normal to $\mathbf{e}_i$ (NO SUM). That is

$$\sigma_{ij} = \mathbf{e}_j \cdot \mathbf{t}(\mathbf{x}, \mathbf{e}_i) \quad \text{(no sum)}. \tag{248}$$

For example

$$\sigma_{12} = \mathbf{e}_2 \cdot \begin{Bmatrix} t_{1,(1)} \\ t_{2,(1)} \\ t_{3,(1)} \end{Bmatrix} = \{0 \quad 1 \quad 0\} \begin{Bmatrix} t_{1,(1)} \\ t_{2,(1)} \\ t_{3,(1)} \end{Bmatrix} = t_{2,(1)} \tag{249}$$

where in this example, the result $t_{2,(1)}$ denotes the 2nd component of the stress vector acting on a surface which is normal to $\mathbf{e}_1$. In three dimensions

$$\begin{bmatrix} \sigma_{11} & \sigma_{12} & \sigma_{13} \\ \sigma_{21} & \sigma_{22} & \sigma_{23} \\ \sigma_{31} & \sigma_{32} & \sigma_{33} \end{bmatrix} = \begin{bmatrix} t_{1,(1)} & t_{2,(1)} & t_{3,(1)} \\ t_{1,(2)} & t_{2,(2)} & t_{3,(2)} \\ t_{1,(3)} & t_{2,(3)} & t_{3,(3)} \end{bmatrix} \quad \text{where } t \sim t_{\text{component of } t, \text{ (direction of vector normal to surface)}}. \tag{250}$$

A visualization of the stress components in 2D is Fig. 2. For example consider the right face. The direction of the vector normal to this surface is $\mathbf{e}_1$ or $x \to t_{\_\_\_\_,(x)}$. Then $\sigma_{xx} = t_{1,(1)}$ is the 1st or $x-$ component of $\mathbf{t}$ for that surface $\to t_{x,(\_\_\_\_)}$. On the other hand the stress component $\sigma_{x,y} = t_{2,(1)}$ is the 2nd or $y-$ component of $\mathbf{t}$ for the face whose normal points in the direction $\mathbf{e}_1$ or $x$. For normal components the stress vector always

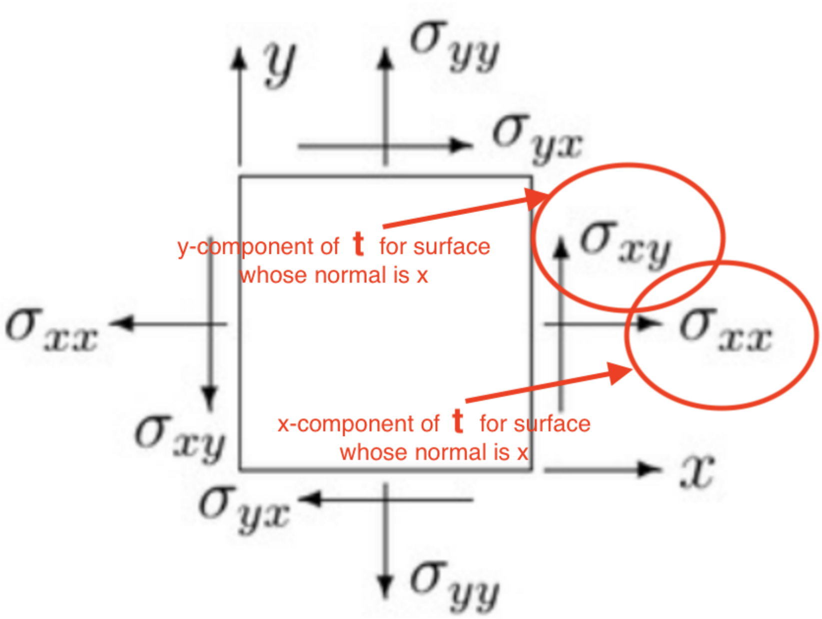



Figure 2: Stress components in 2D and their associated signs (+/-)

points away from the applied surface. Whether or not a shear component is positive is directly related to whether or not the vector normal to the surface is positive. So, the bottom face will have a negative shear component because the face's normal vector points in the direction $-y$. However the right face will have a positive shear component because the face's normal vector points in the direction $+x$.

Thus far we have only considered $\mathbf{n} = \{\mathbf{e}_1, \ \mathbf{e}_2, \ \mathbf{e}_3\}$, but we wish to establish a general relationship between $\mathbf{t}(\mathbf{x}, \mathbf{n})$ and $\sigma_{ij}$ (no sum) for any $\mathbf{n}$. Consider Fig. 3, a four faced pyramid shape (which is called a tetrahedron) with vertices $\{\mathcal{O}, (x_1, 0, 0), (0, x_2, 0), (0, 0, x_3)\}$.

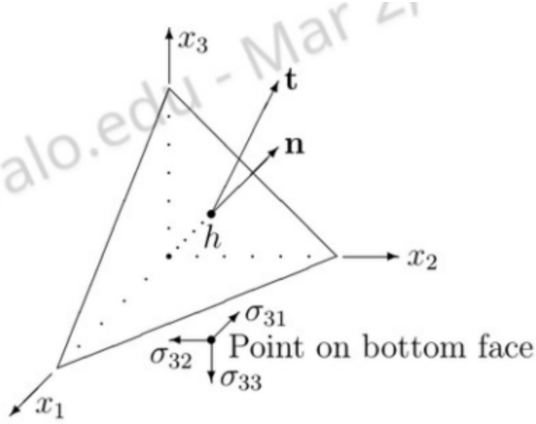



Figure 3: Stresses on trirectangular (three right triangle) tetrahedron

Let the height $h$ $(\neq x_3)$ be the shortest distance between origin $\mathcal{O}$ and the closest point on the "inclined" surface, which is the triangle formed by $(x_1, 0, 0), (0, x_2, 0), (0, 0, x_3)$. If $\Delta S$ is the area of this triangle and $\Delta S_i$ is the area of each of the three right triangles with unit normal $\mathbf{e}_i$, then

$$\Delta S_i = n_i \Delta S \Leftrightarrow \begin{Bmatrix} \text{area of triangle with unit normal } x \\ \text{area of triangle with unit normal } y \\ \text{area of triangle with unit normal } z \end{Bmatrix} = \begin{Bmatrix} \Delta S_1 \\ \Delta S_2 \\ \Delta S_3 \end{Bmatrix} = \begin{Bmatrix} n_1 \Delta S \\ n_2 \Delta S \\ n_3 \Delta S \end{Bmatrix}, \tag{251}$$

where $\mathbf{n}$ is the vector normal to the inclined surface $\Delta S$, and the components of $\mathbf{n}$ are called the direction cosines.

Suppose $h$ is small. In this case the volume is small. We assume density is constant. Therefore mass is also small. Therefore net force is also small. If we assume there is no net force, then using Eq. 246 (net force $= \int_V \rho \mathbf{f} dV + \oint_S \mathbf{t} dS =$ body + surfaces),

$$\Leftrightarrow \rho f_i V + t_i \Delta S -^{?} \underbrace{\sigma_{ji} \Delta S_j}_{i\text{th component of } \mathbf{t} \text{ for surface whose normal is } \mathbf{e}_j} = 0. \tag{252}$$

For all tetrahedra, $V = h\Delta S/3$. Making this substitution as well as Eq. 251 ($\Delta S_j = n_j \Delta S$),

$$\begin{aligned} &\rho f_i h \Delta S/3 + t_i \Delta S - \sigma_{ji}(n_j \Delta S) = 0 \\ &\Leftrightarrow \rho f_i h \Delta S/3 + \Delta S(t_i - \sigma_{ji} n_j) = 0. \end{aligned} \tag{253}$$

We have supposed $h$ is small, so $h \longrightarrow 0$. Because of this,

$$\Delta S(t_i - \sigma_{ji} n_j) = 0 \longrightarrow t_i - \sigma_{ji} n_j = 0$$

$$\longrightarrow t_i = \sigma_{ji} n_j \Leftrightarrow \begin{Bmatrix} t_1 \\ t_2 \\ t_3 \end{Bmatrix} = \begin{Bmatrix} \sigma_{11} n_1 + \sigma_{21} n_2 + \sigma_{31} n_3 \\ \sigma_{12} n_1 + \sigma_{22} n_2 + \sigma_{32} n_3 \\ \sigma_{13} n_1 + \sigma_{23} n_2 + \sigma_{33} n_3 \end{Bmatrix}. \tag{254}$$

The way to interpret Eq. 254 is this. On an arbitrary surface, the components of the stress vector for that surface is determined by the set of the individual stress components on that surface and the direction in which the vector normal to the surface points. This relationship is important because even though it is easy to infer $\sigma_{ij}$ based on $\mathbf{t}(\mathbf{x}, \mathbf{e}_i)$, we are not usually given $\mathbf{t}$. Instead we are usually given $[\sigma_{ij}] \Leftrightarrow \boldsymbol{\sigma}$ and from that are needing to figure out $\mathbf{t}(\mathbf{x}, \mathbf{n})$.

## 3.1 Momentum equation

The net force on the body is the product of the body's mass and acceleration, according to Newton. Getting the net force from Eq. 246,

$$\int_V \rho \mathbf{f} dV + \oint_S \mathbf{t} dS = m\mathbf{a} = \int_V \ddot{\mathbf{u}} \rho dV. \tag{255}$$

In index notation,

$$\int_V \rho f_i dV + \oint_S \underbrace{t_i} dS = \int_V \ddot{u}_i \rho dV. \tag{256}$$

Substituting in Eq. 254,

$$\int_V \rho f_i dV + \oint_S \underbrace{\sigma_{ji} n_j} dS = \int_V \ddot{u}_i \rho dV. \tag{257}$$

Because of the divergence theorem Eq. 95 ($\oint_S [\circ]_j n_j dS = \int_V [\circ]_{j,j} dV$),

$$\int_V \rho f_i dV + \int_V \sigma_{ji,j} dV = \int_V \ddot{u}_i \rho dV. \tag{258}$$

Rearranged,

$$\int_V \rho f_i dV + \int_V \sigma_{ji,j} dV - \int_V \ddot{u}_i \rho dV \tag{259}$$

implies

$$\int_V (\sigma_{ji,j} + \rho f_i - \rho \ddot{u}_i) = 0 \tag{260}$$

implies

$$\sigma_{ji,j} + \rho f_i - \rho \ddot{u}_i = 0 \longrightarrow \sigma_{ji,j} + \rho f_i = \rho \ddot{u}_i. \tag{261}$$

This is called the momentum equation because it arises from the balance of net force, which is related to linear momentum.

## 3.2 Angular momentum

The angular analog to force is torque, and the angular analog to linear momentum is angular momentum. So the sum of the torques implies a balance of angular momentum. Getting the net torque from Eq. 247,

$$\int_V \mathbf{r} \times \rho \mathbf{f} dV + \oint_S \mathbf{r} \times \mathbf{t} dS = \mathbf{r} \times m\mathbf{a} = \mathbf{r} \times m \frac{d}{dt}(\mathbf{v}) = \mathbf{r} \times m \frac{d}{dt}(\dot{\mathbf{u}}) = \frac{d}{dt} \int_V \mathbf{r} \times \rho \dot{\mathbf{u}} dV \tag{262}$$

where $u$ is displacement and $\mathbf{r} = x_i \mathbf{e}_i$. In index notation,

$$\int_V \mathbf{r} \times \rho \mathbf{f} dV + \oint_S \mathbf{r} \times \mathbf{t} dS = \frac{d}{dt} \int_V \mathbf{r} \times \rho \dot{\mathbf{u}} dV$$

$$\iff \int_V \rho x_i f_j \epsilon_{ijk} dV + \oint x_i (t_j) \epsilon_{ijk} dS = \int_V \rho x_i \ddot{u}_j \epsilon_{ijk} dV \tag{263}$$

$$\iff \int_V \rho x_i f_j \epsilon_{ijk} dV + \oint x_i (\sigma_{lj} n_l) \epsilon_{ijk} dS = \int_V \rho x_i \ddot{u}_j \epsilon_{ijk} dV \tag{264}$$

$$\iff \int_V \rho x_i f_j \epsilon_{ijk} dV + \int_V (x_i \sigma_{lj} \epsilon_{ijk})_{,l} dV = \int_V \rho x_i \ddot{u}_j \epsilon_{ijk} dV \tag{265}$$

$$\iff \int_V \rho x_i f_j \epsilon_{ijk} dV + \int_V (x_{i,l} \sigma_{lj} \epsilon_{ijk} + x_i \sigma_{lj,l} \epsilon_{ijk}) dV = \int_V \rho x_i \ddot{u}_j \epsilon_{ijk} dV \tag{266}$$

$$\iff \int_V \rho x_i f_j \epsilon_{ijk} dV + \int_V (\delta_{il} \sigma_{lj} \epsilon_{ijk} + x_i \sigma_{lj,l} \epsilon_{ijk}) dV = \int_V \rho x_i \ddot{u}_j \epsilon_{ijk} dV \tag{267}$$

$$\iff \int_V \rho x_i f_j \epsilon_{ijk} dV + \int_V (\sigma_{ij} \epsilon_{ijk} + x_i \sigma_{lj,l} \epsilon_{ijk}) dV = \int_V \rho x_i \ddot{u}_j \epsilon_{ijk} dV \tag{268}$$

$$\iff \int_V [\rho x_i f_j \epsilon_{ijk} + \sigma_{ij} \epsilon_{ijk} + x_i \sigma_{lj,l} \epsilon_{ijk} - \rho x_i \ddot{u}_j \epsilon_{ijk}] dV = 0 \tag{269}$$

$$\iff \int_V [x_i \epsilon_{ijk} (\rho f_j + \sigma_{lj,l} - \rho \ddot{u}_j) + \sigma_{ij} \epsilon_{ijk}] dV = 0. \tag{270}$$

Then because of Eq. 261 ($\sigma_{lj,l} + \rho f_j = \rho \ddot{u}_j$), the parenthetical term cancels, leaving

$$\iff \int_V \sigma_{ij} \epsilon_{ijk} dV = 0 \longrightarrow \sigma_{ij} \epsilon_{ijk} = 0. \tag{271}$$

Switching indices,

$$\sigma_{ji} \epsilon_{jik} = 0. \tag{272}$$

By definition,

$$\epsilon_{jik} = -\epsilon_{ijk}. \tag{273}$$

Therefore

$$\sigma_{ji} \epsilon_{jik} = -\sigma_{ji} \epsilon_{ijk} = \sigma_{ij} \epsilon_{ijk} \longrightarrow \sigma_{ij} = \sigma_{ji}. \tag{274}$$

This means stress $\boldsymbol{\sigma}$ is symmetric. Reconsidering then Eqs. 261 and 254,

$$\sigma_{ij,j} + \rho f_i = \rho \ddot{u}_i \Leftrightarrow \begin{Bmatrix} \sigma_{xx,x} + \rho f_x + \sigma_{xy,y} + \sigma_{xz,z} \\ \sigma_{yx,x} + \rho f_y + \sigma_{yy,y} + \sigma_{yz,z} \\ \sigma_{zx,x} + \rho f_z + \sigma_{zy,y} + \sigma_{zz,z} \end{Bmatrix} = \begin{Bmatrix} \rho \ddot{u} \\ \rho \ddot{v} \\ \rho \ddot{w} \end{Bmatrix}, \qquad t_i = \sigma_{ij} n_j \Leftrightarrow \mathbf{t} = \boldsymbol{\sigma} \mathbf{n}. \tag{275}$$

## 3.3 Stress as a tensor

Here is proof that $\boldsymbol{\sigma}$ is a rank two tensor. If traction $\mathbf{t} = \boldsymbol{\sigma}\mathbf{n}$ in the original coordinate system $x_i$, then in another coordinate system $\hat{x}_i$,

$$\hat{\mathbf{t}} = \hat{\boldsymbol{\sigma}}\hat{\mathbf{n}}. \tag{276}$$

For rank one tensors,

$$\hat{\mathbf{t}} = \mathbf{R}\mathbf{t}, \quad \hat{\mathbf{n}} = \mathbf{R}\mathbf{n} \longrightarrow \mathbf{R}^T\hat{\mathbf{t}} = \mathbf{t}, \quad \mathbf{R}^T\hat{\mathbf{n}} = \mathbf{n}. \tag{277}$$

Substituting back into Eq. 275b ($\mathbf{t} = \boldsymbol{\sigma}\mathbf{n}$),

$$\mathbf{R}^T\hat{\mathbf{t}} = \boldsymbol{\sigma}\mathbf{R}^T\hat{\mathbf{n}} \longrightarrow \mathbf{R}\mathbf{R}^T\hat{\mathbf{t}} = \hat{\mathbf{t}} = \mathbf{R}\boldsymbol{\sigma}\mathbf{R}^T\hat{\mathbf{n}}. \tag{278}$$

Substituting this result into Eq. 276,

$$\hat{\boldsymbol{\sigma}} = \mathbf{R}\boldsymbol{\sigma}\mathbf{R}^T\hat{\mathbf{n}}, \tag{279}$$

which establishes $\boldsymbol{\sigma}$ as a rank two/second order tensor. $\boldsymbol{\sigma}$ is symmetric and real. As as shown in Sec. 2.9, the two properties that follow from this are (1) the characteristic eigenproblem must yield real eigenvalues and (2) if the eigenvalues are distinct then the eigenvectors are mutually orthogonal. The characteristic eigenproblem to determine principal stresses/eigenvalues $\hat{\sigma}$ is

$$\boldsymbol{\sigma}\mathbf{x} = \hat{\sigma}\mathbf{x}. \tag{280}$$

The stress vector $\mathbf{t} = \boldsymbol{\sigma}\mathbf{n}$ is a vector normal to the surface $\mathbf{n}$ which is transformed by tensor $\boldsymbol{\sigma}$. Therefore this vector $\mathbf{t}$ will not necessarily point in the direction of $\mathbf{n}$. To find out what component of $\mathbf{t}$ points in $\mathbf{n}$, normal stress

$$\sigma_n = \mathbf{n}\cdot\mathbf{t} = \mathbf{n}\cdot\boldsymbol{\sigma}\mathbf{n} = \mathbf{n}^T\boldsymbol{\sigma}\mathbf{n} \tag{281}$$

$$= \begin{Bmatrix} n_1 & n_2 & n_3 \end{Bmatrix} \begin{bmatrix} \sigma_{11} & \sigma_{12} & \sigma_{13} \\ \sigma_{21} & \sigma_{22} & \sigma_{23} \\ \sigma_{31} & \sigma_{32} & \sigma_{33} \end{bmatrix} \begin{Bmatrix} n_1 \\ n_2 \\ n_3 \end{Bmatrix} = \begin{Bmatrix} n_1 & n_2 & n_3 \end{Bmatrix} \begin{Bmatrix} \sigma_{11}n_1 + \sigma_{12}n_2 + \sigma_{13}n_3 \\ \sigma_{21}n_1 + \sigma_{22}n_2 + \sigma_{23}n_3 \\ \sigma_{31}n_1 + \sigma_{32}n_2 + \sigma_{33}n_3 \end{Bmatrix} \tag{282}$$

$$= n_1(\sigma_{11}n_1 + \sigma_{12}n_2 + \sigma_{13}n_3) + n_2(\sigma_{21}n_1 + \sigma_{22}n_2 + \sigma_{23}n_3) + n_3(\sigma_{31}n_1 + \sigma_{32}n_2 + \sigma_{33}n_3) \tag{283}$$

$$\Leftrightarrow n_i\sigma_{ij}n_j = \sigma_n \tag{284}$$

is the normal component of stress on that surface. This calculation of $\boldsymbol{\sigma}$ is permissible for all second order tensors, so normal strain

$$\epsilon_n = n_i\epsilon_{ij}n_j. \tag{285}$$

## 3.4 Mean stress in a deformed body

If for a body subject to Eq. 275a ($\sigma_{ij,j} + \rho f_i = \rho \ddot{u}_i$) there are no body forces and if the body is static, then $\ddot{u} = 0$ and $f_i = 0$, meaning

$$\sigma_{ij,j} = 0. \tag{286}$$

This implies

$$0 = \int_V \sigma_{ij,j} x_k dV = \int_V [(\sigma_{ij} x_k)_{,j} - \sigma_{ij} x_{k,j}] dV \tag{287}$$

$$= \int_V [(\sigma_{ij} x_k)_{,j} - \sigma_{ij} \delta_{kj}] dV \tag{288}$$

$$= \int_V (\sigma_{ij} x_k)_{,j} dV - \int_V \sigma_{ik} dV = 0 \longrightarrow \int_V (\sigma_{ij} x_k)_{,j} dV = \int_V \sigma_{ik} dV \tag{289}$$

$$\longrightarrow \frac{1}{V} \int_V (\sigma_{ij} x_k)_{,j} dV = \frac{1}{V} \int_V \sigma_{ik} dV. \tag{290}$$

We define mean stress over volume as

$$\bar{\sigma}_{ik} = \frac{1}{V} \int_V \sigma_{ik} dV. \tag{291}$$

Substituting,

$$\bar{\sigma}_{ik} = \frac{1}{V} \int_V (\sigma_{ij} x_k)_{,j} dV = \frac{1}{V} \oint_S \sigma_{ij} x_k n_j dS = \frac{1}{V} \oint_S t_i x_k dS. \tag{292}$$

Changing indices,

$$\bar{\sigma}_{ij} = \frac{1}{V} \oint_S t_i x_j dS. \tag{293}$$

Note that the product $t_i x_j \leftrightarrow \mathbf{t} \otimes \mathbf{x}$ produces a second order tensor. Also, we understand that $[\bar{\sigma}_{ij}]$ is symmetric. Therefore the symmetric component of this tensor is the only component, and that is

$$\bar{\sigma}_{ij} = \text{sym}(\bar{\sigma}_{ij}) = \frac{1}{2}\left(\frac{1}{V} \oint_S [(t_i x_j) + (t_i x_j)^T] dS\right) = \frac{1}{2V} \oint_S (t_i x_j + t_j x_i) dS = \bar{\sigma}_{ij}. \tag{294}$$

The utility of this is that you can solve for the mean value of the stress tensor using only surface tractions.

## 3.5 Fluid structure interface condition

The stress normal to an elastic structure in contact with an inviscid fluid is just the inward pressure. That is,

$$\sigma_n = -p. \tag{295}$$

# 4 Equations of elasticity

## 4.1 Hooke's law

It is necessary to relate the forces at hand, or stress, to the resulting kinematics, or strain. Some materials obey Hooke's Law, which is

$$\sigma = E\epsilon, \tag{296}$$

where $E$ is called Young's modulus. In general in three dimensions, there are nine stress/strain components. The generalized Hooke's Law is then

$$\sigma_{ij} + C_{ijkl}\epsilon_{kl} \tag{297}$$

where the elastic constants are contained in $C$. If $i, j$ are fixed, then there are nine terms. To prove $C_{ijkl}$ is a fourth order tensor, let

$$\bar{\boldsymbol{\sigma}} = \mathbf{R}\boldsymbol{\sigma}\mathbf{R}^T \longrightarrow \boldsymbol{\sigma} = \mathbf{R}^T\bar{\boldsymbol{\sigma}}\mathbf{R} \tag{298}$$

$$\longrightarrow \sigma_{ij} = \underbrace{R_{ki}\bar{\sigma}_{kl}R_{lj}} = \underbrace{C_{ijkl}R_{mk}R_{nl}\bar{\epsilon}_{mn}} = C_{ijkl}\epsilon_{kl} \tag{299}$$

$$\longrightarrow R_{oi}R_{pj}\underbrace{R_{ki}\bar{\sigma}_{kl}R_{lj}} = R_{oi}R_{pj}\underbrace{R_{mk}R_{nl}C_{ijkl}\bar{\epsilon}_{mn}}. \tag{300}$$

Note that indices switch with the transpose operator. Therefore if I were to multiply $\mathbf{RR}$ the result is $R_{ij}R_{jk}$, but if I multiply $\mathbf{RR}^T = \mathbf{I}$ it is $R_{ij}R_{kj} = \delta_{ik}$. Applying this idea to Eq. 300,

$$\delta_{ok}\delta_{pl}\bar{\sigma}_{kl} = R_{oi}R_{pj}R_{mk}R_{nl}C_{ijkl}\bar{\epsilon}_{mn}. \tag{301}$$

$$\longrightarrow \bar{\sigma}_{op} = \bar{C}_{opmn}\bar{\epsilon}_{mn}. \tag{302}$$

Because $C_{opmn} = R_{oi}R_{pj}R_{mk}R_{nl}C_{ijkl}$, it is a fourth order tensor. There are four indices where each can be one of three numbers. Therefore it possesses $3^4 = 81$ constants. However, note that since stress and strain are symmetric,

$$\sigma_{ij} = \sigma_{ji} \longrightarrow C_{ijkl}\epsilon_{kl} = C_{jikl}\epsilon_{kl} \longrightarrow C_{ijkl} = C_{jikl}, \tag{303}$$

$$C_{ijkl}\epsilon_{kl} = C_{ijlk}\epsilon_{lk} = C_{ijlk}\epsilon_{kl} \longrightarrow C_{ijkl} = C_{ijlk}. \tag{304}$$

The strain and stress tensors possess six unique constants, where the lower left triangle of entries $\sigma_{21}, \sigma_{31}, \sigma_{32}$ are nonunique with respect to the upper right triangle $\sigma_{12}, \sigma_{13}, \sigma_{23}$. $C$ is then tasked with relating six unique stress constants with six unique strain constants. Because stress and strain must be symmetric, that is the most uniqueness possible, and so there is no possible further variation than that. Therefore $C$ cannot contain more than $6^2 = 36$ unique constants.

Consider the impracticality of writing down a fourth order tensor. But because of the generalized Hooke's Law ($\sigma_{ij} = C_{ijkl}\epsilon_{kl} \iff \{6 \text{ constants}\} = \{36\}\{6\} \longrightarrow \{6 \times 1\} =$

$\{6 \times 6\}\{6 \times 1\}$), we can rearrange the unique constants like

$$\begin{Bmatrix} \sigma_{11} \\ \sigma_{22} \\ \sigma_{33} \\ \sigma_{12} \\ \sigma_{23} \\ \sigma_{13} \end{Bmatrix} = \begin{bmatrix} C_{1111} & C_{1122} & C_{1133} & C_{1112} & C_{1123} & C_{1113} \\ C_{2211} & C_{2222} & C_{2233} & C_{2212} & C_{2223} & C_{2213} \\ C_{3311} & C_{3322} & C_{3333} & C_{3312} & C_{3323} & C_{3313} \\ C_{1211} & C_{1222} & C_{1233} & C_{1212} & C_{1223} & C_{1213} \\ C_{2311} & C_{2322} & C_{2333} & C_{2312} & C_{2323} & C_{2313} \\ C_{1311} & C_{1322} & C_{1333} & C_{1312} & C_{1323} & C_{1313} \end{bmatrix} \begin{Bmatrix} \epsilon_{11} \\ \epsilon_{22} \\ \epsilon_{33} \\ 2\epsilon_{12} \\ 2\epsilon_{23} \\ 2\epsilon_{13} \end{Bmatrix} = [\mathbf{C}] \begin{Bmatrix} \epsilon_{11} \\ \epsilon_{22} \\ \epsilon_{33} \\ \gamma_{12} \\ \gamma_{23} \\ \gamma_{13} \end{Bmatrix}. \tag{305}$$

Note the duplication on off diagonal terms. This is because $C_{ijkl} = C_{ijlk},\ \epsilon_{kl} = \epsilon_{lk}$, so for some unique $ij$, $\epsilon/C$ terms will accumulate. For example

$$\begin{aligned} \sigma_{12} = C_{1211}\epsilon_{11} + \underbrace{C_{1212}\epsilon_{12}}_{\mathbf{I}} + \underbrace{C_{1213}\epsilon_{13}}_{\mathbf{II}} + \underbrace{C_{1221}\epsilon_{21}}_{\mathbf{I}} + C_{1222}\epsilon_{22} + \underbrace{C_{1223}\epsilon_{23}}_{\mathbf{III}} + \underbrace{C_{1231}\epsilon_{31}}_{\mathbf{II}} + \underbrace{C_{1232}\epsilon_{32}}_{\mathbf{III}} + C_{1233}\epsilon_{33} \\ = C_{1211}\epsilon_{11} + 2C_{1212}\epsilon_{12} + 2C_{1213}\epsilon_{13} + C_{1222}\epsilon_{22} + 2C_{1223}\epsilon_{23} + C_{1233}\epsilon_{33} \\ = C_{1211}\epsilon_{11} + C_{1222}\epsilon_{22} + C_{1233}\epsilon_{33} + C_{1212}(2\epsilon_{12}) + C_{1223}(2\epsilon_{23}) + C_{1213}(2\epsilon_{13}). \end{aligned} \tag{306}$$

The same can be shown for any $i, j$.

Because Eq. 305 takes the shape of some $a_i = M_{ij}b_j$, it is not unreasonable to rename the components as

$$\begin{Bmatrix} \sigma_1 \\ \sigma_2 \\ \sigma_3 \\ \sigma_4 \\ \sigma_5 \\ \sigma_6 \end{Bmatrix} = \begin{bmatrix} C_{11} & C_{12} & C_{13} & C_{14} & C_{15} & C_{16} \\ C_{21} & C_{22} & C_{23} & C_{24} & C_{25} & C_{26} \\ C_{31} & C_{32} & C_{33} & C_{34} & C_{35} & C_{36} \\ C_{41} & C_{42} & C_{43} & C_{44} & C_{45} & C_{46} \\ C_{51} & C_{52} & C_{53} & C_{54} & C_{55} & C_{56} \\ C_{61} & C_{62} & C_{63} & C_{64} & C_{65} & C_{66} \end{bmatrix} \begin{Bmatrix} \epsilon_1 \\ \epsilon_2 \\ \epsilon_3 \\ \epsilon_4 \\ \epsilon_5 \\ \epsilon_6 \end{Bmatrix} \tag{307}$$

where $2\epsilon_{12} = \epsilon_4$, NOT $2\epsilon_{12} = 2\epsilon_4$.

## 4.2 Strain energy

Hooke's law essentially says that in a spring (elastomer), the relationship between stress/-force and strain/displacement is linear. A rod in uniaxial tension has elastic behavior and so acts like a spring. Another way to write Hooke's law is

$$F = ku, \tag{308}$$

where $F$ is force, $u$ is displacement, and $k$ is a spring constant. As the displacement $u$ increases, so does $F$, with a slope of $k$. This is visualized in Fig. 4. As is shown in the figure, let us define $F(\bar{u}) = \bar{F} = k\bar{u}$. Now the work done in moving from $u = 0$ to $u = \bar{u}$ is

$$W = F\Delta u = \int_0^{\bar{u}} F(u)du = \int_0^{\bar{u}} kudu = \frac{1}{2}ku^2\Big|_0^{\bar{u}} = \frac{1}{2}k\bar{u}^2 = \frac{1}{2}(k\bar{u})\bar{u} = \frac{1}{2}\bar{F}\bar{u}. \tag{309}$$

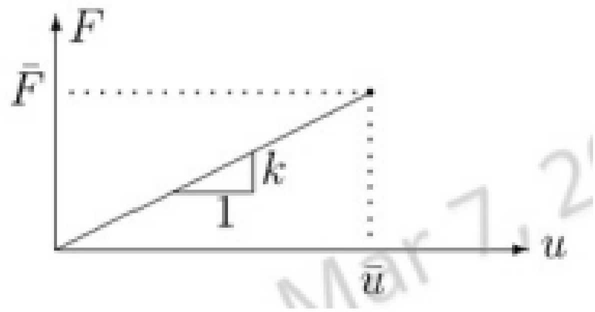



Figure 4: Force displacement curve for rod in uniaxial tension

Recall that stress is force per area $\sigma = \bar{F}/A$ and strain is change in length over original length $\epsilon = \bar{u}/L$. So,

$$W = \frac{1}{2}\bar{F}\bar{u} = \frac{1}{2}(\sigma A)(\epsilon L) = \frac{1}{2}\sigma\epsilon V, \tag{310}$$

where $V = AL$ in that the rod volume is its cross sectional area times its length.

Work is defined as the energy transferred to an object by applying a force across a displacement. So it is also a measure of internal energy, which we call strain energy. Then strain energy density

$$w = W/V = \frac{1}{2}\sigma\epsilon \tag{311}$$

is work/strain energy per unit volume. In just 1D,

$$\frac{\partial w}{\partial \epsilon_{xx}} = \frac{\partial}{\partial \epsilon_{xx}}\left(\frac{1}{2}\sigma_{xx}\epsilon_{xx}\right) = \frac{\partial}{\partial \epsilon_{xx}}\left(\frac{1}{2}E\epsilon_{xx}^2\right) = E\epsilon_{xx} = \sigma_{xx}. \tag{312}$$

This means the derivative of strain energy with respect to the strain is the stress. In other words, the way in which the density of energy changes in the rod based on changes in its shape is a rating of how much stress is being applied. A greater magnitude of stress will cause the energy density to change more rapidly as the shape deforms. Now in 2- or 3D,

$$w = \frac{1}{2}\sigma_{ij}\epsilon_{ij} \Leftrightarrow \frac{1}{2}\boldsymbol{\sigma} : \boldsymbol{\epsilon}, \tag{313}$$

or

$$w = \underbrace{\frac{1}{2}C_{ijkl}\epsilon_{kl}\epsilon_{ij}}_{\text{Eq. 305}} \Longleftrightarrow w = \underbrace{\frac{1}{2}C_{ij}\epsilon_i\epsilon_j}_{\text{Eq. 307}} = \frac{1}{2}\epsilon_i C_{ij}\epsilon_j \Leftrightarrow \frac{1}{2}\epsilon \cdot \mathbf{C}\boldsymbol{\epsilon}. \tag{314}$$

Because of Eq. 73 ($\mathbf{x} \cdot \mathbf{M}\mathbf{x} = \mathbf{x} \cdot \text{sym}[\mathbf{M}]\mathbf{x}$), $C_{ij}$ must be symmetrical in Eq. 314. This means

$$\underbrace{C_{ij} = C_{ji}}_{\text{Eq. 307}} \Longleftrightarrow \underbrace{C_{ijkl} = C_{klij}}_{\text{Eq. 305}}. \tag{315}$$

Differentiating Eq. 314 with respect to some strain component,

$$\frac{\partial w}{\partial \epsilon_{mn}} = \frac{1}{2}C_{ijkl}\frac{\partial \epsilon_{kl}}{\partial \epsilon_{mn}}\epsilon_{ij} + \frac{1}{2}C_{ijkl}\epsilon_{kl}\frac{\partial \epsilon_{ij}}{\partial \epsilon_{mn}} = \frac{1}{2}C_{ijkl}\delta_{km}\delta_{ln}\epsilon_{ij} + \frac{1}{2}C_{ijkl}\epsilon_{kl}\delta_{im}\delta_{jn}$$

$$= \frac{1}{2}C_{ijmn}\epsilon_{ij} + \frac{1}{2}C_{mnkl}\epsilon_{kl} = \frac{1}{2}C_{mnij}\epsilon_{ij} + \frac{1}{2}C_{mnij}\epsilon_{ij} = \underbrace{C_{mnij}\epsilon_{ij}} = \sigma_{mn}. \tag{316}$$

Differentiating again,

$$\frac{\partial^2 w}{\partial\epsilon_{mn}\partial\epsilon_{kl}} = \frac{\partial}{\partial\epsilon_{kl}}[C_{mnij}\epsilon_{ij}] = C_{mnij}\frac{\partial\epsilon_{ij}}{\partial\epsilon_{kl}} = C_{mnij}\delta_{ik}\delta_{jl} = C_{mnkl}. \tag{317}$$

Differentiating in the opposite order,

$$\frac{\partial^2 w}{\partial\epsilon_{kl}\partial\epsilon_{mn}} = C_{klmn} \longrightarrow C_{mnkl} = C_{klmn}. \tag{318}$$

To summarize, if there exists a $w$ that obeys

$$\frac{\partial w}{\partial\epsilon_{ij}} = \sigma_{ij}, \tag{319}$$

then Eq. 318 is true. So the existence of strain energy implies the symmetry of $C$. So since strain energy exists, then $C_{ij}$ is symmetric and so there are at most 6+5+4+3+2+1=21 unique upper triangular elastic constants in Eq. 307.

## 4.3 Material symmetry

The most anisotropic material possesses 21 elastic constants, but further symmetries of some materials permit further reduction. For instance consider a material which is symmetric in the $xy$ plane. This means some coordinate transformations such as

$$\begin{Bmatrix} \hat{x}_1 = x_1 \\ \hat{x}_2 = x_2 \\ \hat{x}_3 = -x_3 \end{Bmatrix} \longrightarrow \mathbf{R} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{bmatrix} \Leftrightarrow R_{ij} = \begin{Bmatrix} 1 \\ 1 \\ -1 \end{Bmatrix} \delta_{ij} \text{ (NO SUM)} = a_i\delta_{ij}. \tag{320}$$

would not vary in the elastic constants of that material. For any $C$,

$$C_{ijkl} = R_{im}R_{jn}R_{ko}R_{lp}C_{mnop} = a_i\delta_{im}a_j\delta_{jn}a_k\delta_{ko}a_l\delta_{lp}C_{mnop} = a_ia_ja_ka_lC_{ijkl} \text{ (NO SUM)} \tag{321}$$

implies either

$$a_ia_ja_ka_l = 1 \text{ or } C_{ijkl} = 0. \tag{322}$$

Since $\mathbf{a} = \{1\ 1\ -1\}^T$, the only way where a product of any number of the internal components is negative is if one is multiplying an odd number of the third component, which is -1. So for example

$$a_1a_1a_2a_3 = 1 * 1 * 1 * -1 = -1 \neq 1 \longrightarrow C_{1123} = 0 \tag{323}$$

while

$$a_2a_1a_3a_3 = 1 * 1 * -1 * -1 = 1 \longrightarrow C_{2133} \neq 0. \tag{324}$$

This reduces the total number of elastic constants from 21 down to 13, in that

$$[\mathbf{C}] = \begin{bmatrix} C_{1111} & C_{1122} & C_{1133} & C_{1112} & C_{1123} & C_{1113} \\ C_{2211} & C_{2222} & C_{2233} & C_{2212} & C_{2223} & C_{2213} \\ C_{3311} & C_{3322} & C_{3333} & C_{3312} & C_{3323} & C_{3313} \\ C_{1211} & C_{1222} & C_{1233} & C_{1212} & C_{1223} & C_{1213} \\ C_{2311} & C_{2322} & C_{2333} & C_{2312} & C_{2323} & C_{2313} \\ C_{1311} & C_{1322} & C_{1333} & C_{1312} & C_{1323} & C_{1313} \end{bmatrix} = \begin{bmatrix} C_{1111} & C_{1122} & C_{1133} & C_{1112} & 0 & 0 \\ C_{2211} & C_{2222} & C_{2233} & C_{2212} & 0 & 0 \\ C_{3311} & C_{3322} & C_{3333} & C_{3312} & 0 & 0 \\ C_{1211} & C_{1222} & C_{1233} & C_{1212} & 0 & 0 \\ 0 & 0 & 0 & 0 & C_{2323} & C_{2313} \\ 0 & 0 & 0 & 0 & C_{1323} & C_{1313} \end{bmatrix}$$

$$= \begin{bmatrix} C_{1111} & C_{1122} & C_{1133} & C_{1112} & 0 & 0 \\ & C_{2222} & C_{2233} & C_{2212} & 0 & 0 \\ & & C_{3333} & C_{3312} & 0 & 0 \\ & & & C_{1212} & 0 & 0 \\ & & & & C_{2323} & C_{2313} \\ \text{sym} & & & & & C_{1313} \end{bmatrix} = \begin{bmatrix} C_{11} & C_{12} & C_{13} & C_{14} & 0 & 0 \\ & C_{22} & C_{23} & C_{24} & 0 & 0 \\ & & C_{33} & C_{34} & 0 & 0 \\ & & & C_{44} & 0 & 0 \\ & & & & C_{55} & C_{56} \\ \text{sym} & & & & & C_{66} \end{bmatrix}. \tag{325}$$

A material with orthotropic symmetry has three planes of symmetry $xy$, $xz$, $yz$, meaning all of the following coordinate transformations

$$\begin{Bmatrix} \hat{x}_1 = x_1 \\ \hat{x}_2 = x_2 \\ \hat{x}_3 = -x_3 \end{Bmatrix}, \quad \begin{Bmatrix} \hat{x}_1 = x_1 \\ \hat{x}_2 = -x_2 \\ \hat{x}_3 = x_3 \end{Bmatrix}, \quad \begin{Bmatrix} \hat{x}_1 = -x_1 \\ \hat{x}_2 = x_2 \\ \hat{x}_3 = x_3 \end{Bmatrix} \tag{326}$$

Do not affect the elastic constants of the material. These mean that an odd number of the first, second, or third component of the $a$ vectors will yield the product $-1$, meaining the corresponding $C$ value must compensate by being itself zero. Therefore for orthotropic materials,

$$[\mathbf{C}] = \begin{bmatrix} C_{1111} & C_{1122} & C_{1133} & C_{1112} & C_{1123} & C_{1113} \\ C_{2211} & C_{2222} & C_{2233} & C_{2212} & C_{2223} & C_{2213} \\ C_{3311} & C_{3322} & C_{3333} & C_{3312} & C_{3323} & C_{3313} \\ C_{1211} & C_{1222} & C_{1233} & C_{1212} & C_{1223} & C_{1213} \\ C_{2311} & C_{2322} & C_{2333} & C_{2312} & C_{2323} & C_{2313} \\ C_{1311} & C_{1322} & C_{1333} & C_{1312} & C_{1323} & C_{1313} \end{bmatrix} = \begin{bmatrix} C_{1111} & C_{1122} & C_{1133} & 0 & 0 & 0 \\ C_{2211} & C_{2222} & C_{2233} & 0 & 0 & 0 \\ C_{3311} & C_{3322} & C_{3333} & 0 & 0 & 0 \\ 0 & 0 & 0 & C_{1212} & 0 & 0 \\ 0 & 0 & 0 & 0 & C_{2323} & 0 \\ 0 & 0 & 0 & 0 & 0 & C_{1313} \end{bmatrix}$$

$$= \begin{bmatrix} C_{11} & C_{12} & C_{13} & 0 & 0 & 0 \\ & C_{22} & C_{23} & 0 & 0 & 0 \\ & & C_{33} & 0 & 0 & 0 \\ & & & C_{44} & 0 & 0 \\ & & & & C_{55} & 0 \\ \text{sym} & & & & & C_{66} \end{bmatrix}. \tag{327}$$

This leaves a total of just nine elastic constants.

## 4.4 Isotropic materials

There does not exist an isotropic tensor off odd rank. As for rank 2, if a tensor is isotropic then it must take the form $c\delta_{ij}$. If it is rank 4, it must take the form $a\delta_{ij}\delta_{kl}+b\delta_{ik}\delta_{jl}+c\delta_{il}\delta_{jk}$.

Specifically for $C_{ijkl}$,

$$C_{ijkl} = \lambda\delta_{ij}\delta_{kl} + \mu\delta_{ik}\delta_{jl} + \beta\delta_{il}\delta_{jk}. \tag{328}$$

Since $C_{ijkl} = C_{jikl}$,

$$\cancel{\lambda\delta_{ij}\delta_{kl}} + \mu\delta_{ik}\delta_{jl} + \beta\delta_{il}\delta_{jk} = \cancel{\lambda\delta_{ji}\delta_{kl}} + \mu\delta_{jk}\delta_{il} + \beta\delta_{jl}\delta_{ik} \tag{329}$$

$$\rightarrow (\mu - \beta)(\delta_{ik}\delta_{jl}) + (\beta - \mu)(\delta_{il}\delta_{jk}) = 0. \tag{330}$$

For this to be true of all $i, j, k, l$, $\beta = \mu$. Therefore,

$$C_{ijkl} = \lambda\delta_{ij}\delta_{kl} + \mu(\delta_{ik}\delta_{jl} + \delta_{il}\delta_{jk}). \tag{331}$$

Here $\lambda$ and $\mu$ are called the Lamé constants of elasticity. With Hooke's law,

$$\sigma_{ij} = C_{ijkl}\epsilon_{kl} = \lambda\delta_{ij}\delta_{kl}\epsilon_{kl} + \mu\delta_{ik}\delta_{jl}\epsilon_{kl} + \mu\delta_{il}\delta_{jk}\epsilon_{kl} \tag{332}$$

$$= \lambda\epsilon_{kk}\delta_{ij} + \mu\epsilon_{ij} + \mu\epsilon_{ji} = \lambda\epsilon_{kk}\delta_{ij} + 2\mu\epsilon_{ij} = \sigma_{ij}. \tag{333}$$

Expanding this,

$$\begin{Bmatrix} \sigma_{11} \\ \sigma_{22} \\ \sigma_{33} \\ \sigma_{12} \\ \sigma_{23} \\ \sigma_{31} \end{Bmatrix} = \begin{bmatrix} 2\mu+\lambda & \lambda & \lambda & 0 & 0 & 0 \\ \lambda & 2\mu+\lambda & \lambda & 0 & 0 & 0 \\ \lambda & \lambda & 2\mu+\lambda & 0 & 0 & 0 \\ 0 & 0 & 0 & \mu & 0 & 0 \\ 0 & 0 & 0 & 0 & \mu & 0 \\ 0 & 0 & 0 & 0 & 0 & \mu \end{bmatrix} \begin{Bmatrix} \epsilon_{11} \\ \epsilon_{22} \\ \epsilon_{33} \\ 2\epsilon_{12} \\ 2\epsilon_{23} \\ 2\epsilon_{31} \end{Bmatrix}. \tag{334}$$

This structure emerges from Eq. 333, because for a diagonal component, e.g.

$$\sigma_{11} = \lambda(\epsilon_{11} + \epsilon_{22} + \epsilon_{33})\delta_{11} + 2\mu\epsilon_{11} = (2\mu + \lambda)\epsilon_{11} + \lambda\epsilon_{22} + \lambda\epsilon_{33}, \tag{335}$$

and for an off diagonal component, e.g.

$$\sigma_{23} = \cancel{\lambda(\epsilon_{11} + \epsilon_{22} + \epsilon_{33})\delta_{23}} + 2\mu\epsilon_{23} = 2\mu\epsilon_{23} = \mu(2\epsilon_{23}). \tag{336}$$

Eq. 333 is a way to get stress from the strains. We wish also to have a way to find strains based on the stress. Using contraction on Eq. 333 ($\sigma_{ij} = \lambda\epsilon_{kk}\delta_{ij} + 2\mu\epsilon_{ij}$),

$$\sigma_{kk} = \lambda\epsilon_{kk}\delta_{kk} + 2\mu\epsilon_{kk} = 3\lambda\epsilon_{kk} + 2\mu\epsilon_{kk} = (3\lambda + 2\mu)\epsilon_{kk} \longrightarrow \epsilon_{kk} = \frac{\sigma_{kk}}{3\lambda + 2\mu}. \tag{337}$$

Substituting this back into Eq. 333,

$$\sigma_{ij} = \lambda\frac{\sigma_{kk}}{3\lambda + 2\mu}\delta_{ij} + 2\mu\epsilon_{ij} \Longrightarrow \epsilon_{ij} = \frac{1}{2\mu}\left[\sigma_{ij} - \lambda\frac{\sigma_{kk}}{3\lambda + 2\mu}\delta_{ij}\right]. \tag{338}$$

# 5 Simplest problems of elastostatics

Eqs. 275 are the momentum and stress equations. They are

$$\sigma_{ij,j} + \rho f_i = \rho \ddot{u}_i, \qquad t_i = \sigma_{ij} n_j. \tag{339}$$

If the body is not moving, there is no acceleration. Therefore, the momentum equation reduces to

$$\sigma_{ij,j} = -\rho f_i. \tag{340}$$

If there are no body forces, such as gravity, then

$$\sigma_{ij,j} = 0. \tag{341}$$

## 5.1 Simple shear

Consider the deformation map

$$\begin{cases} u_1 = kx_2 = x_2 \tan\theta \\ u_2 = 0 \\ u_3 = 0. \end{cases} \tag{342}$$

The corresponding displacement gradient and strain tensor (where the latter is the

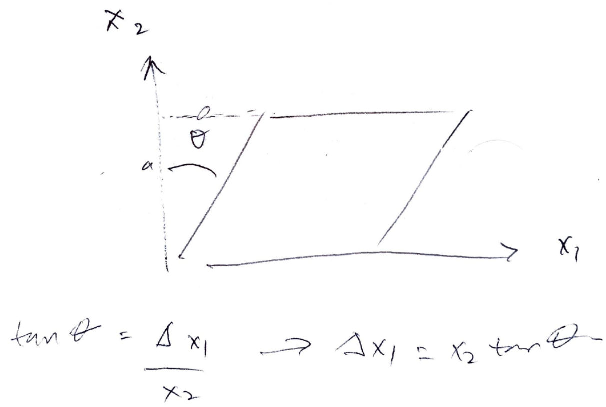



Figure 5: Simple shear: $\tan\theta$ = opposite/adjacent = $o/a$ = $\Delta x_1/x_2 \to \Delta x_1 = u_1 = x_2 \tan\theta$

symmetric part of the displacement gradient) is

$$[\mathbf{u}] = \begin{bmatrix} 0 & k & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \qquad [\boldsymbol{\epsilon}] = \begin{bmatrix} 0 & k/2 & 0 \\ k/2 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}. \tag{343}$$

If the material is isotropic then we use 333 ($\lambda\epsilon_{kk}\delta_{ij} + 2\mu\epsilon_{ij} = \sigma_{ij}$) to calculate the stresses. Here $\epsilon_{kk} = 0$, so

$$\sigma_{12} = \lambda\epsilon_{kk}\delta_{12} + 2\mu\epsilon_{12} = 2\mu(k/2) = \mu k = \sigma_{21}. \tag{344}$$

Otherwise, $\sigma_{ij} = 0$. This is to say that there are only two nonzero components of stress.

Recall that $\gamma_{ij} = 2\epsilon_{ij}$ is the engineering shear strain. Therefore, $\gamma_{12} = 2\epsilon_{12} = 2(k/2) = k$, meaning $\sigma_{12} = \mu k = \mu\gamma_{12}$. Here $\mu$ is the shear modulus and also a Lamé constant.

## 5.2 Simple tension

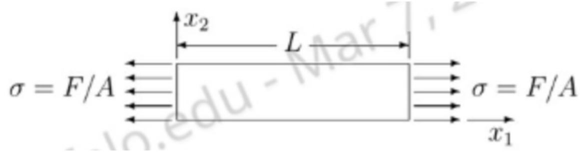



Figure 6: Simple tension

The stress tensor corresponding to simple tension is

$$[\boldsymbol{\sigma}] = \begin{bmatrix} \sigma & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \tag{345}$$

meaning $\sigma_{kk} = \sigma_{11} + \sigma_{22} + \sigma_{33} = \sigma$ and, from Eq. 338,

$$\epsilon_{11} = \frac{1}{2\mu}\left[\sigma_{11} - \lambda\frac{\sigma_{kk}}{3\lambda + 2\mu}\delta_{11}\right] = \frac{1}{2\mu}\left[\sigma - \frac{\lambda\sigma}{3\lambda + 2\mu}\right] = \frac{\sigma}{2\mu}\left[\frac{3\lambda + 2\mu + \lambda}{3\lambda + 2\mu}\right] = \frac{\sigma}{\mu}\frac{\lambda + \mu}{3\lambda + 2\mu}. \tag{346}$$

Young's modulus

$$E = \sigma/\epsilon = \sigma \bigg/ \left[\frac{\sigma}{\mu}\frac{\lambda + \mu}{3\lambda + 2\mu}\right] = \mu\frac{3\lambda + 2\mu}{\lambda + \mu}. \tag{347}$$

Also from Eq. 338 we can derive the other on-diagonal strains. They are

$$\epsilon_{22} = \frac{-\lambda}{2}\frac{\sigma}{\mu}\left[\frac{1}{3\lambda + 2\mu}\right] = \frac{-\lambda}{2}\frac{1}{\lambda + \mu}\epsilon_{11} = \epsilon_{33}. \tag{348}$$

Now, we let Poisson's ratio be the negative of the ratio between transverse strain to axial strain, or

$$\nu = -\epsilon_{22}/\epsilon_{11} = \frac{\lambda}{2(\lambda + \mu)}. \tag{349}$$

Poisson's ratio is a measure of how things shrink in the perpendicular direction with respect to how things stretch in the axial direction. For example if you stretch out a piece of gum it will also become thinner. The extent to which the gum slice becomes thinner (shrinkage in the transverse direction) as the slices stretches (stretch in the loading direction) is $\nu$.

Now notice that $E$ can be written as

$$E = \mu\frac{\lambda + 2\lambda + 2\mu}{\lambda + \mu} = \mu(\frac{\lambda}{\lambda + \mu} + 2) = \mu(2\nu + 2) = 2\mu(1 + \nu). \tag{350}$$

Therefore

$$\mu = \frac{E}{2(1 + \nu)} = G \longrightarrow 2\mu = \frac{E}{1 + \nu}, \tag{351}$$

where $G$ is how shear modulus $\mu$ is represented sometimes in engineering applications. To solve for the other Lame constant $\lambda$ in terms of $E$ and $\nu$, we isolate $\lambda$ in Eq. 349, as in

$$\lambda = 2\lambda\nu + 2\mu\nu \rightarrow \lambda(1-2\nu) = 2\mu\nu \rightarrow \lambda = \mu\frac{2\nu}{1-2\nu} = \frac{E}{1+\nu}\frac{\nu}{1-2\nu}. \tag{352}$$

With knowledge of the Lame constants $\mu, \lambda$ in terms of the engineering constants $E, \nu$, we can represent the generalized Hooke's law (how to find stress and strain) in terms of the elastic constants instead of the Lame constants. Solving for stress using Eq. 333,

$$\sigma_{ij} = \lambda\epsilon_{kk}\delta_{ij} + 2\mu\epsilon_{ij} = \left[\frac{E}{1+\nu}\frac{\nu}{1-2\nu}\epsilon_{kk}\delta_{ij}\right] + \left[\frac{E}{1+\nu}\epsilon_{ij}\right] = \frac{E}{1+\nu}\left[\frac{\nu}{1-2\nu}\epsilon_{kk}\delta_{ij} + \epsilon_{ij}\right]. \tag{353}$$

Solving for strain using Eq. 338,

$$\epsilon_{ij} = \frac{1}{2\mu}\left[\sigma_{ij} - \lambda\frac{\sigma_{kk}}{3\lambda+2\mu}\delta_{ij}\right] = \frac{1+\nu}{E}\left[\sigma_{ij} - \frac{E}{1+\nu}\frac{\nu}{1-2\nu}\frac{\sigma_{kk}}{\left[3\frac{E}{1+\nu}\frac{\nu}{1-2\nu} + 2\frac{E}{1+\nu}\right]}\delta_{ij}\right]$$

$$= \frac{1+\nu}{E}\left[\sigma_{ij} - \frac{\nu}{1+\nu}\sigma_{kk}\delta_{ij}\right]$$

$$= \frac{1}{E}\left[\sigma_{ij}(1+\nu) - \nu\sigma_{kk}\delta_{ij}\right] = \epsilon_{ij}.$$

Expanding the subscript $k$,

$$\epsilon_{ij} = \frac{1}{E}\left[\sigma_{ij}(1+\nu) - \nu(\sigma_{11}+\sigma_{22}+\sigma_{33})\delta_{ij}\right]. \tag{354}$$

Solving for diagonal strain components,

$$\epsilon_{11} = \frac{1}{E}[\sigma_{11} - \nu\sigma_{22} - \nu\sigma_{33}] = \frac{\sigma_{11}}{E} - \frac{\sigma_{22}}{E}\nu - \frac{\sigma_{33}}{E}\nu, \tag{355}$$

$$\epsilon_{22} = -\frac{\sigma_{11}}{E}\nu + \frac{\sigma_{22}}{E} - \frac{\sigma_{33}}{E}\nu, \tag{356}$$

$$\epsilon_{33} = -\frac{\sigma_{11}}{E}\nu - \frac{\sigma_{22}}{E}\nu + \frac{\sigma_{33}}{E}. \tag{357}$$

Because of the definitions of $\nu$ and $E$ this is not at all unexpected. Remember, elastic modulus is stress over strain, and Poisson's ratio is the negative transverse strain relative to axial strain.

## 5.3 Uniform compression

The displacement gradient associated with uniform compression is

$$[\mathbf{u}] \leftrightarrow u_{i,j} = \begin{bmatrix} \epsilon & 0 & 0 \\ 0 & \epsilon & 0 \\ 0 & 0 & \epsilon \end{bmatrix} = \epsilon\mathbf{I} = \boldsymbol{\epsilon}. \tag{358}$$

This means there is an application of uniform stress in $xx, yy, andzz$ with the same magnitude, leading to the same magnitude of strain on all sides. The volumetric strain, which in this case is called dilatation, is

$$\Delta = \frac{\Delta V}{V} = u_{i,i} = \mathrm{tr}\boldsymbol{\epsilon} = 3\epsilon. \tag{359}$$

From Hooke's law Eq. 333 ($\sigma_{ij} = \lambda\epsilon_{kk}\delta_{ij} + 2\mu\epsilon_{ij}$),

$$\sigma_{ij} = 3\epsilon\lambda\delta_{ij} + 2\mu(\epsilon\delta_{ij}) = 3\epsilon\delta_{ij}(\lambda + 2\mu/3) = \Delta(\lambda + 2\mu/3)\delta_{ij}. \tag{360}$$

If $-p$ is the average normal stress, then

$$-p = \frac{1}{3}\sigma_{ii}. \tag{361}$$

Then

$$-p = \sigma_{11} = \sigma_{22} = \sigma_{33} = \Delta(\lambda + 2\mu/3). \tag{362}$$

Then we define the bulk modulus, also called the modulus of compression, as

$$\begin{aligned} K = \frac{-p}{\Delta} = \lambda + \frac{2}{3}\mu &= \frac{E}{1+\nu}\frac{\nu}{1-2\nu} + \frac{2}{3}\frac{E}{2(1+\nu)} \\ &= \frac{3E\nu + E(1-2\nu)}{3(1+\nu)(1-2\nu)} = \frac{E(\nu+1)}{3(1+\nu)(1-2\nu)} = \frac{E}{3(1-2\nu)}. \end{aligned} \tag{363}$$

Note that $\nu > 1/2$ implies $K < 0$. It is not possible for the volume to increase. Therefore

$$\nu \leq \frac{1}{2}. \tag{364}$$

The special case $\nu = 1/2$ implies an infinite bulk modulus which means that the material is incompressible, or does not increase or decrease in volume. Rubber is nearly incompressible.

Substituting $K = \lambda + 2\mu/3$ into Eq. 360,

$$\sigma_{ij} = K\delta_{ij}\Delta = K\epsilon_{kk}\delta_{ij}. \tag{365}$$

## 5.4 Stress and strrain deviators

Recall the considerations of this chapter. Simple shear is a change in shape without a change in volume. Uniform compression is a volume change without a shape change. In simple shear, the stress field is

$$\sigma_{ij} = 2\mu\epsilon_{ij},$$

and for uniform compression it is

$$\sigma_{ij} = K\epsilon_{kk}\delta_{ij}.$$

Uniform compression is also called hydrostatic or isotropic compression.

Any deformation is a combination of pure shape change (simple shear) and pure volume change (uniform compression). Let express the strain tensor as

$$\epsilon_{ij} = \epsilon_{ij} - \frac{1}{3}\epsilon_{kk}\delta_{ij} + \frac{1}{3}\epsilon_{kk}\delta_{ij} = e_{ij} + \frac{1}{3}\epsilon_{kk}\delta_{ij}, \tag{366}$$

where $e_{ij}$ is the simple shear component, since the sum of the diagonals is zero, as evidenced by

$$\epsilon_{11} = \underbrace{\epsilon_{11} - \frac{1}{3}\epsilon_{11} - \frac{1}{3}\epsilon_{22} - \frac{1}{3}\epsilon_{33}}_{e_{ij}} + \frac{1}{3}\epsilon_{kk}\delta_{ij} = \frac{1}{3}\epsilon_{kk}\delta_{ij} \quad (\epsilon_{kk} = 3\epsilon,\ \epsilon_{11} = \epsilon_{22} = \epsilon_{33} = \epsilon),$$

and the last term is a hydrostatic compression component. The first term and the second term are respectively called the deviatoric and isotropic terms. Similarly for stress,

$$\sigma_{ij} = s_{ij} + \frac{1}{3}\sigma_{kk}\delta_{ij} = s_{ij} - p\delta_{ij}, \tag{367}$$

where mean normal stress $-p = \sigma_{kk}/3$ and

$$s_{ij} = \sigma_{ij} - \frac{1}{3}\sigma_{kk}\delta_{ij}. \tag{368}$$

Substituting Eqs. 367,366 into Eq. 333 ($\sigma_{ij} = \lambda\epsilon_{kk}\delta_{ij} + 2\mu\epsilon_{ij}$),

$$s_{ij} - p\delta_{ij} = \lambda\epsilon_{kk}\delta_{ij} + 2\mu\left(e_{ij} + \frac{1}{3}\epsilon_{kk}\delta_{ij}\right). \tag{369}$$

Using contraction such that $j = i$, $s_{ii} = 0$, $e_{ii} = 0$ by definition and $\sum_i \delta_{ii} = 3$. Therefore

$$-3p = 3\lambda\epsilon_{kk} + 2\mu\left(0 + \epsilon_{kk}\right) = (3\lambda + 2\mu)\epsilon_{kk} = 3K\epsilon_{kk}. \tag{370}$$

Therefore,

$$-p = K\epsilon_{kk}, \tag{371}$$

which we already knew. Substituting this back into Eq. 369,

$$s_{ij} + \left(\lambda + \frac{2}{3}\mu\right)\epsilon_{kk}\delta_{ij} = \lambda\epsilon_{kk}\delta_{ij} + 2\mu\left(e_{ij} + \frac{1}{3}\epsilon_{kk}\delta_{ij}\right), \tag{372}$$

which implies very simply

$$s_{ij} = 2\mu e_{ij}. \tag{373}$$

Therefore Hooke's law can be written simply as

$$\begin{cases} s_{ij} = 2\mu e_{ij}, & i \neq j \\ \sigma_{ii}/3 = K\epsilon_{ii}, & i = j. \end{cases} \tag{374}$$

These are not conditional statements as though they are not both true always. It is just that if $i \neq j$, the second equation will be zero on both sides and so not relevant. If $i = j$, then both sides of the first equation will be zero and likewise irrelevant.

## 5.5 Stable reference states

Strain energy density is Eq. 313, which is

$$w = \frac{1}{2}c_{ijkl}\epsilon_{kl}\epsilon_{ij} = \frac{1}{2}\sigma_{ij}\epsilon_{ij}.$$

Substituting in 366,367,

$$w = \frac{1}{2}\left(s_{ij} + \frac{1}{3}\sigma_{kk}\delta_{ij}\right)\left(e_{ij} + \frac{1}{3}\epsilon_{ll}\delta_{ij}\right)$$

$$= \frac{1}{2}\left(s_{ij}e_{ij} + \frac{1}{3}\epsilon_{ll}s_{ij}\delta_{ij} + \frac{1}{3}\sigma_{kk}e_{ij}\delta_{ij} + \frac{1}{9}\sigma_{kk}\epsilon_{ll}\delta_{ij}\delta_{ij}\right)$$

$$= \frac{1}{2}\left(s_{ij}e_{ij} + \frac{1}{3}\epsilon_{ll}s_{ii} + \frac{1}{3}\sigma_{kk}e_{ii} + \frac{1}{3}\sigma_{kk}\epsilon_{ll}\right)$$

$$= \frac{1}{2}\left(s_{ij}e_{ij} + \frac{1}{3}\epsilon_{kk}s_{ii} + \frac{1}{3}\sigma_{kk}e_{ii} + \frac{1}{3}\sigma_{kk}\epsilon_{ll}\right) = w. \tag{375}$$

If the material is isotropic, this means Hooke's law Eq. 369 applies. This means

$$w = \mu e_{ij}e_{ij} + \frac{1}{2}K\epsilon_{ii}\epsilon_{ll}$$

$$\mu e_{ij}e_{ij} + \frac{1}{2}K\epsilon_{ii}\epsilon_{ii} = w. \tag{376}$$

We define a stable reference state as

$$w = c_{ijkl}\epsilon_{kl}\epsilon_{ij} > 0. \tag{377}$$

For this to be true, $c_{ijkl}$ must be positive definite such that the expression is positive for all $\boldsymbol{\epsilon}$. If the material is isotropic such that the strain energy density is given by Eq. 376, then

$$w = \mu e_{ij}e_{ij} + \frac{1}{2}K\epsilon_{ii}\epsilon_{ii} > 0. \tag{378}$$

If the deformation is pure volume change with no shape change, then deviatoric strain $e_{ij} = 0$, meaning $K > 0 \leftrightarrow w > 0$. If the deformation is pure shear with no volume change, then isotropic strain $\epsilon_{ii} = 0$ meaning $\mu > 0 \leftrightarrow w > 0$. Together,

$$K > 0, \quad \mu > 0 \tag{379}$$

which imply for isotropic mateirals that

$$E > 0, \qquad -1 < \nu < 1/2 \tag{380}$$

because of the definitions for bulk and shear modulus Eqs. 363 and 351.

# 6 Boundary value problems in elastostatics

The basic equations of elastostatics are

$$\sigma_{ij,j} + \rho f_i = 0, \tag{381}$$

$$\sigma_{ij} = c_{ijkl}\epsilon_{kl}, \tag{382}$$

$$\epsilon_{ij} = \frac{1}{2}(u_{i,j} + u_{j,i}), \tag{383}$$

$$t_i = \sigma_{ij}n_j. \tag{384}$$

Substituting Eq. 383 into Eq. 382,

$$\sigma_{ij} = \frac{1}{2}c_{ijkl}(u_{k,l} + u_{l,k}) = \frac{1}{2}c_{ijkl}u_{k,l} + \frac{1}{2}c_{ijkl}u_{l,k} \tag{385}$$

$$= \frac{1}{2}c_{ijkl}u_{k,l} + \frac{1}{2}c_{ijlk}u_{l,k} \tag{386}$$

$$= \frac{1}{2}c_{ijkl}u_{k,l} + \frac{1}{2}c_{ijkl}u_{k,l} \quad \text{(dummy index)} \tag{387}$$

$$= c_{ijkl}u_{k,l} = \sigma_{ij}. \tag{388}$$

Substituting this into Eq. 381,

$$(c_{ijkl}u_{k,l})_{,j} + \rho f_i = 0. \tag{389}$$

This system of three PDEs is the Navier equations of elasticity. The unknowns here are the three Cartesian displacement components. These equations apply to anisotropic materials. For isotropic materials,

$$\sigma_{ij} = \lambda\epsilon_{kk}\delta_{ij} + 2\mu\epsilon_{ij} = \lambda u_{k,k}\delta_{ij} + \mu(u_{i,j} + u_{j,i}). \tag{390}$$

Differentiating with respect to $x_j$,

$$\sigma_{ij,j} = \lambda u_{k,kj}\delta_{ij} + \mu(u_{i,jj} + u_{j,ij}) = \lambda u_{k,ki} + \mu(u_{i,jj} + u_{j,ij}) = (\lambda + \mu)u_{j,ji} + \mu u_{i,jj}. \tag{391}$$

We have assumed that $\lambda, \mu \sim E, \nu$ are constants and not dependent on position. In doing so we assume that the material is homogeneous. A nonhomogeneous material has position dependent properties. So for an isotropic homogeneous material the Navier equations are

$$(\lambda + \mu)u_{j,ji} + \mu u_{i,jj} + \rho f_i = 0. \tag{392}$$

A boundary value problem or BVP is dedicated to finding the stress/displacement distributions in the interior of an elastic body in equilibrium with specified boundary conditions and boundary tractions. A purely displacement problem has a unique existing solution and is therefore said to be well-posed. However a boundary traction problem is not well posed because its displacement solution is not unique. However the stress solution is unique because stress is derivative of displacement gradients[?].

## 6.1 Uniqueness

In a displacement BVP the goal is to find **u** such that

$$\begin{cases} (c_{ijkl}u_{k,l})_{,j} + \rho f_i = 0 \longrightarrow (c_{ijkl}u_{k,l})_{,j} = -\rho f_i, & V, \\ u_i = U_i, & S, \end{cases} \tag{393}$$

where $S$ is the surface and $V$ is the volume. $U_i$ is the set of prescribed displacement boundary conditions on $S$. The two equations must hold at all points in $V$ and $S$.

If the BVP is homogeneous, then every term of the PDE and boundary condition is proportional to **u**. Otherwise, the BVP is nonhomogeneous. Eq. 393 is nonhomogeneous because not every term in the PDE is proportional to **u**.

Let us try to prove that Eq. 393 has a unique solution. To do so let us assume it does not have a unique solution. Suppose there is a second solution $v_i$ such that

$$\begin{cases} (c_{ijkl}v_{k,l})_{,j} = -\rho f_i, & V, \\ v_i = U_i, & S. \end{cases} \tag{394}$$

Subtracting Eq. 394 from Eq. 393 so that $w_i = u_i - v_i$,

$$\begin{cases} (c_{ijkl}w_{k,l})_{,j} = 0, & V, \\ w_i = 0, & S. \end{cases} \tag{395}$$

This is now a homogeneous problem because $\rho f_i$ and $U_i$ were nonhomogeneous terms (as in, they were not proportional to **u**), but they were removed through subtraction. Proving that $w_i = u_i - v_i = 0$ is the only solution to the system Eq. 395 is the same as proving that $u_i := v_i$ and therefore the original system Eq. 393 is the same as Eq. 394 and therefore there is only one solution and so the solution is unique. So to prove the uniqueness of Eq. 393 all we need to do is show that Eq. 395 is true, which can be renamed as

$$\begin{cases} (c_{ijkl}u_{k,l})_{,j} = 0, & V, \\ u_i = 0, & S. \end{cases} \tag{396}$$

This implies that

$$\int_V (c_{ijkl}u_{k,l})_{,j} dV = 0 \tag{397}$$

$$\rightarrow \int_V u_i(c_{ijkl}u_{k,l})_{,j} dV = 0 \tag{398}$$

$$\rightarrow 0 = \int_V [(u_i c_{ijkl}u_{k,l})_{,j} - u_{i,j}c_{ijkl}u_{k,l}]dV \tag{399}$$

$$= \int_V (u_i c_{ijkl}u_{k,l})_{,j} dV - \int_V u_{i,j}c_{ijkl}u_{k,l} dV \tag{400}$$

$$= \oint_S u_i c_{ijkl}u_{k,l}n_j dS - \int_V u_{i,j}c_{ijkl}u_{k,l} dV \tag{401}$$

$$= \oint_S u_i c_{ijkl}\epsilon_{kl} n_j dS - \int_V \epsilon_{i,j} c_{ijkl}\epsilon_{kl} dV \quad (402)$$

$$= \oint_S u_i \sigma_{ij} n_j dS - \int_V 2w dV \quad (403)$$

$$= \cancel{\oint_S u_i t_i dS} - \int_V 2w dV \quad (404)$$

$$= -\int_V 2w dV = 0 \longrightarrow w = 0. \quad (405)$$

The traction term cancels out because $u_i = 0$ on the surface $S$. Note that $w = c_{ijkl}\epsilon_{ij}\epsilon_{kl} = 0$. Therefore, $\boldsymbol{\epsilon} = 0$ at all points in the volume. Zero strain implies no deformation, only rotation and translation. Therefore, $u_i = 0$ for all $V$. Indeed this is what we were trying to prove. Therefore the Eq. 396 is true and therefore we have established uniqueness for the displacement solution that is Eq. 393.

## 6.2 Uniqueness for thr traction problem

In a traction BVP the goal is to find **u** such that Eq. 384 and Eq. 381 are true, meaning

$$\begin{cases} (c_{ijkl}u_{k,l})_{,j} = -\rho f_i, & V, \\ \sigma_{ij}n_j = t_i, & S. \end{cases} \quad (406)$$

In the same way as in the previous section, to prove uniqueness of Eq. 406 we can alternatively prove uniqueness of

$$\begin{cases} (c_{ijkl}u_{k,l})_{,j} = 0, & V, \\ \sigma_{ij}n_j = 0, & S. \end{cases} \quad (407)$$

We arrive at

$$w = 0 \rightarrow \boldsymbol{\epsilon} = 0 \rightarrow u_i = 0 \in V, \quad (408)$$

but recall that there is no such requirement on $S$. Therefore the displacements are not unique. However it is true that the strains are unique in the volume, as shown. Therefore, so are the stresses.

## 6.3 Uniqueness for the mixed problem

Uniqueness follows immediately from Eq. 404 because if $u_i$ or $t_i$ are zero at every point then the whole term vanishes.

# 7 Torsion

## 7.1 Circular shaft

Consider a circular shaft with length $L$ fixed at one end such that $z = 0$ and subjected to shear forces/torques at $z = L$. Instead of specifying tractions at $z = L$ we will instead find the displacement field and see if all required equations are satisfied and if the resulting surface tractions/body forces are reasonable.

Let $\alpha$ be an angle of twist per unit length. That is, $\alpha = \theta/z$. So $\theta$ is the twist angle and $z$ is the length of the shaft. Note that

$$\theta = \alpha z \tag{409}$$

because of our definition of $\alpha$.

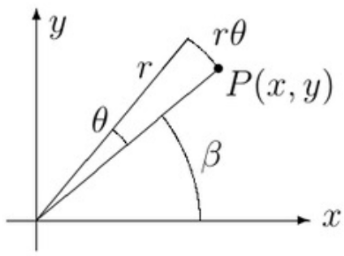



Figure 7: Torsional rotation of cross section of circular shaft in $xy$ plane

Assume that the angle of twist per unit length $\alpha$ is small. Therefore, $\theta$ is small. Therefore, it can be assumed that $\sin\theta \approx \theta$. Therefore, the segment $r\sin\theta \approx r\theta$. Consider also that this segment $r\theta$ runs (almost) perpendicularly to $r$. This means that when $\beta = 0$, $r$ is completely aligned with the $x$ axis, meaning $r\theta$ is aligned with the y$\leftrightarrow$ $x_2$ axis. Conversely if $\beta = 90$, $r$ is competely aligned with the $y$ axis and so $r\theta$ is ANTIPARALLEL with the $x \leftrightarrow x_1$ axis (runs parallel, but the two ends of $r\theta$ go from right to left, whereas the $x$ axis goes from left to right). These statements mean that displacement $u$ is completely vertical when $\beta = 0 \to \cos\beta = 1$, so that $u = u_2$. On the other hand, $u = u1$ when $\beta = 90 \to \sin\beta = 1$. Therefore,

$$\begin{cases} u_1 = -r\theta\sin\beta \\ u_2 = r\theta\cos\beta. \end{cases} \tag{410}$$

Let the $x$ component of $r$ be $x$ and the $y$ component of $r$ be $y$. Then, $r\cos\beta = x$ and $r\sin\beta = y$. ($r^2 = x^2 + y^2$.) This means that

$$\cos\beta = x/r, \quad \sin\beta = y/r. \tag{411}$$

Substituting this in,

$$\begin{cases} u_1 = -r\theta(y/r) = -\theta y = -\alpha z y, \\ u_2 = r\theta(x/r) = \theta x = \alpha z x, \\ u_3 = 0. \end{cases} \tag{412}$$

The corresponding deformation gradient is

$$[u_{i,j}] = \begin{bmatrix} u_{1,1} & u_{1,2} & u_{1,3} \\ u_{2,1} & u_{2,2} & u_{2,3} \\ u_{3,1} & u_{3,2} & u_{3,3} \end{bmatrix} = \begin{bmatrix} 0 & -\alpha z & -\alpha y \\ \alpha z & 0 & \alpha x \\ 0 & 0 & 0 \end{bmatrix}. \tag{413}$$

The symmetric part of this is

$$\boldsymbol{\epsilon} = \frac{1}{2}(\mathbf{u} + \mathbf{u}^T) = \frac{1}{2}\begin{bmatrix} 0 & 0 & -\alpha y \\ 0 & 0 & \alpha x \\ -\alpha y & \alpha x & 0 \end{bmatrix} = \frac{\alpha}{2}\begin{bmatrix} 0 & 0 & -y \\ 0 & 0 & x \\ -y & x & 0 \end{bmatrix}. \tag{414}$$

Assuming an isotropic material so that $\sigma_{ij} = \lambda \cancel{\epsilon_{kk}} \delta_{ij} + 2\mu\epsilon_{ij}$,

$$[\sigma_{ij}] = \alpha\mu \begin{bmatrix} 0 & 0 & -y \\ 0 & 0 & x \\ -y & x & 0 \end{bmatrix}. \tag{415}$$

Note that the trace of the strain tensor $\epsilon_{kk}$ indicates no volume change.

Recall Fig. 2, which says that a stress component $\sigma_{mn}$ is the $n$-component of traction $\mathbf{t}$ for the surface whose normal is $x_m$. Therefore, the $n$-components of $\mathbf{t}$ for the surface whose normal is $z = x_3$ are

$$\sigma_{3n} = \sigma_{3j} = \alpha\mu\{-y \;\; x \;\; 0\}^T. \quad \text{(third row)}. \tag{416}$$

Note that this vector is orthogonal to the vector $\{x \;\; y \;\; 0\}$ because

$$\alpha\mu\{-y \;\; x \;\; 0\}^T \cdot \{x \;\; y \;\; 0\}^T = -\alpha\mu yx + \alpha\mu xy = 0. \tag{417}$$

The vector $\{x, y, 0\}$ represents a radial vector outward from the center of a cross section from $z = 0$ to $z = r$ (because $x^2 + y^2 = r^2$.) This means that the vector $\{\sigma_{3i}\}$, which indicates the tractions on the top face, will always be orthogonal/perpendicular to the radial vector. In other words, for every radial vector, there is a perpendicular traction components vector. This means that in the same way the radial vectors are symmetrical, the tractions on the top face also have rotational symmetry, as in Fig. 8. This is called a shear stress vector since the diagonal stress component $\sigma_{33} = 0$.

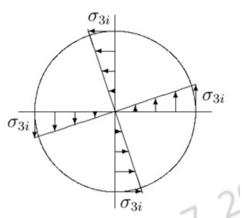



Figure 8: Rotational symmetry of shear stress $(\sigma_{31}, \sigma_{32})$ on the top face

If the rod has physical radius $R$, then the unit normal along the radius of the shaft is

$$\mathbf{n} = \{x/R \;\; y/R \;\; 0\}^T \rightarrow x^2/R^2 + y^2/R^2 = R^2/R^2 = 1. \tag{418}$$

Then, the traction vector is

$$\mathbf{t} = \boldsymbol{\sigma}\mathbf{n} = \frac{\alpha\mu}{R}\begin{bmatrix} 0 & 0 & -y \\ 0 & 0 & x \\ -y & x & 0 \end{bmatrix}\begin{Bmatrix} x \\ y \\ 0 \end{Bmatrix} = \frac{\alpha\mu}{R}\begin{Bmatrix} 0x + 0y - 0y \\ 0x + 0y + 0x \\ -yx + xy + 0 \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \\ 0 \end{Bmatrix}. \tag{419}$$

As for the traction components on the surface – that is, where $r = R$ – they vanish. This is a requirement for the torsional problem because there is an absence of surface forces assumed. It is also assumed that there is an absence of body forces, meaning for the equilibrium equation Eq. 381 to be satisfied, then

$$\sigma_{ij,j} + \cancel{\rho f_i} = \sigma_{ij,j} = 0. \tag{420}$$

Here, $\sigma_{1j,j} = \sigma_{11,1} + \sigma_{12,2} + \sigma_{13,3} = 0 + 0 - \frac{d}{dz}y = 0$. Likewise, $\sigma_{2j,j} = \sigma_{3j,j} = 0$. Therefore the equilibrium equation is satisfied in the absence of body forces, which is a proper assumption. Als, the compatibility equations need not be validated because we were given displacement to start with and not strain.

Finally, note that the magnitude of this shear stress is proportional to $r$. That is, at $r = 0$ (the center) the stres is zero and at $r = R$ the stress is maximized. Therefore the stress written in polar coordinates is some

$$\sigma_{z\theta} = \alpha\mu r. \tag{421}$$

The relationship between torque/moment $M$ and twist angle $\alpha$ is

$$M = r \times F = r\sigma A = \int\int r\sigma dA = \int\int r\sigma_{z\theta} r dr d\theta \tag{422}$$

where $dA = rdrd\theta$ and the cross product vanishes because the angle between the radius and the stress, as shown in the figure earlier, is always 90 degrees. Then

$$M = \int_0^{2\pi}\int_0^R r^3\alpha\mu dr d\theta = \mu\alpha(2\pi)R^4/4 = \mu\pi R^4\alpha/2 = \mu\alpha I = G\alpha I = GI\theta/L, \tag{423}$$

where moment of inertia for a circle??? $I = \pi R^4/2$ and $L = z$ is the outer circle. Then

$$\theta = \frac{ML}{IG} \tag{424}$$

solves for the twist angle. Uniqueness implies that the solution derived is unique and correct.

## 7.2 Noncircular shaft

Above was a circular torsion case. But suppose the cross section of the body is not circular. This leads to the same derivation as the previous section until Eq. 418, where the normal vector $\mathbf{n}$ was always assumed to be $\{x/R, y/R, 0\}$, which points from the center of the circle to a point outside of the circle $(x, y)$ and has length 1. Instead the normal cannot always assume to point radially outward in this way. Instead we remove this assumption and let the normal vector be some general

$$\mathbf{n} = \{n_1, n_2, 0\}. \tag{425}$$

Then from Eq. 419 we get

$$\mathbf{t} = \boldsymbol{\sigma}\mathbf{n} = \frac{\alpha\mu}{R}\begin{bmatrix} 0 & 0 & -y \\ 0 & 0 & x \\ -y & x & 0 \end{bmatrix}\begin{Bmatrix} n_1 \\ n_2 \\ 0 \end{Bmatrix} = \begin{bmatrix} 0 \\ 0 \\ -yn_1 + xn_2 \end{bmatrix}. \tag{426}$$

Because of our lack of knowledge of the traction vector in $z$ we thus have to remove the assumption that plane sections remain in the plane. Therefore we have to assume there could be some $z$-component of displacement, and this can be represented as

$$\begin{cases} u_1 = -\alpha z y \\ u_2 = \alpha z x \\ u_3 = \alpha\phi(x, y), \end{cases} \tag{427}$$

where $\phi(x, y)$ is the warping function. Exactly to what magnitude the shape is warped could be unique to each point in the plane which is why the function depends on plane dimensions $x, y$. Then the displacement gradient is

$$[u_{i,j}] = \begin{bmatrix} 0 & -\alpha z & -\alpha y \\ \alpha z & 0 & \alpha x \\ \alpha\phi_{,x} & \alpha\phi_{,y} & 0 \end{bmatrix}. \tag{428}$$

The strain matrix is the symmetric part of the displacement gradient $(u)i, j = u_{j,i})^T/2$ or

$$[\epsilon_{ij}] = \frac{1}{2}\begin{bmatrix} 0 & 0 & \alpha(\phi_{,x} - y) \\ 0 & 0 & \alpha(\phi_{,y} + x) \\ \alpha(\phi_{,x} - y) & \alpha(\phi_{,y} + x) & 0 \end{bmatrix}. \tag{429}$$

If the material is isotropic then

$$\sigma_{ij} = 2\mu\epsilon_{ij} + \delta_{ij}\cancel{\epsilon_{kk}}\lambda \to [\sigma_{ij}] = \mu\alpha\begin{bmatrix} 0 & 0 & (\phi_{,x} - y) \\ 0 & 0 & (\phi_{,y} + x) \\ (\phi_{,x} - y) & (\phi_{,y} + x) & 0 \end{bmatrix}. \tag{430}$$

Then traction

$$\mathbf{t} = \boldsymbol{\sigma}\mathbf{n} = \mu\alpha\begin{bmatrix} 0 & 0 & (\phi_{,x} - y) \\ 0 & 0 & (\phi_{,y} + x) \\ (\phi_{,x} - y) & (\phi_{,y} + x) & 0 \end{bmatrix}\begin{Bmatrix} n_1 \\ n_2 \\ 0 \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \\ (\phi_{,x} - y)n_1 + (\phi_{,y} + x)n_2 \end{Bmatrix} \tag{431}$$

if the normal is $n = \{n_1, n_2, 0\}$. This applies to some point on the side of the shaft where $n$ goes from the center to $x, y$. Now the top face of the shaft is perpendicular to this (as a cylinder's top face is perpendicular to its side). Normal is always of length 1 and so the normal vector going from the center to the top face is $\mathbf{n} = \{0, 0, 1\}$, meaning

$$\mathbf{t} = \boldsymbol{\sigma}\mathbf{n} = \mu\alpha \begin{bmatrix} 0 & 0 & (\phi_{,x} - y) \\ 0 & 0 & (\phi_{,y} + x) \\ (\phi_{,x} - y) & (\phi_{,y} + x) & 0 \end{bmatrix} \begin{Bmatrix} 0 \\ 0 \\ 1 \end{Bmatrix} = \begin{Bmatrix} \phi_{,x} - y \\ \phi_{,y} + x \\ 0 \end{Bmatrix}. \tag{432}$$

As enforced in the circular problem by Eq. 419, traction must be zero on the sides because the absence of surface forces is assumed. Therefore, the last component of the side traction

$$(\phi_{,x} - y)n_1 + (\phi_{,y} + x)n_2 = 0 \rightarrow \phi_{,x}n_1 + \phi_{,y}n_2 = n_1 y - n_2 x. \tag{433}$$

The left hand side is the same as

$$\phi_{,x}n_1 + \phi_{,y}n_2 = \begin{Bmatrix} \phi_{,x} & \phi_{,y} & 0 \end{Bmatrix} \begin{Bmatrix} n_1 \\ n_2 \\ 0 \end{Bmatrix} = \nabla\phi \cdot \mathbf{n} = \frac{\partial \phi}{\partial n}. \tag{434}$$

Therefore, substituting this in,

$$\frac{\partial \phi}{\partial n} = n_1 y - n_2 x. \tag{435}$$

Recall that the equilibrium equation

$$\sigma_{ij,j} = 0 \tag{436}$$

must be satisfied, as the absence of body forces is also assumed. Taking stress components from Eq. 430,

$$\sigma_{1j,j} = (\phi_{,x} - y)_{,z} = 0, \tag{437}$$

$$\sigma_{2j,j} = (\phi_{,y} + x)_{,z} = 0, \tag{438}$$

$$\sigma_{3j,j} = (\phi_{,x} - y)_{,x} + (\phi_{,y} + x)_{,y} = \phi_{,xx} + \phi_{,yy} = 0 = \nabla^2\phi. \tag{439}$$

Therefore the warping function must satisfy the 2D Laplace equation. Solutions of these will be harmonic functions. Therefore the solution to the noncircular shaft reduces to solving the equations

$$\begin{cases} \nabla^2\phi = 0, \\ \phi_{,x}n_1 + \phi_{,y}n_2 = \partial\phi/\partial n = n_1 y - n_2 x. \end{cases} \tag{440}$$

PDEs whose boundary conditions specify the value of the unknown function are Dirichlet problems. However, PDEs whose BCs are given by gradients of functions are called Neumann problems. This is a Neumann problem.

The resulting moment on top and sides of the shaft is ultimately desired. To do this first of all we calculate force in $x$, which is

$$F_x = t_x A = \int \mu\alpha(\phi_{,x} - y)dA. \tag{441}$$

Notice that

$$\phi_{,x} - y = \phi_{,x} - y + \overset{0}{(\phi_{,xx} + \phi_{,yy})} = \phi_{,x} - y + x(\phi_{,xx} + \phi_{,yy}) \tag{442}$$

$$= (-y + x\phi_{,xx} + \phi_{,x}) + (x\phi_{,yy}) = \frac{\partial}{\partial x}(-yx + x\phi_{,x}) + \frac{\partial}{\partial y}(x^2 + x\phi_{,y}) \tag{443}$$

$$= \frac{\partial}{\partial x}(x(-y + \phi_{,x})) + \frac{\partial}{\partial y}(x(x + \phi_{,y})). \tag{444}$$

Therefore

$$F_x = \int \mu\alpha \Big\{ \frac{\partial}{\partial x}(x(-y + \phi_{,x})) + \frac{\partial}{\partial y}(x(x + \phi_{,y})) \Big\} dA. \tag{445}$$

With the divergence theorem,

$$F_x = \mu\alpha \oint_S \Big\{ n_1(x(-y + \phi_{,x})) + n_2(x(x + \phi_{,y})) \Big\} dS \tag{446}$$

$$= \mu\alpha \oint_S x\Big\{ n_1(-y + \phi_{,x}) + n_2(x + \phi_{,y}) \Big\} dS = F_x = 0, \tag{447}$$

because 2d area is to 1d curve as 3d volume is to 2d surface area. The quantity is zero because again the force is related to the traction and the traction components must be zero. This result is similar to that of $F_y = 0$.

Moment on the top face is

$$M = \underbrace{r_x \times F_y + r_y \times F_x}_{\text{perpendicular components}} = \underbrace{r_x \times F_y - r_y \times F_x}_{F_x = 0} = \int_A (xt_2 - yt_1) dA \tag{448}$$

$$= \mu\alpha \int_A [(\phi_{,y} + x)x - (\phi_{,x} - y)y] dA = \mu\alpha \int_A (\phi_{,y}x - y\phi_{,x} + x^2 + y^2) dA. \tag{449}$$

Recall from the previous section that moment of inertia is

$$I = \pi R^4/2 = \int_0^{2\pi} \int_0^R r^3 dr d\theta = \int_0^{2\pi} \int_0^R (r^2) r dr d\theta = \int \int (x^2 + y^2) dA. \tag{450}$$

Therefore,

$$M = \mu\alpha \Big( I + \int_A (\phi_{,y}x - y\phi_{,x}) dA \Big) = \mu\alpha J, \tag{451}$$

where

$$I + \int_A (\phi_{,y}x - y\phi_{,x}) dA = J \tag{452}$$

is called the torsional constant. Note that if $\phi = 0$ then $J = I$. From Eq. 451,

$$J - I = \int_A (\phi_{,y}x - y\phi_{,x}) dA = \int_A [\frac{\partial}{\partial y}(\phi x) - \frac{\partial}{\partial x}(\phi y)] dA \tag{453}$$

$$= \oint_C (\phi x n_2 - \phi y n_1) dS = \oint_C (x n_2 - y n_1) \phi dS. \tag{454}$$

Sustituting in Eq. 435 ($\frac{\partial\phi}{\partial n} = n_1 y - n_2 x$),

$$J - I = \oint_C (\phi x n_2 - \phi y n_1) dS = -\oint_C \frac{\partial\phi}{\partial n}\phi dS = -\oint_C (\phi_{,x} n_1 + \phi_{,y} n_2)\phi dS \tag{455}$$

$$= -\oint_C (\nabla\phi\cdot\mathbf{n})\phi dS \leftrightarrow -\oint_C (\phi\phi_{,i}) n_i dS = -\int_A (\phi\phi_{,i})_{,i} dA \tag{456}$$

$$= -\int_A (\phi_{,i}\phi_{,i} + \phi\phi_{,ii}) dA \leftrightarrow -\int_A (\nabla\phi\cdot\nabla\phi + \phi\nabla^2\phi) dA = J - I. \tag{457}$$

Because of Eq. 440a,

$$J - I = -\int_A (\nabla\phi\cdot\nabla\phi) dA \longrightarrow J = I - \int_A (\nabla\phi\cdot\nabla\phi) dA. \tag{458}$$

## 7.3 Uniqueness of warrping function in torsion problem

To solve the noncircular shaft problem we have to solve for the warping function $\phi$ such that

$$\begin{cases} \nabla^2\phi = 0, \\ \partial\phi/\partial n = n_1 y - n_2 x, \end{cases} \tag{459}$$

as stated in Eq. 440. Now consider the logic used to prove uniqueness in Sec 6.1. It states that if uniqueness of the solution was NOT true then there would be some other warping function $\psi$ such that $\partial\psi/\partial n = n_1 y - n_2 x$. Then the difference of these two would be the system $\nabla^2\lambda = 0,\ \partial\lambda/\partial n = 0$. This can be renamed as

$$\begin{cases} \nabla^2\phi = 0 \\ \partial\phi/\partial n = 0. \end{cases} \tag{460}$$

So the uniqueness of this problem is the same as the uniqueness of the earlier problem because if this is true then it follows that $\partial\lambda/\partial n = 0 = \partial\phi/\partial n - \partial\psi/\partial n \longrightarrow \partial\phi/\partial n = \partial\psi/\partial n \longrightarrow$ the solutions are the same and so there is only one solution. From the first equation of the system,

$$0 = \nabla^2\phi = \phi_{,ii} \longrightarrow 0 = \int \phi\phi_{,ii} dA = \int (\phi\phi_{,i})_{,i} dA - \int \phi_{,i}\phi_{,i} dA \tag{461}$$

$$= \oint \phi(\phi_{,i} n_i) dS - \int_A \nabla\phi\cdot\nabla\phi dA = \oint \phi(\cancel{\partial\phi/\partial n}) dS - \int_A \nabla\phi\cdot\nabla\phi dA = 0. \tag{462}$$

Therefore

$$\int \nabla\phi\cdot\nabla\phi dA = \int |\nabla\phi|^2 dA = 0. \tag{463}$$

This implies $\nabla\phi = \mathbf{0}$ and so $\phi$ is a constant. This means that the difference between $\phi$ and $\psi$ can be at most a constant.

## 7.4 Existence of warping function in torsion problem

The solution to the problem of torsion a nonciruclar shaft is related to the solution of $\phi$ in

$$\begin{cases} \nabla^2\phi = 0 \\ \partial\phi/\partial n = \phi_{,x}n_1 + \phi_{,y}n_2 = yn_1 - xn_2. \end{cases} \tag{464}$$

Following the first equation,

$$0 = \int_A \nabla\phi^2 dA = \int_A \nabla\cdot\nabla\phi dA = \oint_S \nabla\phi\cdot\mathbf{n}dS = \oint_S \partial\phi/\partial n dS. \tag{465}$$

$$= \oint_S (n_1 y - n_2 x)dS = \int_A y_{,1} - x_{,2}dA = \int_A \frac{\partial}{\partial x}y - \frac{\partial}{\partial y}x dA = 0. \tag{466}$$

## 7.5 Some properties of harmonic functions

Functions that satisfy Laplace's equation are called harmonic functions. Harmonic functions

- (inside a circular domain) have a center value equal to the average of the surrounding values, and
- achieve their maxima and minima on the boundary.

Physically the first statement is seen in a steady state heat conduction on a circular plate. All boundary points have equal influence on the center point so the temperature at the center is the average of the surroundings.

## 7.6 Stress function for torsion

Recall the stress tensor is

$$[\boldsymbol{\sigma}] = \mu\alpha \begin{bmatrix} 0 & 0 & \phi_{,x} - y \\ 0 & 0 & \phi_{,y} + x \\ \phi_{,x} - y & \phi_{,y} + x & 0 \end{bmatrix}. \tag{467}$$

For a certain $i$ (row) the stress components must obey the equilibrium equation

$$\sigma_{ij,j} = 0$$

if the absence of body forces is assumed. For row three,

$$\sigma_{3j,j} = \mu\alpha(\phi_{,x} - y)_{,x} + \mu\alpha(\phi_{,y} + x)_{,y} = \mu\alpha(\phi_{,xx} + \phi_{,yy}) = \mu\alpha\nabla^2\phi = 0 \longrightarrow \nabla^2\phi = 0 \tag{468}$$

must be true. But instead of the expansion carried out as such let us instead assume a stress function $\psi$ so that

$$\sigma_{31} = \psi_{,y}, \quad \sigma_{32} = -\psi_{,x}. \tag{469}$$

Then

$$\sigma_{31,1} = \psi_{,yx}, \quad \sigma_{32,2} = -\psi_{,xy}. \tag{470}$$

This leads to the automatic satisfaction of the equilibrium equation in that

$$\sigma_{31,1} + \sigma_{32,2} = \psi_{,yx} - \psi_{,xy} = 0 \tag{471}$$

without any further assumption. Substituting in the appropriate values for $\sigma$ in Eq. 469,

$$\psi_{,y} = \mu\alpha(\phi_{,x} - y), \quad -\psi_{,x} = \mu\alpha(\phi_{,y} + x). \tag{472}$$

Then

$$\psi_{,yy} + \psi_{,xx} = \nabla^2\psi = \mu\alpha(\phi_{,xy} - 1 - \phi_{,yx} - 1) = -2\mu\alpha = \nabla^2\psi. \tag{473}$$

Recall that Laplacain is divergence of gradient and so maintains rank. Therefore the curves $\psi(x, y)$ are constants, just as $-2\mu\alpha$ is a constant. So, their derivatives are zero. Taking the derivative of $\psi(x, y)$ with respect specifically to $x$,

$$\frac{d}{dx}\psi(x, y) = \frac{\partial\psi}{\partial x} + \frac{\partial\psi}{\partial y}\frac{\partial y}{\partial x} = 0. \tag{474}$$

Rearranged,

$$\frac{\partial y}{\partial x} = -\frac{\partial\psi}{\partial x}\Big/\frac{\partial\psi}{\partial y}. \tag{475}$$

Substituting in Eq. 469,

$$\frac{\partial y}{\partial x} = \frac{\sigma_{32}}{\sigma_{31}} = \frac{\sigma_{zy}}{\sigma_{zx}}. \tag{476}$$

This is the change in $y$ relative to the change in $x$ of the curve $\psi$. In other words it is its slope. So the tangent vector is

$$\sigma_{zy}\mathbf{e}_y + \sigma_{zx}\mathbf{e}_x = \mathbf{T}. \tag{477}$$

Recall also that the $z \leftrightarrow 3$ component of the traction vector is

$$t_3 = \sigma_{31}n_1 + \sigma_{32}n_2 = \mathbf{T}\cdot\mathbf{n} = 0. \tag{478}$$

This is to say that the dot product of the slope of $\psi$ and the normal vector $\mathbf{n}$ is zero, meaning the stress function is perpendicular to the boundary and so is constant at the boundary. For convenience we let $\psi = 0$ because a constant shift in the stress function is not meaningful since the derivatives of $\psi$ are $\sigma$, and this will be zero regardless of what constant $\psi$ is.

So the two conditions are

$$\begin{cases} \nabla^2\psi = -2\mu\alpha, & \text{body,} \\ \psi = 0, & \text{boundary,} \end{cases} \tag{479}$$

where

$$\psi_{,y} = \sigma_{31}, \quad \psi_{,x} = \sigma_{32}. \tag{480}$$

Note that

$$\nabla\psi \cdot \mathbf{T} = \sigma_{32}\sigma_{31} - \sigma_{31}\sigma_{32} = 0, \tag{481}$$

meaning the rate of change of $\psi$ is perpendicular to the tangent vector. Therefore $\nabla\psi$ happens most when it is perpendicular to $\psi$ (technically the line vector that is tangent to $\psi$ at a certain point).

Torque/moment at the top face ($\mathbf{n} = \{0, 0, 1\}$) is

$$M = \int_A (xt_2 - yt_1) = \int_A (x\sigma_{23}n_3 - y\sigma_{13}n_3)dA = \int_A x\sigma_{32} - y\sigma_{31}dA = \int_A -x\psi_{,x} - y\psi_{,y}dA \tag{482}$$

$$= -\int\int x\frac{\partial\psi}{\partial x}dxdy - \int\int y\frac{\partial\psi}{\partial y}dxdy \tag{483}$$

$$= -\int\left(\int x\frac{\partial\psi}{\partial x}dx\right)dy - \int\left(\int y\frac{\partial\psi}{\partial y}dy\right)dx. \tag{484}$$

Integration by parts is

$$\int udv = uv - \int vdu. \tag{485}$$

Looking at the first term, if $u = x \longrightarrow du = dx$ and $dv = (\partial\psi/\partial x)dx, \longrightarrow v = \psi$, then

$$M = -\int\left(\cancel{\psi x} - \int \psi dx\right)dy - \int\left(\cancel{\psi y} - \int \psi dy\right)dx \tag{486}$$

$$= 2\int\int \psi dxdy = 2\int_A \psi dA = M. \tag{487}$$

If it is also true from Eq. 451 that

$$M = \alpha\mu J, \tag{488}$$

then

$$J = \frac{2}{\mu\alpha}\int_A \psi dA. \tag{489}$$

## 7.7 Torsion of elliptical cylinder

The ellipse equation is

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1 \longrightarrow \frac{x^2}{a^2} + \frac{y^2}{b^2} - 1 = 0. \tag{490}$$

Note a few things, that $a = b = 1$ leads to a circle equation with $r = 1$ and that an increase in $|a|$ leads to stretchiness in $x$ while an increase in $|b|$ leads to strethiness in $y$.

The boundary of a elliptical cylinder is clearly an ellipse. Also remmeber that the stress function at the boundary is $\psi = 0$. This is given by Eq. 479. Therefore,

$$0 = \psi = c\left(\frac{x^2}{a^2} + \frac{y^2}{b^2} - 1\right). \tag{491}$$

Here $c$ is a constant. Remember the other condition contained in Eq. 479 which is that $\nabla^2\psi = -2\mu\alpha$. Therefore

$$-2\mu\alpha = \nabla^2\psi = c\left(\frac{2}{a^2} + \frac{2}{b^2}\right). \tag{492}$$

Therefore,

$$c = -\mu\alpha \Big/ \left(\frac{1}{a^2} + \frac{1}{b^2}\right). \tag{493}$$

Substituting this into Eq. 491,

$$\psi = -\mu\alpha\left(\frac{x^2}{a^2} + \frac{y^2}{b^2} - 1\right) \Big/ \left(\frac{1}{a^2} + \frac{1}{b^2}\right) \tag{494}$$

$$= \mu\alpha\left(1 - \frac{x^2}{a^2} - \frac{y^2}{b^2}\right) \Big/ \left(\frac{b^2}{a^2b^2} + \frac{a^2}{b^2a^2}\right) \tag{495}$$

$$= \mu\alpha\left(1 - \frac{x^2}{a^2} - \frac{y^2}{b^2}\right)\left(\frac{a^2b^2}{a^2+b^2}\right) \tag{496}$$

$$= \frac{\mu\alpha a^2b^2}{a^2+b^2}\left(1 - \frac{x^2}{a^2} - \frac{y^2}{b^2}\right) = \psi. \tag{497}$$

From Eq. 489,

$$J = \frac{2}{\mu\alpha}\int_A \psi dA \tag{498}$$

$$= \frac{2a^2b^2}{a^2+b^2}\int_A \left(1 - \frac{x^2}{a^2} - \frac{y^2}{b^2}\right) dA \tag{499}$$

$$= \frac{2a^2b^2}{a^2+b^2}\left[A - \frac{1}{a^2}\int_A (x^2+0)dA - \frac{1}{b^2}\int_A (0+y^2)dA\right] \tag{500}$$

$$= \frac{2a^2b^2}{a^2+b^2}\left[A - \frac{1}{a^2}\int_0^{2\pi}\int_0^a r^3 dr d\theta - \frac{1}{b^2}\int_0^b r^3 dr d\theta\right] \tag{501}$$

$$=^{\text{see end of chapter}} \frac{2a^2b^2}{a^2+b^2}\left[\pi ab - \frac{\pi}{4}ab - \frac{\pi}{4}ba\right] = \frac{\pi a^3b^3}{a^2+b^2} = J. \tag{502}$$

Using Eq. 497,

$$\psi_{,y} = \sigma_{31} = \frac{\mu\alpha a^2b^2}{a^2+b^2}\left(-\frac{2y}{b^2}\right) = -\frac{2\mu\alpha a^2 y}{a^2+b^2}, \tag{503}$$

$$-\psi_{,x} = \sigma_{32} = -\frac{\mu\alpha a^2b^2}{a^2+b^2}\left(-\frac{2x}{a^2}\right) = \frac{2\mu\alpha b^2 x}{a^2+b^2}. \tag{504}$$

At the boundary,

$$\sigma_{31}\Big|_{y=b} = -\frac{2\mu\alpha a^2 b}{a^2+b^2} = -\left(\frac{2\mu\alpha ab}{a^2+b^2}\right)a, \tag{505}$$

$$\sigma_{32}\Big|_{x=a} = \left(\frac{2\mu\alpha ba}{a^2+b^2}\right)b. \tag{506}$$

## 7.8 Torsion of rectangular bars: warping function

The torsion problem can be solved in terms of either the warping function or the stress function: $\phi$ or $\psi$, just like the cylindrical bar. Here we start with $\phi$. Consider a rectangle with side lengths $2a = L_x$ and $2b = L_y$. The origin is the center so that the corners are $(a, -b), (a, b), (-a, b), (-a, -b)$, if starting from the bottom right and going counter-clockwise. Then, also starting from the bottom right and going CCW, the corresponding stress functions are

$$\begin{cases} x = a, & \partial\phi/\partial n = \partial\phi/\partial x = y \\ y = b, & \partial\phi/\partial n = \partial\phi/\partial y = -x \\ x = -a, & \partial\phi/\partial n = \partial\phi/\partial x = -y \\ y = -b, & \partial\phi/\partial n = \partial\phi/\partial y = x \\ \nabla^2\phi = 0, & \text{body.} \end{cases} \tag{507}$$

So the change in stress function is commensurate with the point on the boundary. That is if the boundary point is on the right boundary then $\partial\phi/\partial n$ increases as you go "up" it, i.e. as $y$ increases.

Notice that these are all odd functions of $x$ and $y$. That is because they mirror each other along $y = x$. That is, $f(-x) = -f(x)$. Because of this the solution to this system must be antisymmetric in $y$ and $x$. This means that $\partial\phi/\partial n$ in between the two boundaries must be exactly zero, from which it can be concluded that $\phi$ is a constant here, which we set to zero. Then a reduced system is

$$\begin{cases} x = a, & \partial\phi/\partial n = \partial\phi/\partial x = y \\ y = b, & \partial\phi/\partial n = \partial\phi/\partial y = -x \\ x = 0, & \phi = 0, \\ y = 0, & \phi = 0, \\ \nabla^2\phi = 0, & \text{body.} \end{cases} \tag{508}$$

We then introduce the transformation function $w = w(x, y)$ such that

$$\phi(x, y) = xy - w(x, y) \tag{509}$$

$$\longrightarrow \frac{\partial\phi}{\partial x} = y - \frac{\partial w}{\partial x}, \qquad \frac{\partial\phi}{\partial y} = x - \frac{\partial w}{\partial y}. \tag{510}$$

Then,

$$\frac{\partial w}{\partial x} = y - \frac{\partial\phi}{\partial x}, \qquad \frac{\partial w}{\partial y} = x - \frac{\partial\phi}{\partial y}. \tag{511}$$

Then, the system becomes

$$\begin{cases} x = a, & \partial w/\partial x = y - y = 0, \\ y = b, & \partial w/\partial y = x - (-x) = 2x, \\ x = 0, & w = 0, \\ y = 0, & w = 0, \\ \text{body,} & \nabla^2 w = 0. \end{cases} \tag{512}$$

Using separation of variables the solution to the function $w(x, y)$ can be represented as some

$$X(x)Y(y) = w(x, y). \tag{513}$$

Then

$$\nabla^2 w = \frac{\partial^2}{\partial x^2} w + \frac{\partial^2}{\partial y^2} w = X''(x)Y(y) + X(x)Y(y)'' = 0, \tag{514}$$

which implies

$$\frac{X''}{X} = -\frac{Y''}{Y} = -\lambda^2, \tag{515}$$

where $\lambda$ is some so-called separation constant. The form of Eq. 515 can be used to form the two separate ODEs

$$X'' = -\lambda^2 X, \quad Y'' = \lambda^2 Y. \tag{516}$$

These have the solutions

$$X(x) = A \sin \lambda x + B \cos \lambda x \leftrightarrow X = Ae^{i\lambda x} + Be^{-i\lambda x}, \tag{517}$$

$$Y(y) = C \sinh \lambda y + D \cosh \lambda y. \tag{518}$$

Since the stress function is odd across the axes $x, y$, the established boundary conditions were $w(0, y) = w(x, 0) = 0$. Therefore,

$$X(0)Y(y) = X(x)Y(0) = 0 \longrightarrow X(0) = Y(0) = 0 \tag{519}$$

$$\longrightarrow X(0) = A \sin \lambda 0 + B \cos \lambda 0 = B = 0, \tag{520}$$

$$Y(0) = C \sinh \lambda 0 + D \cosh \lambda 0 = D = 0. \tag{521}$$

Therefore the solutions reduce to

$$X(x) = A \sin \lambda x, \quad Y(y) = C \sinh \lambda y. \tag{522}$$

The other boundary conditions are $\partial w/\partial y|_{y=b} = 2x, \quad \partial w/\partial x|_{x=a} = 0$. Because of the latter statement,

$$X'(a)Y(y) = 0 \longrightarrow 0 = X'(a) = \lambda A \cos \lambda x \rightarrow \lambda_n = (2n-1)\frac{\pi}{2}. \tag{523}$$

This is because $\cos 90 = \cos 180 = \cos 270 \leftrightarrow \cos(\pi/2) = \cos(3\pi/2) = \cos(5\pi/2) = \cos((2n-1)\pi/2) = 0$. Note that this implies the existence of a series of $\lambda$'s with index $n$, called $\lambda_n$. Therefore,

$$X(x) = A \sin \lambda_n x, \quad Y(y) = C \sinh \lambda_n y, \tag{524}$$

and altogether,

$$w = XY = \sum_n^\infty A_n \sin \lambda_n x \sinh \lambda_n y, \tag{525}$$

where $AC$ has been combined (as constants are arbitrarily named) and indexed for each $\lambda_n$. This series satisfies $\nabla^2 w = 0$, which was the original statement given by Eq. 514. Using the final boundary condition $y = b \rightarrow w = \partial\phi/\partial y = 2x$,

$$2x = XY' = \sum_{n}^{\infty} A_n \lambda_n \sin \lambda_n x \cosh \lambda_n b. \tag{526}$$

Multiplying on both sides,

$$\int_0^a 2x \sin \lambda_m x dx = \sum_{n}^{\infty} A_n \lambda_n \cosh \lambda_n b \int_0^a \sin \lambda_m x \sin \lambda_n x dx, \tag{527}$$

where on the right side,

$$\int_0^a \sin \lambda_m x \sin \lambda_n x dx = \begin{cases} 0, & n \neq m, \\ a/2, & n = m, \end{cases} \tag{528}$$

and on the left side,

$$\int_0^a x \sin \lambda_m x dx = \frac{1}{\lambda_m^2}(-1)^{m+1}. \tag{529}$$

Since only $n = m$ remains in the series, it no longer becomes an infinite series, and

$$\phi = xy - w = xy - \frac{32a^2}{\pi^3} \sum_{n=1}^{\infty} \frac{-1^{n+1}}{(2n-1)^3} \frac{1}{\cosh \lambda_n b} \sin \lambda_n x \sinh \lambda_n y m, \tag{530}$$

and the torsional constant is estimated as

$$J = I + \int_{-b}^{b} \int_{-a}^{a} \left( x \frac{\partial \phi}{\partial y} - y \frac{\partial \phi}{\partial x} \right) dx dy \tag{531}$$

$$\approx \frac{16}{3} a^3 b \left[ 1 - \frac{192a}{\pi^5 b} \sum_{n=1}^{\infty} \frac{\tanh \lambda_n b}{(2n-1)^5} \right] = \kappa a^3 b \approx J, \tag{532}$$

where $\kappa$ depends on the $b/a$ ratio. Particularly,

$$\kappa = \begin{cases} 2.249, & b/a = 1.0, \\ 3.659, & b/a = 2.0, \\ 4.661, & b/a = 5.0, \\ 5.333, & b/a \rightarrow \infty. \end{cases} \tag{533}$$

Ignoring higher order terms of the series, another reasonable approximation for the torsional constant is

$$J \approx \frac{16}{3} a^3 b \left( 1 - \frac{192a}{\pi^5 b} \tanh \frac{\pi b}{2a} \right). \tag{534}$$

Some steps of this derivation are not shown (Li pp. 74-75).

## 7.9 Torsion of rectangular bars: stress function

We now move to solving the torsion problem in terms of $\psi$.

The two conditions on the stress function are Eq. 479, which are

$$\begin{cases} \nabla^2\psi = -2\mu\alpha, & \text{body,} \\ \psi = 0, & \text{boundary.} \end{cases}$$

where

$$-\psi_{,x} = \sigma_{32} = \sigma_{zy}, \qquad \psi_{,y} = \sigma_{31} = \sigma_{zx}.$$

One solution to the first equation is

$$\psi = -\mu\alpha x^2 + C. \tag{535}$$

Then, $\nabla^2\psi = \partial^2\psi/\partial x^2 = -2\mu\alpha$. Then, let

$$C = \mu\alpha a^2 + \mu\alpha v. \tag{536}$$

Substituting this in,

$$\psi = \mu\alpha(a^2 - x^2) + \mu\alpha v. \tag{537}$$

The other boundary conditions are satisfied if

$$\begin{cases} v = 0, & x = \pm a \to \psi = 0 + \cancel{\mu\alpha v}, \\ v = x^2 - a^2, & y = \pm b \to \psi = \mu\alpha(a^2 - x^2) + \mu\alpha v, \\ \nabla^2 v = 0, & \text{body} \to \nabla^2\psi = -2\mu\alpha + \cancel{\mu\alpha\nabla^2 v}. \end{cases} \tag{538}$$

We observe that these boundary conditions are even along axes $x$ and $y$ because $f(x) = f(-x)$ and likewise $f(y) = f(-y)$. ... The eventual result is

$$\psi = \mu\alpha(a^2 - x^2) + \mu\alpha\sum_{n=1}^{\infty} A_n \cos\lambda_n x \cosh\lambda_n y, \tag{539}$$

$$J = \frac{16}{3}a^3 b - a^4\left(\frac{4}{\pi}\right)^5 \sum_{n=1}^{\infty} \frac{\tanh\lambda_n b}{(2n-1)^5}. \tag{540}$$

Eq. 540 is similar to Eq. 532.

## Note

Letting

$$y = br\sin\theta, \quad x = ar\cos\theta \longleftarrow \frac{x^2}{a^2} + \frac{y^2}{b^2} = r^2(\sin^2\theta + \cos^2\theta) = r^2 = 1 \leftrightarrow r = 1, \tag{541}$$

the conversion from the Cartesian space to the polar space is

$$\int_\Omega f(x,y)dA = \int\int f(r,\theta)\det\mathbf{J}drd\theta. \tag{542}$$

See jjmarzia-mae529 pp. 17. The determinant of the Jacobian is

$$\det \mathbf{J} = \det \begin{bmatrix} \partial x/\partial r & \partial y/\partial r \\ \partial x/\partial\theta & \partial y/\partial\theta \end{bmatrix} = \det \begin{bmatrix} a\cos\theta & b\sin\theta \\ -ar\sin\theta & br\cos\theta \end{bmatrix} = abr\cos^2\theta + abr\sin^2\theta = abr. \tag{543}$$

Therefore

$$\int_A y^2 dA = \int_0^{2\pi}\int_0^1 (b^2r^2\sin^2\theta)abrdrd\theta = ab^3\int_0^{2\pi}\sin^2\theta d\theta\int_0^1 r^3dr. \tag{544}$$

The trig identity

$$\cos 2\theta = \cos^2\theta - \sin^2\theta = (1-\sin^2\theta) - \sin^2\theta = 1 - 2\sin^2\theta \tag{545}$$

$$\rightarrow \sin^2\theta = \frac{1}{2} - \frac{1}{2}\cos 2\theta. \tag{546}$$

Substituting this in,

$$\int_A y^2 dA = ab^3\int_0^{2\pi}\left(\frac{1}{2} - \frac{1}{2}\cos 2\theta\right)d\theta\int_0^1 r^3dr \tag{547}$$

$$= ab^3(\pi - \frac{1}{4}\sin 2\theta|_0^{2\pi})\frac{1}{4} = \frac{1}{4}\pi ab^3. \tag{548}$$

Likewise,

$$\int_A x^2 dA = \frac{1}{4}\pi a^3 b. \tag{549}$$

# 8 Plane deformation

## 8.1 Plane strain

A body is in plane strain if $u_1 = u_1(x, y), u_2 = u_2(x, y), u_3 = 0$. This means that for each $xy$ plane with a fixed $z$, there is no $z$ displacement. In other words, cross sections do not overlap. For cylindrical bodies this is true if they either have infinite length or have finite length with fixed ends. Under plane strain conditions,

$$\epsilon_{31} = \epsilon_{32} = \epsilon_{33} = 0 \leftrightarrow \epsilon_{3j} = 0, \tag{550}$$

because

$$\epsilon_{ij} = \frac{1}{2}(u_{i,j} + u_{j,i}) \tag{551}$$

$$\rightarrow \epsilon_{31} = \frac{1}{2}(u_{1,3} + u_{3,1}) = 0 + 0 = 0, \tag{552}$$

etc. Now, recall Eq. 357, which is strictly a way to compute diagonal strain elements of isotropic materials. It is

$$\epsilon_{33} = -\frac{\sigma_{11}}{E}\nu - \frac{\sigma_{22}}{E}\nu + \frac{\sigma_{33}}{E}.$$

This implies

$$0 = -\sigma_{11}\nu - \sigma_{22}\nu + \sigma_{33} \longrightarrow \sigma_{33} = \nu(\sigma_{11} + \sigma_{22}). \tag{553}$$

Using Eq. 333 ($\sigma_{ij} = \delta_{ij}\epsilon_{kk}\lambda + 2\mu\epsilon_{ij}$), it is known that $\sigma_{ij} = \sigma_{ij}(x, y)$ because this is true of $\epsilon_{ij} = \epsilon_{ij}(x, y)$, and this is the case because of the displacements. Therefore

$$\begin{cases} \sigma_{11} = \sigma_{11}(x_1, x_2) \\ \sigma_{22} = \sigma_{22}(x_1, x_2) \\ \sigma_{33} = \nu(\sigma_{11} + \sigma_{22}) \\ \sigma_{12} = \sigma_{12}(x_1, x_2) \\ \sigma_{23} = 0 \\ \sigma_{13} = 0. \end{cases} \tag{554}$$

Recall the equilibrium equation Eq. 275, which comes from Eq. 261 → Eq. 275. It assumes the absence of body acceleration. It is

$$\sigma_{ij,j} + \rho f_i = 0.$$

Using this,

$$\begin{cases} 0 = \sigma_{11,1} + \sigma_{12,2} + \sigma_{13,3} + \rho f_1 = \sigma(x, y) + f_1 = 0 \rightarrow f_1 = f_1(x, y), \\ 0 = \sigma_{21,1} + \sigma_{22,2} + \sigma_{23,3} + \rho f_2 = \sigma(x, y) + f_2 = 0 \rightarrow f_2 = f_2(x, y), \\ 0 = \sigma_{3j,j} + f_3 = 0 \rightarrow f_3 = 0. \end{cases} \tag{555}$$

So there is no $z$ component of the body force, only $x, y$ components. Of the compatibility equations Eq. 190 there is only one nontrivial iteration, and that is

$$2\epsilon_{21,12} = \epsilon_{11,22} + \epsilon_{22,11}$$

because it does not involve any 3−components.

## 8.2 Plane stress

Recall that plane strain implies $\epsilon_{3j} = 0$.Very similarly, a body is in a state of plane stress if

$$\sigma_{3j} = 0 \leftrightarrow \sigma_{31} = \sigma_{32} = \sigma_{33} = 0. \tag{556}$$

Hooke's law is Eq. 333 ($\sigma_{ij} = \lambda\epsilon_{kk}\delta_{ij} + 2\mu\epsilon_{ij}$), and so

$$\begin{cases} \sigma_{31} = 0 = 2\mu\epsilon_{31} \to \epsilon_{31} = 0, \\ \sigma_{32} = 0 = 2\mu\epsilon_{32} \to \epsilon_{32} = 0, \\ \sigma_{33} = 0 = \lambda\epsilon_{11} + \lambda\epsilon_{22} + (\lambda + 2\mu)\epsilon_{33}, \end{cases} \tag{557}$$

the last statement implying

$$\epsilon_{33} = -\frac{\lambda}{\lambda + 2\mu}(\epsilon_{11} + \epsilon_{22}). \tag{558}$$

If Eq. 349 is true ($\nu = \lambda/(2\lambda + 2\mu)$), then

$$-\frac{\lambda}{\lambda + 2\mu} = \frac{-\lambda/(2\lambda + 2\mu)}{\lambda + 2\mu/(2\lambda + 2\mu)} \tag{559}$$

$$= \frac{-\nu}{-\lambda + 2\lambda + 2\mu/(2\lambda + 2\mu)} = \frac{-\nu}{-\nu + 1}. \tag{560}$$

Therefore,

$$\epsilon_{33} = -\frac{\nu}{1 - \nu}(\epsilon_{11} + \epsilon_{22}). \tag{561}$$

Again because of Eq. 357, if the material is isotropic then

$$\epsilon_{33} = -\frac{\sigma_{11}}{E}\nu - \frac{\sigma_{22}}{E}\nu + \cancel{\frac{\sigma_{33}}{E}} = \frac{-\nu}{E}(\sigma_{11} + \sigma_{22}). \tag{562}$$

The static equilibrium equations Eq. 275, or

$$\sigma_{ij,j} + \rho f_i = 0 \tag{563}$$

imply

$$\begin{cases} 0 = \sigma_{11,1} + \sigma_{12,2} + \sigma_{13,3} + \rho f_1, \\ 0 = \sigma_{21,1} + \sigma_{22,2} + \sigma_{23,3} + \rho f_2, \\ 0 = \sigma_{31,1} + \sigma_{32,2} + \sigma_{33,3} + \rho f_3 = \rho f_3 \to f_3 = 0. \end{cases} \tag{564}$$

Of the compatibility equations Eq. 190 there is only one nontrivial iteration, and that is

$$2\epsilon_{21,12} = \epsilon_{11,22} + \epsilon_{22,11}$$

because it does not involve any 3−components.

## 8.3 Formal equivalence between plane stress/plane strain

Restating the inverse Hooke's law of Eq. 357,

$$\epsilon_{xx} = \frac{1}{E}\sigma_{xx} - \frac{\nu}{E}\sigma_{yy} - \frac{\nu}{E}\sigma_{zz}, \tag{565}$$

$$\epsilon_{yy} = -\frac{\nu}{E}\sigma_{xx} + \frac{1}{E}\sigma_{yy} - \frac{\nu}{E}\sigma_{zz}, \tag{566}$$

$$\epsilon_{zz} = -\frac{\nu}{E}\sigma_{xx} - \frac{\nu}{E}\sigma_{yy} + \frac{1}{E}\sigma_{zz}. \tag{567}$$

### 8.3.1 Plane stress

In plane stress conditions where $\sigma_{zz} = 0$,

$$\epsilon_{xx} = \frac{1}{E}\sigma_{xx} - \frac{\nu}{E}\sigma_{yy} \rightarrow \begin{cases} \sigma_{xx} = E\epsilon_{xx} + \nu\sigma_{yy} \\ \sigma_{yy} = -\frac{E}{\nu}\epsilon_{xx} + \frac{1}{\nu}\sigma_{xx} \end{cases} \tag{568}$$

$$\epsilon_{yy} = \frac{-\nu}{E}\sigma_{xx} + \frac{1}{E}\sigma_{yy} \rightarrow \begin{cases} \sigma_{xx} = \frac{E}{-\nu}\epsilon_{yy} + \frac{1}{\nu}\sigma_{yy} \\ \sigma_{yy} = E\epsilon_{yy} + \nu\sigma_{xx}. \end{cases} \tag{569}$$

Substituting the first branched equation into the fourth,

$$\sigma_{yy} = E\epsilon_{yy} + \nu(E\epsilon_{xx} + \nu\sigma_{yy}) = E\epsilon_{yy} + \nu E\epsilon_{xx} + \nu^2\sigma_{yy} \longrightarrow \sigma_{yy} = \frac{E}{1-\nu^2}(\epsilon_{yy} + \nu\epsilon_{xx}). \tag{570}$$

Substituting the second branch into the third,

$$\sigma_{xx} = -\frac{E}{\nu}\epsilon_{yy} + \frac{1}{\nu}(-\frac{E}{\nu}\epsilon_{xx} + \frac{1}{\nu}\sigma_{xx}) = -\frac{E}{\nu}\epsilon_{yy} - \frac{E}{\nu^2}\epsilon_{xx} + \frac{1}{\nu^2}\sigma_{xx}$$

$$\rightarrow \sigma_{xx}\left(\frac{1}{\nu^2} - 1\right) = E(\frac{1}{\nu}\epsilon_{yy} + \frac{1}{\nu^2}\epsilon_{xx})$$

$$\rightarrow \sigma_{xx}\left(\frac{1-\nu^2}{\nu^2}\right) = E(\frac{1}{\nu}\epsilon_{yy} + \frac{1}{\nu^2}\epsilon_{xx})$$

$$\rightarrow \sigma_{xx} = \frac{E}{1-\nu^2}(\nu\epsilon_{yy} + \epsilon_{xx}). \tag{571}$$

Then substituting these results into Eq. 567,

$$\epsilon_{zz} = -\frac{\nu}{E}(\sigma_{xx} + \sigma_{yy}) = -\frac{\nu}{E}\frac{E}{1-\nu^2}(\epsilon_{yy} + \nu\epsilon_{xx} + \nu\epsilon_{yy} + \epsilon_{xx}) \tag{572}$$

$$= -\frac{\nu}{(v+1)(-v+1)}((v+1)(\epsilon_{yy} + \epsilon_{xx})) \tag{573}$$

$$= -\frac{\nu}{-v+1}(\epsilon_{yy} + \epsilon_{xx}) = \epsilon_{zz}. \tag{574}$$

### 8.3.2 Plane strain

Also using Eq. 567 but assuming instead plane strain conditions or $\epsilon_{zz} = 0$,

$$0 = -\frac{\nu}{E}\sigma_{xx} - \frac{\nu}{E}\sigma_{yy} + \frac{1}{E}\sigma_{zz} \tag{575}$$

$$\longrightarrow \sigma_{zz} = \nu(\sigma_{xx} + \sigma_{yy}) \tag{576}$$

which implies, using Eqs. 565-566,

$$\epsilon_{xx} = \frac{1}{E}\sigma_{xx} - \frac{\nu}{E}\sigma_{yy} - \frac{\nu^2}{E}(\sigma_{xx} + \sigma_{yy}) = \frac{1-\nu^2}{E}\sigma_{xx} - \frac{\nu+\nu^2}{E}\sigma_{yy}, \tag{577}$$

$$\epsilon_{yy} = -\frac{\nu}{E}\sigma_{xx} + \frac{1}{E}\sigma_{yy} - \frac{\nu^2}{E}(\sigma_{xx} + \sigma_{yy}) = -\frac{\nu+\nu^2}{E}\sigma_{xx} + \frac{1-\nu^2}{E}\sigma_{yy}. \tag{578}$$

Let the constants

$$\frac{1}{\bar{E}} = \frac{1-\nu^2}{E}, \qquad \frac{\bar{\nu}}{\bar{E}} = \frac{\nu+\nu^2}{E} \longrightarrow \bar{\nu} = \frac{\nu(\nu+1)}{E}\frac{E}{1-\nu^2} = \frac{\nu(\nu+1)}{(\nu+1)(-\nu+1)} \tag{579}$$

### 8.3.3 Conversion

$$= \frac{\nu}{-\nu+1} = \bar{\nu}, \qquad \bar{E} = \frac{E}{-\nu^2+1}. \tag{580}$$

This is the conversion between plane stress results and plane strain results. So if one obtains results for plane stress they can substitute $E, \nu$ with $\bar{E}, \bar{\nu}$ and in doing so obtain results for plane strain. This is because if this is substituted into Eqs. 577,578, then

$$\epsilon_{xx} = \frac{1}{\bar{E}}\sigma_{xx} - \frac{\bar{\nu}}{\bar{E}}\sigma_{yy}, \tag{581}$$

$$\epsilon_{yy} = -\frac{\bar{\nu}}{\bar{E}}\sigma_{xx} + \frac{1}{\bar{E}}\sigma_{yy}, \tag{582}$$

and this in stead of $E, \nu$ perfectly matches up with $\epsilon_{xx}, \epsilon_{yy}$ in plane stress conditions, represented by Eqs. 568,569.

## 8.4 Compatibility equation in terms of stress

Recall that for 2d plane stress/plane strain problems, the only nontrivial compatibility equation which remains is

$$2\epsilon_{21,12} = \epsilon_{11,22} + \epsilon_{22,11}. \tag{583}$$

If the material is isotropic then Eq. 338 is used to obtain

$$\epsilon_{11} = \frac{1}{E}\sigma_{11} - \frac{\nu}{E}\sigma_{22}, \tag{584}$$

$$\epsilon_{22} = -\frac{\nu}{E}\sigma_{11} + \frac{1}{E}\sigma_{22}, \tag{585}$$

$$\epsilon_{12} = \frac{1}{2\mu}\sigma_{12} = \frac{1+\nu}{E}\sigma_{12} \longrightarrow 2\epsilon_{12} = 2\frac{1+\nu}{E}\sigma_{12}, \tag{586}$$

where the last equation is obtained from the conversion between $\mu$ and $E$ which is introduced in Eq. 351. Substituting these into Eq. 583,

$$2\frac{1+\nu}{E}\sigma_{12,12} = \frac{1}{E}\sigma_{11,22} - \frac{\nu}{E}\sigma_{22,22} - \frac{\nu}{E}\sigma_{11,11} + \frac{1}{E}\sigma_{22,11} \tag{587}$$

implies

$$2(1+\nu)\sigma_{12,12} = \sigma_{11,22} - \nu\sigma_{22,22} - \nu\sigma_{11,11} + \sigma_{22,11}. \tag{588}$$

At the same time, static equilibrum conditions with zero body force is assumed so that

$$\begin{cases} \sigma_{11,1} + \sigma_{12,2} = 0, \\ \sigma_{21,1} + \sigma_{22,2} = 0, \end{cases} \tag{589}$$

meaning

$$\sigma_{12,21} = -\sigma_{11,11}, \tag{590}$$

$$\sigma_{21,12} = -\sigma_{22,22}. \tag{591}$$

By the arbitrariness of the order in which partial derivatives are taken, and by virtue of the stress tensor being symmetric,

$$\sigma_{12,21} = \sigma_{21,12}. \tag{592}$$

Therefore,

$$\sigma_{12,12} = -\sigma_{11,11} = -\sigma_{22,22}. \tag{593}$$

Therefore,

$$\sigma_{12,12} + \sigma_{12,12} = -\sigma_{11,11} - \sigma_{22,22} = 2\sigma_{12,12}. \tag{594}$$

Substituting this into Eq. 588,

$$-(1+\nu)(\sigma_{11,11} + \sigma_{22,22}) = -\nu(\sigma_{11,11} + \sigma_{22,22}) + (\sigma_{11,22} + \sigma_{22,11}) \tag{595}$$

$$\longrightarrow -(\sigma_{11,11} + \sigma_{22,22}) - \cancel{\nu(\sigma_{11,11} + \sigma_{22,22})} = \cancel{-\nu(\sigma_{11,11} + \sigma_{22,22})} + (\sigma_{11,22} + \sigma_{22,11}). \tag{596}$$

This leaves

$$\sigma_{11,11} + \sigma_{11,22} + \sigma_{22,11} + \sigma_{22,22} = 0, \tag{597}$$

or

$$(\sigma_{11} + \sigma_{22})_{,11} + (\sigma_{11} + \sigma_{22})_{,22} = \nabla^2(\sigma_{11} + \sigma_{22}) = 0. \tag{598}$$

This is the stress compatibility equation which is valid both for isotropic, static materials under no body forces and under plane stress and plane strain conditions (as there is no dependence on material constants).

## 8.5 Airy stress function

A so-called Airy stress function $U$ automatically solves the static equilibrium equations if it is defined as

$$\sigma_{11} = U_{,22}, \quad \sigma_{22} = U_{,11}, \quad \sigma_{12} = -U_{,12} = -U_{,21}. \tag{599}$$

This is because

$$\sigma_{ij,j} = 0 \rightarrow \begin{cases} \sigma_{11,1} + \sigma_{12,2} = U_{,221} - U_{,212} = 0 \\ \sigma_{21,1} + \sigma_{22,2} = -U_{,121} + U_{,112} = 0. \end{cases} \tag{600}$$

That is because the partial derivative order is arbitrary. Substituting $U$ into the stress compatibility equation Eq. 598,

$$0 = \nabla^2(U_{,11} = U_{,22}) = U_{,1111} + U_{,1122} + U_{,2211} + U_{,2222} \tag{601}$$

$$= \left(\frac{\partial^2}{\partial^2 x} + \frac{\partial^2}{\partial^2 y}\right)\left(\frac{\partial^2}{\partial^2 x} + \frac{\partial^2}{\partial^2 y}\right)U = \nabla^2\nabla^2 U = \nabla^4 U = 0. \tag{602}$$

This is called the biharmonic equation, which $U$ satisfies in 2d. The Airy function is useful if you

- Let $U$ be different polynomials of various degrees to see what problem is solved (Sec. 8.6), or
- pose a problem of interest and attempt to solve it using the Airy function definition (Sec. 8.7).

In accordance with the above citations, the next sections are dedicated to applying these methods.

## 8.6 Polynomial solutions of the biharmonic equation

If

$$U = \frac{a}{2}x^2 - bxy + \frac{c}{2}y^2 \tag{603}$$

Then

$$U_{,xx} = \sigma_{yy} = a, \quad U_{,yy} = \sigma_{xx} = c, \quad -U_{,xy} = \sigma_{12} = b. \tag{604}$$

This means that all the stress components are constant. $a, c$ are constant diagonal elements, meaning they are uniform tension/compression parameters. Constant $b$ is a uniform shear parameter.

If

$$U = \frac{1}{6}ax^3 + \frac{1}{2}bx^2y + \frac{1}{2}cxy^2 + \frac{1}{6}dy^3 \tag{605}$$

then

$$U_{,xx} = \sigma_{yy} = ax + by, \quad U_{,yy} = \sigma_{xx} = +cx + dy, \quad -U_{,xy} = \sigma_{xy} = -bx - cy. \tag{606}$$

Different problems emerge from how the constants are set.

For example if only $d \neq 0$ but $a, b, c = 0$, then only the term $\sigma_{xx} = dy$ remains and it is a case of uniform bending where, because of the linear form, $y < 0 \rightarrow \sigma < 0, y > 0 \rightarrow \sigma > 0$. So the body is bent where the top half is a tension while the bottom half is a compression.

If only $b \neq 0$ but $a, c, d = 0$, then only $\sigma_{yy} = by, \sigma_{xy} = -bx$ remain. So the stress has four different overall behaviors based on the quadrants $(x, y), (-x, y), (x, -y), (-x, -y)$, if the origin is the center.

Letting $U$ be a fourth order polynomial or high does not necessarily satisfy the equation $\nabla^4 U = 0$. That is, constants remain. They are not all eliminated in the process of taking derivatives. Therefore the coefficients can not be picked arbitrarily.

## 8.7 Bending of a narrow cantilever of rectangular cross section under end load

Consider a beam fixed at $x = L$ and free at $x = 0$ (so that the free end is the origin) and depth $d$. If the thickness of the beam in the $z$ direction is $h \ll d$ then we can assume plane stress conditions because then the stress does not change virtually at all across the depth. So, the beam can be treated in 2 dimensions as a plane. We are solving for displacement due to the applied load $P$.

The stress in $x$ will be proportional to $x$ and $y$; particularly it will be higher as both $x$ and $y$ increase. So the quantity

$$\sigma_{xx} = c_1 xy = U_{,yy} \tag{607}$$

is chosen. Taking antiderivatives,

$$U_{,y} = \frac{1}{2} c_1 x y^2 + f_1(x), \tag{608}$$

$$U = \frac{1}{6} c_1 x y^3 + y f_1(x) + f_2(x). \tag{609}$$

It is known that $U$ satisfies the biharmonic equation and so

$$0 = \nabla^4 U = U_{,xxxx} + 2U_{,xxyy} + U_{,yyyy} = y f_1''''(x) + f_2''''(x). \tag{610}$$

The only way for this to be true of all $y$ is the trivial solution, which is $f_1''''(x), f_2''''(x) = 0$. That is because if $f_2''''$ is some constant, then $y$ can change and make the product $y f_1''''$ change, but $f_2''''$ cannot change. So both must go to zero so that a change in $y$ does not change the relationship. Therefore,

$$f_1(x) = c_2 x^3 + c_3 x^2 + c_4 x + c_5, \tag{611}$$

$$f_2(x) = c_6 x^3 + c_7 x^2 + c_8 x + c_9. \tag{612}$$

Substituting this in,

$$U = \frac{1}{6} c_1 x y^3 + y(c_2 x^3 + c_3 x^2 + c_4 x + c_5) + (c_6 x^3 + c_7 x^2 + c_8 x + c_9). \tag{613}$$

Then the other stress components are automatically supported by the equilibrium equation and are

$$\sigma_{yy} = U_{,xx} = 6c_2yx + 2c_3y + 6xc_6 + 2c_7, \tag{614}$$

$$\sigma_{xy} = -U_{,xy} = -\frac{1}{2}c_1y^2 - 3c_2x^2 - 2c_3x - c_4. \tag{615}$$

There is an assumption of a lack of external stresses, meaning the boundary is traction free, and this means that $\sigma_{yy}(y = \pm d/2) = 0$. Therefore,

$$0 = 3c_2dx + c_3d + 6xc_6 + 2c_7, \quad 0 = -3c_2dx - c_3d + 6xc_6 + 2c_7. \tag{616}$$

Setting these equal,

$$3c_2dx + c_3d + 6xc_6 + 2c_7 = -3c_2dx - c_3d + 6xc_6 + 2c_7 \tag{617}$$

$$\rightarrow 3c_2dx + c_3d = -3c_2dx - c_3d \rightarrow c2 = c_3 = 0. \tag{618}$$

$$\rightarrow 6xc_6 + 2c_7 = 0 \rightarrow c_6 = c_7 = 0. \tag{619}$$

So all the coefficients $c_2, c_3, c_6, c_7 = 0$ and therefore the system becomes

$$\sigma_{xx} = c_1xy, \quad \sigma_{yy} = 0, \quad \sigma_{xy} = -\frac{1}{2}c_1y^2 - c_4. \tag{620}$$

The same assumption implies a lack of shear on the boundary, so that $\sigma_{xy}(y = \pm d/2) = 0$. Therefore,

$$0 = -\frac{1}{2}c_1\frac{d^2}{4} - c_4, \tag{621}$$

meaning

$$c_4 = -\frac{1}{8}c_1d^2. \tag{622}$$

Substituting this in,

$$\sigma_{xx} = c_1xy, \quad \sigma_{yy} = 0, \quad \sigma_{xy} = -\frac{1}{2}c_1y^2 + \frac{1}{8}c_1d^2 = -\frac{1}{8}c_1(4y^2 - d^2). \tag{623}$$

Suppose we are analyzing the load/force of magnitude $P$, based on the stress experienced by a cross section of thickness $h << d$ and depth $d$. Then the force

$$-P = \int_{-d/2}^{d/2} \sigma_{xy} h dy, \tag{624}$$

where the shear component is used because the load $P$ in the diagram is applied in the $y$ direction on a surface whose normal is $x$. The sign of $P$ is negative because of the sign convention of shear stress.

**Note**

(At $x < 0$, $\sigma_{xy}$ pointing down is positive. The axes usually goes from left to right with the origin at the fixed end, meaning the bar normally goes from $x = 0$ (fixed) to $x = L$ (free). However in this example the axes go from right to left so that $x = -L$ (free) and $x = 0$ (fixed). ALSO, the origin is reset to the free end so that $x = 0$ (free and $x = L$ (fixed). So that is why $P$ is negative despite the application of load being at $x = 0$ by name. Conventionally it would be $x = -L$.)

Substituting in $\sigma_{xy}$,

$$-P = \int_{-d/2}^{d/2} (-\frac{1}{2}c_1 y^2 + \frac{1}{8}c_1 d^2) h dy = -\frac{1}{8}c_1 h \int_{-d/2}^{d/2} (4y^2 - d^2) dy \tag{625}$$

$$= -\frac{1}{4}c_1 h(\frac{4}{3}(d/2)^3 - d^2(d/2)) = -\frac{1}{4}c_1 h(d^3/6 - 3d^3/6) = \frac{1}{12}c_1 h d^3 \tag{626}$$

$$\longrightarrow P = -\frac{1}{12}c_1 h d^3. \tag{627}$$

Rearranging,

$$c_1 = -\frac{12P}{hd^3} = -\frac{12}{hd^3}P = -P/I, \tag{628}$$

where moment of inertia

$$I = \frac{hd^3}{12} \tag{629}$$

Substuting this into the system,

$$\sigma_{xx} = -\frac{P}{I}xy, \quad \sigma_{yy} = 0, \quad \sigma_{xy} = -\frac{P}{8I}(4y^2 - d^2). \tag{630}$$

Now that the stresses are computed, the strain/stress Inverse Hooke Law Eq. 357 can be summoned, and that is

$$\epsilon_{xx} = \frac{1}{E}\sigma_{xx} - \cancel{\frac{\nu}{E}\sigma_{yy}} = -\frac{Pxy}{IE}, \tag{631}$$

$$\epsilon_{yy} = -\frac{\nu}{E}\sigma_{xx} + \cancel{\frac{1}{E}\sigma_{yy}} = \frac{\nu PI}{xyE} \tag{632}$$

$$2\epsilon_{xy} = \frac{\sigma_{xy}}{\mu} = \left(\frac{1+\nu}{E}\right)\left(-\frac{P}{8I}(4y^2 - d^2)\right) = \frac{(1+\nu)P}{8EI}(d^2 - 4y^2) \tag{633}$$

Of course,

$$\epsilon_{xx} = \frac{\partial u}{\partial x}, \quad \epsilon_{yy} = \frac{\partial v}{\partial y}, \quad \epsilon_{xy} = \frac{1}{2}\left(\frac{\partial u}{\partial y} + \frac{\partial v}{\partial x}\right). \tag{634}$$

Then, the procedure of Li pg. 90-92 can be followed, which is analogous to that of Sec. 2.7. The eventual result is

$$u = -\frac{P}{2EI}x^2 y + \frac{P}{3EI}(1 + \nu/2)y^3 + \frac{P}{2EI}\left[L^2 - (1+\nu)\frac{d^2}{2}\right]y, \tag{635}$$

$$v = \frac{\nu P x y^2}{2EI} + \frac{P x^3}{6EI} - \frac{P L^2 x}{2EI} + \frac{P L^3}{3EI}, \tag{636}$$

and

$$u(L, y) \ll u(0, y) \approx \frac{P L^2 y}{2EI} = \mathcal{O}(L^2 y). \tag{637}$$

It is important to note that this solution requires the assumption in elementary beam theory that $u(L,0) = v(L,0) = \frac{\partial v}{\partial x}(L,0) = 0$, remembering that $x = L$ is the fixed end. So, it is assumed that the fixed end in the middle of the bar moves none whatsoever. These boundary conditions can be applied after finding $u$ and $v$ in terms of unknown constants, in order to solve for those constants.

**Page dedicated to working out the rest of the cantilever problem by hand.**

## 8.8 Bending of a beam by uniform load

Consider the uniformly loaded beam of length $2L$ and depth $2d$. If $q$ is the force per length then the total force is $2qL$, and the reaction to this is shared by the two ends each with magnitude $qL$. If thickness $h \ll d$ then we can assume plane stress conditions because it can be assumed that there is no change in stress across the beam, and so the beam can be treated like a 2 dimensional plane.

Recall the sign conventions for shear components in Fig. 2. On the left side, positive is down. On the right side, positive is up. Similarly, on the top side positive is to the right and on the bottom side positive is to the left. So since the shear stress on both sides is pointing down, the left side will be positive and the right side will be negative. Particularly,

$$\int_{-d}^{d} \sigma_{xy}(\pm L, y)dy = \mp \frac{qL}{h}. \tag{638}$$

With that in mind, the other nonzero boundary condition is

$$\sigma_{yy}(x, -d) = -\frac{q}{h}. \tag{639}$$

It is negative because the bottom of the body is pushing back up, and up is negative on the bottom side.

Finally other boundaries are set to zero. Those are stress on the top side, and shear components on both the top and the bottom. So,

$$\sigma_{xy}(x, \pm d) = 0, \tag{640}$$

$$\sigma_{yy}(x, d) = 0. \tag{641}$$

Finally, there is no net moment at the left and right ends. This means

$$\int_{-d}^{d} \sigma_{xx}(\pm L, y)dy = 0, \tag{642}$$

$$r \times F = r \times \sigma A \sim \int_{-d}^{d} y\sigma_{xx}(\pm L, y)dy = 0. \tag{643}$$

A fifth order polynomial $U$ with a total of 6+5+4+3=18 terms is utilized for the Airy function. That is, 6 fifth-order terms $x^5t^0, x^4y^1, \ldots, x^0y^5$, 5 fourth-order terms, etc. down to second order terms. Linear terms and constants do not affect the stress field since the second derivative of the Airy function is how the stress field is defined. Therefore, they are set to zero.

To solve this problem, it is not much different than the method used in Sec. 8.7. It is known that Using

$$\nabla^4 U = 0, \tag{644}$$

and also that

$$\sigma_{xx} = U_{,yy}, \quad \sigma_{yy} = U_{,xx}, \quad \sigma_{xy} = -U_{,xy}. \tag{645}$$

Then, the problem is simplified in the fact that because $\sigma_{-x,y} = -\sigma_{x,y}$, it is an odd function and so

The moment of inertia of the cross sectional area in this example with respect to the z axis is

$$I = \frac{2}{3}hd^3, \tag{646}$$

and the stress field is solved as

$$\sigma_{xx} = -\frac{q}{2I}(L^2 - x^2)y + \frac{q}{I}(\frac{1}{3}y^3 - \frac{1}{5}d^2y), \tag{647}$$

$$\sigma_{yy} = -\frac{q}{6I}y^3 + \frac{qd^2}{2I}y - \frac{qd^3}{3I}, \tag{648}$$

$$\sigma_{xy} = -\frac{q}{2I}(d^2 - y^2)x. \tag{649}$$

**Page dedicated to working out the rest of the cantilever problem by hand.**

# 9 General theorems of infinitesimal elastostatics

## 9.1 Work theorem

A stress field is statitically admissible if

$$\begin{cases} \sigma_{ij,j} + \rho f_i = 0, \\ \sigma_{ij} n_j = T_i. \end{cases} \tag{650}$$

Here $T_i$ is the surface force/traction vector. Static admissibility requires smoothness of the stress field.

A strain field is kinematically admissible if

$$\begin{cases} \epsilon_{ij} = \frac{1}{2}(u_{i,j} + u_{j,i}) \\ u_i = U_i, \end{cases} \tag{651}$$

where $U_i$ are the boundary displacements. Kinematic admissibility assumes smoothness of the displacement fields.

Note that these stress and strain fields need not necessarily be related to the same problem.

The work done by surface tractions is

$$\oint_S T_i u_i dS = \oint_S \sigma_{ij} n_j u_i dS = \int (\sigma_{ij} u_i)_{,j} dV = \int_V \sigma_{ij,j} u_i + \sigma_{ij} u_{i,j} dV \tag{652}$$

$$= \int_V -\rho f_i u_i + \sigma_{ij} \epsilon_{ij} dV = \oint_S T_i u_i dS. \tag{653}$$

Rearranging,

$$\oint_S T_i u_i dS + \int_V \rho f_i u_i dV = \int_V \sigma_{ij} \epsilon_{ij} dV. \tag{654}$$

The way to interpret Eq. 654 is this. Work done by surface tractions and body forces is equal to a strain energy computing the stress and strain of the two, not necessarily identical problems. Eq. 654 is called the work theorem.

The work theorem can be reverse engineered. That is, given $\sigma_{ij}$, $f_i$, and $T_i$, if the work theorem is true for these then $\sigma_{ij}$ is also statically admissible. Starting with the RHS of Eq. 654,

$$\int_V \sigma_{ij} \epsilon_{ij} dV = \int_V \sigma_{ij} u_{i,j} dV = \int_V [(\sigma_{ij} u_i)_{,j} - \sigma_{ij,j} u_i] dV = \oint_S \sigma_{ij} u_i n_j dS - \int_V \sigma_{ij,j} u_i dV. \tag{655}$$

Substituting this into the LHS,

$$\oint_S T_i u_i dS + \int_V \rho f_i u_i dV = \oint_S \sigma_{ij} u_i n_j dS - \int_V \sigma_{ij,j} u_i dV. \tag{656}$$

Rearranged,

$$\oint_S (\sigma_{ij} n_j - T_i) u_i dS = \int_V (\rho f_i + \sigma_{ij,j}) u_i dV. \tag{657}$$

If the work theorem holds for all displacements then this means it holds in particular if the surface displacements are zero, meaning the surface term goes to zero and

$$\int_V (\rho f_i + \sigma_{ij,j}) u_i dV = \oint_S \cancelto{0}{(...)} = 0, \tag{658}$$

meaning

$$\rho f_i + \sigma_{ij,j} = 0 \tag{659}$$

always, in order for the LHS to hold for all volumetric body displacements $u_i$. Then releasing the earlier surface displacements assumption,

$$0 = \oint_S (\sigma_{ij} n_j - T_i) u_i dS, \tag{660}$$

meaning

$$\sigma_{ij} n_j + T_i. \tag{661}$$

## 9.2 Betti's reciprocal theorem

Suppose we have knowledge of some $\sigma_{ij}, \rho f_i, T_i, u_i, \epsilon_{ij}$ for one problem, and suppose we also know some other set $\bar{\sigma}_{ij}, \rho \bar{f}_i, \bar{T}_i, \bar{u}_i, \bar{\epsilon}_{ij}$ for a second elasticity problem with the same body. (That is, a different displacement/strain/stress response that the same body exhibits in response to two different conditions.) Betti's theorem is that given these two sets, the work done by the first system of forces acting through the displacements of the second system is equal to the second system of forces acting through the displacements of the first system. That is, $F_{(1)} u_{(2)} = F_{(2)} u_{(1)} = W$. More specifically,

$$\int_V \rho f_i \bar{u}_i dV + \oint_S T_i \bar{u}_i dS = \int_V \rho \bar{f}_i u_i dV + \oint_S \bar{T}_i u_i dS. \tag{662}$$

To prove this we substitute in the work theorem Eq. 654 in for the LHS to get

$$\int_V \rho f_i \bar{u}_i dV + \oint_S T_i \bar{u}_i dS = \int_V \sigma_{ij} \bar{\epsilon}_{ij} dV. \tag{663}$$

Substituting it in for the RHS,

$$\int_V \rho \bar{f}_i u_i dV + \oint_S \bar{T}_i u_i dS = \int_V \bar{\sigma}_{ij} \epsilon_{ij} dV. \tag{664}$$

These two sides become equal because

$$\sigma_{ij} \bar{\epsilon}_{ij} = c_{ijkl} \epsilon_{kl} \bar{\epsilon}_{ij} = c_{klij} \epsilon_{ij} \bar{\epsilon_{kl}} = \bar{\sigma}_{ij} \epsilon_{ij}. \tag{665}$$

## 9.3 Variational principles

If function

$$F(\mathbf{x}) = \mathbf{x} \cdot \mathbf{M}\mathbf{x} - 2\mathbf{x} \cdot \mathbf{y}, \tag{666}$$

then the minimum of this function is located (not uniquely) by where $F'(\mathbf{x}) = \delta F(\mathbf{x}) = 0$, or

$$0 = \delta\mathbf{x} \cdot \mathbf{M}\mathbf{x} + \mathbf{x} \cdot \mathbf{M}\delta\mathbf{x} - 2\delta\mathbf{x} \cdot \mathbf{y} \tag{667}$$

$$\leftrightarrow \delta x_i M_{ij} x_j + x_i M_{ij} \delta x_j - 2\delta x_i y_i \tag{668}$$

$$= \delta x_i M_{ij} x_j + \delta x_j M_{ji} x_i - 2\delta x_i y_i \tag{669}$$

$$\leftrightarrow \delta\mathbf{x} \cdot \mathbf{M}\mathbf{x} + \delta\mathbf{x} \cdot \mathbf{M}^T\mathbf{x} - 2\delta\mathbf{x} \cdot \mathbf{y} \tag{670}$$

$$= \delta\mathbf{x} \cdot (\mathbf{M}\mathbf{x} + \mathbf{M}^T\mathbf{x} - 2\mathbf{y}) \tag{671}$$

$$= 2\delta\mathbf{x} \cdot (\frac{1}{2}(\mathbf{M} + \mathbf{M}^T)\mathbf{x} - \mathbf{y}) = 0 \tag{672}$$

$$\rightarrow \frac{1}{2}(\mathbf{M} + \mathbf{M}^T)\mathbf{x} - \mathbf{y} = 0 \longrightarrow \mathbf{y} = \frac{1}{2}(\mathbf{M} + \mathbf{M}^T)\mathbf{x} \tag{673}$$

for any $\delta\mathbf{x}$ and therefore for any $\mathbf{x}$. If $\mathbf{M}$ is symmetric, then

$$\mathbf{M}\mathbf{x} = \mathbf{y}. \tag{674}$$

So the function $\mathbf{y}$ determines the minimization of $F$.

Let us consider how this applies to strain energy

$$W(\mathbf{u}) = \frac{1}{2}\int_V c_{ijkl}\epsilon_{kl}\epsilon_{ij} dV = \frac{1}{2}\int_V c_{ijkl} u_{k,l} u_{i,j} dV. \tag{675}$$

Let $\bar{\mathbf{u}} \leftrightarrow \mathbf{y}$ be the minimizing displacement, and let a slightly different displacement field be $\bar{\mathbf{u}}(\mathbf{x}) + \epsilon\boldsymbol{\eta}(\mathbf{x})$ such that

$$f(\epsilon) = W(\bar{\mathbf{u}} + \epsilon\boldsymbol{\eta}) > W(\bar{\mathbf{u}}) \tag{676}$$

by virtue of $\bar{\mathbf{u}}$ being a minimum. Then

$$f(\epsilon) = \frac{1}{2}\int_V c_{ijkl}[u_{i,j} + \epsilon\eta_{i,j}][u_{k,l} + \epsilon\eta_{k,l}] dV \tag{677}$$

$$= \frac{1}{2}\int_V c_{ijkl} u_{i,j} u_{k,l} + \frac{1}{2}\int_V c_{ijkl} u_{i,j} \epsilon\eta_{k,l} dV + \frac{1}{2}\int_V c_{ijkl}\epsilon\eta_{i,j} u_{k,l} dV + \frac{1}{2}\int_V c_{ijkl}\epsilon^2\eta_{i,j}\eta_{k,l} dV \tag{678}$$

$$= W(\bar{\mathbf{u}}) + \epsilon\left(\int_V c_{ijkl} u_{i,j}\eta_{k,l} dV\right) + \epsilon^2 W(\boldsymbol{\eta}) = W(\bar{\mathbf{u}}) + \epsilon Q(\bar{\mathbf{u}}, \boldsymbol{\eta}) + \epsilon^2 W(\boldsymbol{\eta}) = f(\epsilon). \tag{679}$$

Then

$$f'(\epsilon) = Q(\bar{\mathbf{u}}, \boldsymbol{\eta}) + 2\epsilon W(\boldsymbol{\eta}). \tag{680}$$

Minimizing,

$$0 = f'(0) = Q(\bar{\mathbf{u}}, \boldsymbol{\eta}) = \int_V c_{ijkl}\bar{u}_{i,j}\eta_{k,l}dV = \int_V c_{klij}\bar{u}_{i,j}\eta_{k,l}dV = \int_V \bar{\sigma}_{kl}\eta_{k,l}dV = \int_V \bar{\sigma}_{ij}\eta_{i,j}dV \tag{681}$$

$$= \int_V [(\bar{\sigma}_{ij}\eta_i)_{,j} - \bar{\sigma}_{ij,j}\eta_i]dV = \oint_S \bar{\sigma}_{ij}\eta_i n_j dS - \int_V \bar{\sigma}_{ij,j}\eta_i dV. \tag{682}$$

Because of the boundary conditions $u_i = U_i$, therefore $\eta_i = \mathbf{0}$ on the surface and so

$$\oint_S \bar{\sigma}_{ij}\eta_i n_j dS = 0. \tag{683}$$

Substituting this in,

$$0 = f'(0) = -\int_V \bar{\sigma}_{ij,j}\eta_i dV \tag{684}$$

which, if this is to be true for all $\eta$, implies

$$\bar{\sigma}_{ij,j} = 0. \tag{685}$$

What this means is that the stress corresponding to the minimized displacements satisfies the static equlibrium equation.

## 9.4 Theorem of minimum potential energy

In the previous section it was assumed that there are no body forces, and the displacements are prescribed over the entire surface. A more general prescription is that the surface has some amount of displacement boundary conditions and some amount of traction boundary conditions. Recall that an admissible displacement field is one which satisfies the strain displacement relation $\epsilon_{ij} = \text{sym}(u_{i,j})$, the boundary displacements $u_i = U_i$, and that the displacement fields are smooth enough to yield strains. This set of conditions assumes

- $c_{ijkl} = c_{klij}$,
- $c_{ijkl}\epsilon_{kl}\epsilon_{ij} > 0$,
- the region $S_u$ on which the set of surface displacements $U_i$ are imposed is not a line.

This is the theory of minimal potential energy, and it states that if $\mathbf{u}$ is admissible then total work done

$$W(\mathbf{u}) = \frac{1}{2}\int_V c_{ijkl}u_{i,k}u_{k,l}dV - \int_V \rho f_i u_i dV - \int_{S_t} T_i u_i dS \tag{686}$$

implies

$$\begin{cases} (c_{ijkl}u_{k,l})_{,j} + \rho f_i = 0, \\ \bar{u}_i = \bar{U}_i, \\ \bar{\sigma}_{ij}n_j = (c_{ijkl}u_{k,l})_{,j}n_j = T_i, \\ W(\bar{\mathbf{u}}) < W(\mathbf{u}). \end{cases} \tag{687}$$

A way to state it is this. If a displacement field can satisfy the displacement boundary conditions, potential energy $W$ achieves its minimum for displacement fields that correspond to a state of equilibrium. That is, the corresponding stresses obtained using Hooke's Law satisfy the equilibrium and traction boundary conditions. To prove this we find the minimum of $W$, which is setting to zero the derivatiev

$$0 = \delta W(\mathbf{u}) = \frac{1}{2}\int_V [c_{ijkl}\delta u_{i,j}u_{k,l} + c_{ijkl}u_{i,j}\delta u_{k,l}]dV - \int_V \rho f_i \delta u_i dV - \int_{St} T_i \delta u_i dS \quad (688)$$

$$= \int_V c_{ijkl}\delta u_{i,j}u_{k,l}dV - \int_V \rho f_i \delta u_i dV - \int_{St} T_i \delta u_i dS \quad (689)$$

$$= \int_V [(c_{ijkl}\delta u_i u_{k,l})_{,j} - \delta u_i (c_{ijkl}u_{k,l})_{,j}]dV - \int_V \rho f_i \delta u_i dV - \int_{St} T_i \delta u_i dS \quad (690)$$

$$\oint_S c_{ijkl}\delta u_i u_{k,l}n_j dS - \int_V [\delta u_i (c_{ijkl}u_{k,l})_{,j} - \rho f_i \delta u_i]dV - \int_{St} T_i \delta u_i dS = 0 \quad (691)$$

$$\longrightarrow \oint_S [c_{ijkl}u_{k,l}n_j - T_i]dS - \int_V [(c_{ijkl}u_{k,l})_{,j} - \rho f_i]dV = 0. \quad (692)$$

This implies

$$\begin{cases} c_{ijkl}u_{k,l}n_j - T_i = 0 \longrightarrow \sigma_{ij}n_j = T_i, \\ (c_{ijkl}u_{k,l})_{,j} + \rho f_i = 0 \longrightarrow \sigma_{ij,j} + \rho f_i = 0, \end{cases} \quad (693)$$

which is already known, and so the first three statements Eq. 687 of the MPE theorem Eq. 686 is proved. To prove the fourth, consider some displacement field slightly greater than the equilibrium displacement field

$$\bar{\mathbf{u}}(\mathbf{x}) + \epsilon \boldsymbol{\eta}(\mathbf{x}) > \bar{\mathbf{u}}(\mathbf{x}) \quad (694)$$

where

$$\boldsymbol{\eta} = \mathbf{0} \text{ on } S_u. \quad (695)$$

Then

$$f(\epsilon) = W(\bar{\mathbf{u}}+\epsilon\boldsymbol{\eta}) = \frac{1}{2}\int_V c_{ijkl}[u_{i,j}+\epsilon\eta_{i,j}][u_{k,l}+\epsilon\eta_{k,l}]dV - \int_V \rho f_i [u_i - \epsilon\eta_i]dV - \oint_{S_t} T_i[u_i - \epsilon\eta_i]dS \quad (696)$$

$$= \frac{1}{2}\left(\int_V c_{ijkl}u_{i,j}u_{k,l} + c_{ijkl}u_{i,j}\epsilon\eta_{k,l} + c_{ijkl}\epsilon\eta_{i,j}u_{k,l} + \epsilon^2 c_{ijkl}\eta_{i,j}\eta_{k,l}\right) - \int_V \rho f_i[u_i - \epsilon\eta_i]dV - \oint_{S_t} T_i[u_i - \epsilon\eta_i]dS \quad (697)$$

$$= W(\bar{\mathbf{u}}) + \epsilon\left[Q(\bar{\mathbf{u}}, \boldsymbol{\eta}) - \int_V \rho f_i \eta_i dV - \oint_S T_i \eta_i dS\right] + \frac{1}{2}\epsilon^2 \int_V c_{ijkl}c_{ijkl}\eta_{i,j}\eta_{k,l}dV. \quad (698)$$

The minimum is where $f'(0) = 0$ and $f''(0) > 0$. That is

$$0 = \int_V c_{ijkl}\eta_{i,j}u_{k,l}dV - \int_V \rho f_i \eta_i dV - \oint_S T_i \eta_i dS + 0\cancel{\int_V c_{ijkl}\eta_{i,j}\eta_{k,l}dV}, \quad (699)$$

$$= \int_V \sigma_{ij}\eta_{i,j}dV - \int_V \rho f_i \eta_i dV - \oint_{S_t} T_i \eta_i dS \quad (700)$$

$$= \int_V [(\sigma_{ij}\eta_i)_{,j} - \sigma_{ij,j}\eta_i]dV - \int_V \rho f_i \eta_i dV - \oint_{S_t} T_i \eta_i dS \tag{701}$$

$$= \oint_{S_t} \sigma_{ij}\eta_i n_j dS - \int_V \eta_i(\sigma_{ij,j} + \rho f_i)dV - \oint_{S_t} T_i \eta_i dS \tag{702}$$

$$\longrightarrow 0 = \oint_{S_t} (\sigma_{ij}n_j - T_i)dS - \int_V (\sigma_{ij,j} + \rho f_i)dV \tag{703}$$

$$\rightarrow \sigma_{ij}n_j = T_i, \quad \sigma_{ij,j} + \rho f_i = 0, \tag{704}$$

and

$$0 < \int_V c_{ijkl}\eta_{i,j}\eta_{k,l}dV, \tag{705}$$

meaning

$$f(\epsilon) = W(\bar{\mathbf{u}} + \epsilon\boldsymbol{\eta}) = W(\bar{\mathbf{u}}) + \text{ term greater than zero.} \tag{706}$$

Therefore for any $\boldsymbol{\eta}$, $\epsilon$, it causes the potential energy $W$ to increase. Therefore $W(\bar{\mathbf{u}})$ is an absolute minimum and the fourth statement in Eq. 687 is proven.

## 9.5 Minimum complementary energy

A stress field $\sigma_{ij}$ is statically admissible if

$$\begin{cases} \sigma_{ij,j} + \rho f_i = 0, \\ \sigma_{ij}n_j = T_i, \end{cases} \tag{707}$$

where

$$\sigma_{ij} = c_{ijkl}\epsilon_{kl}. \tag{708}$$

Strain energy density

$$w = \frac{1}{2}\sigma_{ij}\epsilon_{ij} = \frac{1}{2}c_{ijkl}\epsilon_{kl}\epsilon_{ij} > 0. \tag{709}$$

The inverse hooke law can be written as

$$\epsilon_{ij} = s_{ijkl}\sigma_{kl}, \tag{710}$$

where $\mathbf{s} = \mathbf{c}^{-1}$. Then strain energy density

$$w = \frac{1}{2}s_{ijkl}\sigma_{kl}\sigma_{ij} > 0. \tag{711}$$

Now let us define complementary energy $\hat{W}$ as the sum of the internal strain energy and the potential of the boundary forces acting through the prescribed displacements (force*distance=work). Then

$$\hat{W} = \int_V \frac{1}{2}s_{ijkl}\sigma_{kl}\sigma_{ij} - \oint_S \sigma_{ij}n_j U_i dS. \tag{712}$$

Here, $U_i$ are the set of prescribed displacements on the boundary.

The minimum complementary energy theorem is this. The complementary energy achieves its absolute minimum for the stress field which is that of the equilibrium state. That is, a stress field which satisfies the compatibility equations. To prove this, suppose $\bar{\sigma}_{ij}$ is the equilibrium state. Then let

$$\sigma_{ij} = \bar{\sigma}_{ij} + \Delta\sigma_{ij}. \tag{713}$$

We want to prove that

$$\hat{W}(\bar{\sigma}_{ij} + \Delta\sigma_{ij}) > \hat{W}(\bar{\sigma}_{ij}). \tag{714}$$

To do this we analyze the quantity

$$\hat{W}(\bar{\sigma}_{ij} + \Delta\sigma_{ij}) - \hat{W}(\bar{\sigma}_{ij}) \tag{715}$$

$$= \int_V \frac{1}{2} s_{ijkl}(\bar{\sigma}_{ij}+\Delta\sigma_{ij})(\bar{\sigma}_{kl}+\Delta\sigma_{kl})dV - \oint_S (\bar{\sigma}_{ij}+\Delta\sigma_{ij})n_j U_i dS - \int_V \frac{1}{2} s_{ijkl}\bar{\sigma}_{kl}\bar{\sigma}_{ij} - \oint_S \bar{\sigma}_{ij} n_j U_i dS. \tag{716}$$

$$= \int_V s_{ijkl}\bar{\sigma}_{ij}\Delta\sigma_{kl}dV + \underbrace{\int_V \frac{1}{2} s_{ikjl}\Delta\sigma_{ij}\Delta\sigma_{kl}dV}_{\text{strain energy definition: must be>0}} - \oint_S \Delta\sigma_{ij} n_j U_i dS = \hat{W}(\bar{\sigma}_{ij}+\Delta\sigma_{ij}) - \hat{W}(\bar{\sigma}_{ij}). \tag{717}$$

The middle term, as written, must be a positive number as it is a strain energy density term by definition. Therefore,

$$\hat{W}(\bar{\sigma}_{ij} + \Delta\sigma_{ij}) - \hat{W}(\bar{\sigma}_{ij}) > \int_V s_{ijkl}\bar{\sigma}_{ij}\Delta\sigma_{kl}dV - \oint_S \Delta\sigma_{ij} n_j U_i dS \tag{718}$$

$$= \int_V \bar{\epsilon}_{kl}\Delta\sigma_{kl}dV - \oint_S \Delta\sigma_{ij} n_j U_i dS \tag{719}$$

$$= \int_V \bar{u}_{i,j}\Delta\sigma_{ij}dV - \oint_S \Delta\sigma_{ij} n_j U_i dS \tag{720}$$

$$= \int_V [(\Delta\sigma_{ij}\bar{u}_i)_{,j} - \Delta\sigma_{ij,j}\bar{u}_i]dV - \oint_S \Delta\sigma_{ij} n_j U_i dS \tag{721}$$

$$= \oint_S \Delta\sigma_{ij}\bar{u}_i n_j dS - \int_V \Delta\sigma_{ij,j}\bar{u}_i dV - \oint_S \Delta\sigma_{ij} n_j U_i dS. \tag{722}$$

The middle term vanishes because if

$$\sigma_{ij} - \bar{\sigma}_{ij} = \Delta\sigma_{ij} \tag{723}$$

and if

$$\sigma_{ij,j} + \rho f_i = 0, \qquad \bar{\sigma}_{ij,j} + \rho f_i = 0 \tag{724}$$

then

$$\Delta\sigma_{ij,j} = \sigma_{ij,j} - \bar{\sigma}_{ij,j} = -\rho f_i + \rho f_i = 0. \tag{725}$$

Then,

$$\hat{W}(\bar{\sigma}_{ij} + \Delta\sigma_{ij}) - \hat{W}(\bar{\sigma}_{ij}) = \oint_S \Delta\sigma_{ij}\bar{u}_i n_j dS - \cancel{\int_V \Delta\sigma_{ij,j}\bar{u}_i dV} - \oint_S \Delta\sigma_{ij} n_j U_i dS. \tag{726}$$

As $\bar{u}_i = U_i \rightarrow \bar{u}_i - U_i = 0$,

$$\hat{W}(\bar{\sigma}_{ij} + \Delta\sigma_{ij}) - \hat{W}(\bar{\sigma}_{ij}) > 0. \tag{727}$$