Research statement

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March 22, 2021

My research is concerned with the spectral and inverse spectral theory of differential operators and their relationship to integrable nonlinear wave equations. My work has two goals: First, to uncover the dynamical behavior of solutions to integrable nonlinear PDEs in certain asymptotic limits [1]. Second, to develop the spectral and inverse spectral theory of differential operators arising in mathematical physics. [2].

1 Background

Nonlinear waves. Waves are found throughout nature. Familiar examples include water waves, electromagnetic waves, acoustics, plasmas, quantum physics, and cosmology. Wave phenomena are governed by linear and nonlinear partial differential equations (PDEs). A ubiquitous feature of linear wave equations is dispersion, namely the fact that different Fourier modes inside a wave packet travel at different speeds, leading to spreading and decay of a local disturbance [3, 4]. Conversely, nonlinear effects in the governing equation often lead to wave breaking and the formation of shocks. Remarkably, dispersion and nonlinearity can sometimes interact to form a stable structure referred to as a solitary wave (i.e., a localized traveling wave solution). In a seminal work in 1965, Zabusky and Kruskal [5] conducted a numerical investigation of solutions of the Korteweg-de Vries (KdV) equation and demonstrated the formation of a train of solitary waves which interacted elastically. They named these objects *solitons*. Many weakly nonlinear dispersive PDEs of physical interest admit soliton solutions. Since many of these equations appear as universal models in physics the theoretical aspects of applied mathematical research on nonlinear waves are relevant to subjects as diverse as fluid mechanics, nonlinear optics, and Bose-Einstein condensates,

Integrable systems. Nonlinear wave equations are mathematically interesting due to the remarkable structures present in their solutions. Integrable equations are an important subclass of nonlinear wave equations. Examples include the above-mentioned KdV equation, the nonlinear Schrödinger (NLS) equation, the sine-Gordon equation, the Toda lattice, and many others [6]. These equations are also completely integrable infinite dimensional Hamiltonian systems. Importantly, the initial-value problem (IVP) for all of these equations is solvable by the inverse scattering transform (IST). The IST is one of the major achievements in mathematical physics over the past 50 years. It was first developed to solve the the KdV equation [7]. Shortly afterward, Lax developed an operator-theoretic way to generalize the method [8]. In theory, any evolution equation that serves as the compatibility of the overdetermined linear system

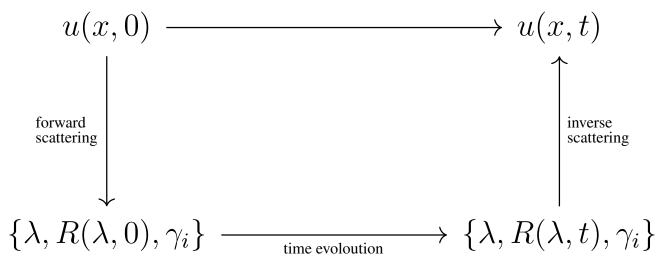
$$\mathcal{L}v = \lambda v\,, \tag{1.1a}$$

$$\partial_t v = Bv\,, \tag{1.1b}$$

where $\{\mathcal{L}(t), B(t)\}$ are time-dependent linear operators acting on a fixed Hilbert space, and satisfying

$$\partial_t \mathcal{L} = [B, \mathcal{L}], \tag{1.2}$$

where $[B, \mathcal{L}] := B\mathcal{L} - \mathcal{L}B$ is solvable using the IST. The set $\{\mathcal{L}(t), B(t)\}$ is referred to as a *Lax pair*. Importantly, the scattering problem i.e, (1.1a) is isospectral with respect to time deformations that obey the governing evolution equation. By associating a nonlinear PDE to a linear scattering problem where the scattering data evolves in a trivial manner we have in some sense linearized the problem. The time-evolved scattering data is used to reconstruct the solution at some later time, that is, we solve an inverse scattering problem. It turns out that one can often reduce the inverse scattering problem to a Riemann-Hilbert factorization problem. Thus, we have the commutative diagram:



(1.3)

where $u = u(x,t)$ is the solution to the nonlinear evolution equation, and $\{\lambda, R, \gamma_i\}$ are the scattering data needed to reconstruct the potential, i.e., the eigenvalues, reflection coefficient, and norming constants, respectively.

In 1972 Zakharov and Shabat (ZS) [9] solved the IVP for the NLS equation by the inverse scattering transform. Shorty afterwards, Ablowitz, Kaup, Newell, and Segur (AKNS) [10] clarified the analogy between the IST and the solution of linear PDEs by the Fourier transform. At the same time, various researchers [11, 12, 13, 14, 15, 16] began developing the inverse scattering method in the case of periodic boundary conditions. Then in 1992 a major advancement in the rigorous asymptotic analysis of integrable systems was formulated by Deift and Zhou (DZ) [17] who developed a nonlinear steepest descent method for oscillatory Riemann-Hilbert problems.

2 Preliminaries

Nonlinear Schrödinger equation. The semiclassical focusing NLS equation is the PDE

$$i\epsilon\psi_t + \epsilon^2\psi_{xx} + 2|\psi|^2\psi = 0, \tag{2.1}$$

where $\psi : \mathbb{R} \times \mathbb{R}^+ \to \mathbb{C}$, is the slowly-varying complex envelope of a quasi-monochromatic, weakly dispersive nonlinear wave packet, subscripts x and t denote partial derivatives and the physical meaning of the variables depends on the context. (E.g., in optics, t represents propagation distance while x is a retarded time.) In general, the parameter ϵ quantifies the relative stength of dispersion compared to nonlinearity; letting $\epsilon \downarrow 0$ is referred to as the *semiclassical limit*. (In the quantum-mechanical setting, ϵ is also proportional to Planck's constant $\hbar$.)

Lax pair. Zakharov and Shabat showed [9] that (2.1) is the compatibility of a Lax pair in the form

$$\epsilon v_x = Xv\,, \qquad \epsilon v_t = Tv\,. \tag{2.2}$$

In particular, the first half of (2.2) is the so-called ZS spectral problem,

$$\mathcal{L}v = i\epsilon \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} v_x - \begin{pmatrix} 0 & iq(x) \\ i\overline{q(x)} & 0 \end{pmatrix} v = zv \tag{2.3}$$

where $v(x; z, \epsilon) = (v_1, v_2)^T$ (the superscript T denoting matrix transpose), overbar stands for complex conjugation, $z \in \mathbb{C}$ is a spectral parameter, $0 < \epsilon \leq 1$ is the semiclassical parameter, and $q : \mathbb{R} \to \mathbb{C}$ is a given potential function. Throughout this note we will assume that q has period L, that is,

$$q(x + L) = q(x) \,, \qquad \forall x \in \mathbb{R} \,. \tag{2.4}$$

Lax spectrum. The system (2.3) defines the scattering and inverse scattering transforms of (2.1) (Lax's $\mathcal{L}$ operator for focusing NLS). The solution $\psi = \psi(x, t; \epsilon)$ of (2.1) with initial data given by $\psi(x, 0; \epsilon) = q(x)$ is constructed by computing suitable *scattering data* generated by the potential in (2.3). Note, $\mathcal{L}$ is a one-dimensional non-self-adjoint Dirac operator acting on $L^2(\mathbb{R}, \mathbb{C}^2)$ with dense domain $H^1(\mathbb{R}, \mathbb{C}^2)$. The inverse spectral theory of periodic Dirac operators is concerned with the *Lax spectrum*, i.e., the closure of the set of $z \in \mathbb{C}$ for which a non-trivial solution of (2.3) exists which is bounded for all $x \in \mathbb{R}$. Specifically,

$$\Sigma_{\text{Lax}} := \text{cl}\{z \in \mathbb{C} : \exists v \not\equiv 0 \in AC_{\text{loc}}(\mathbb{R}) \text{ s.t. } \mathcal{L}v = zv \ \& \ \sup_{x\in\mathbb{R}}|v(x)| < \infty\} \,, \tag{2.5}$$

where 'cl' denotes set closure. It is well known that Σ_{Lax} is purely continuous, that is, essential without any eigenvalues and empty residual spectrum [18, 19, 20]. By the theory of linear differential equations with periodic coefficients [19, 20] one gets Σ_{Lax} is comprised of an at most countable collection of smooth arcs, or *bands*, in the spectral z-plane.

Importantly, the scattering data provides information regarding the dynamical behavior of solutions to integrable models. For example, soliton solutions comprise the discrete eigenvalues of a scattering problem with vanishing boundary conditions [6]. The real part of the eigenvalue is the soliton's velocity while the imaginary part is the soliton's amplitude. The study of ZS scattering problems has become an active area of research and a natural starting point for the study of solutions to the focusing NLS equation [21, 22, 23, 24, 25, 26, 27, 28].

3 Research

Semiclassical dynamics in self-focusing nonlinear media. In [1] the semiclassical limit of the focusing NLS equation (2.1) with periodic initial data was studied analytically and numerically.

Motivation. Solutions to (2.1) in the *semiclassical limit*, i.e., as $\epsilon \downarrow 0$ have a remarkable structure and is of interest to applied analysts. Similar phenomena occur in physical applications when dispersive effects are weak when compared to nonlinear ones. These situations, referred to as small-dispersion limits, can produce a wide variety of physical effects such as supercontinuum generation, dispersive shocks and wave turbulence, to name a few (e.g., see [29, 30, 31] and references therein). Indeed, it was the desire to understand the Fermi-Pasta-Ulem (FPU) recurrences via the behavior of solutions in the small-dispersion limit of the KdV equation that led to the discovery of solitons in the first place. Further, the solutions of integrable models have a rich mathematical

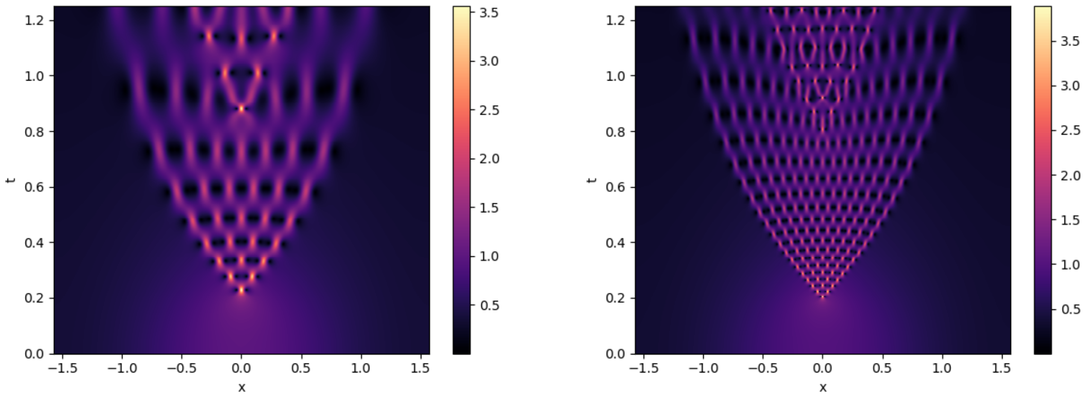


Figure 1: Numerical simulations of (2.1) with $\psi(x,0) = \exp(-\sin^2 x)$. (Left) $\epsilon = 0.06$. (Right) $\epsilon = 0.027$.

structure which becomes more evident as the semiclassical parameter tends to zero. In the defocusing NLS equation with periodic initial data, the semiclassical limit was recently realized in fiber optics experiments, which show fission of dark solitons from periodic breaking points [32]. These results were then characterized analytically in [33].

The focusing NLS equation has several features such as modulational instability, and a non-self-adjoint scattering problem which complicates the analysis significantly. Previous studies have analyzed the semiclassical limit of the focusing NLS equation subject to initial data which goes to zero at infinity [34, 35]. Using the IST machinery rigorous estimates for solutions in the semiclassical limit were obtained in various regions of the (x, t) plane. Since the IST for problems with periodic initial data is more complicated, one must rely on careful numerical simulations and asymptotic calculations of the spectral data.

Results. Recall that $\psi(x, 0; \epsilon) = q(x)$, where $q = q(x)$ is referred to as the potential of (2.3). For simplicity attention was restricted to a class of initial-conditions (ICs) coined as *single-lobe periodic*. Specifically, I defined a single-lobe periodic potential as the periodic extension of a continuous function satisfying: (i) $q(x) > 0 \; \forall \; x \in (-L/2, L/2)$; (ii) $q(-x) = q(x)$; and (iii) $q(x)$ is increasing on $(-L/2, 0)$, and is decreasing on $(0, L/2)$. Importantly, it was shown that the resulting dynamics is characterized by the generation of a large number of nonlinear excitations referred to as *effective solitons*.

Using asymptotic calculations of the spectrum and careful numerical simulations, it was shown that for ICs that are smooth single-lobe periodic there exists universal features of solutions to the focusing NLS equation in the semiclassical limit. Further many of these features coincide with those found for single-lobe ICs that are rapidly decaying at infinity. These features include the occurrence of a gradient catastrophe at some time $t = t_b$ that depends on the potential, regions of $O(\epsilon)$ oscillations, and a series of caustics in the (x, t) plane (see Fig. 1).

Recall that properties of solutions to (2.1) such as the existence of solitons can be obtained by a careful examination of the ZS scattering problem (2.3). The Lax spectrum was calculated numerically using Floquet-Hill's method [36], and analytically using WKB approximation. In the class of single-lobe periodic potentials the Lax spectrum is described in the semiclassical limit. Specifically, we have: (i) a spectral band along the interval $(-iq_{\text{min}}, iq_{\text{min}})$; (ii) spectral bands and gaps along the intervals $(-iq_{\text{max}}, -iq_{\text{min}}) \cup (iq_{\text{min}}, iq_{\text{max}})$; (iii) the region $|\text{Im}z| > q_{\text{max}}$ does not belong to the spectrum; and (iv) bands, or *spines*, intersecting the real z-axis transversely which become negligible as $\epsilon \downarrow 0$. (In [2] I proved that the spines are $O(\epsilon)$ as $\epsilon \downarrow 0$.)

Using a WKB approach to leading order the number of bands in the interval $(iq_{\text{min}}, iq_{\text{max}})$ is

$$N_\epsilon \sim \left\lfloor \frac{S_1(-q_{\text{min}}^2)}{\pi\epsilon} + \frac{1}{2} \right\rfloor \qquad \text{as} \qquad \epsilon \downarrow 0\,, \tag{3.1}$$

where the floor function $\lfloor x \rfloor$ denotes the integer part of a real number x, i.e., the largest integer less than or equal to x. Note $S_1(z) := \int_{-p}^{p} \sqrt{|z^2 + q^2(x)|}\,\mathrm{d}x$, where $p = p(z)$ is a turning point, that is, a root of the equation $z^2 + q^2(x) = 0$. The relative band width to leading order is

$$W_n := \frac{w_n}{w_n + g_n} \sim \frac{4}{\pi}\text{sech}\left(\frac{2S_{2,\epsilon}(z_n)}{\epsilon}\right) \qquad \text{as} \qquad \epsilon \downarrow 0\,, \tag{3.2}$$

where $S_{2,\epsilon}(z) := \int_{p}^{L} \sqrt{|z^2 + q^2(x)|}\,\mathrm{d}x + \epsilon \ln 2/2$, w_n is the band width, g_n is the gap width, and $z = z_n$ is a zero of the trace of a monodromy matrix for $z \in (iq_{\text{min}}, iq_{\text{max}})$. Recall that a monodromy matrix is given by $\Phi(z; L)$, where Φ is a fundamental matrix of (2.3) normalized so that $\Phi(z; 0) \equiv \mathbb{I}$. As in the KdV and defocusing NLS equations, the relative band width W_n is a physically important quantity which governs the characteristic features of periodic nonlinear excitations [33, 37]. In particular, when $W_n \to 1$ to the corresponding nonlinear excitation reduces to a constant background, whereas when $W_n \to 0$ the excitation becomes a soliton. Thus, setting a threshold $\kappa \ll 1$ we define a nonlinear excitation to be an effective soliton when $W_n < \kappa$. Let $\lambda = z^2$ and assume that $W_n < \kappa$ for $\lambda \in (-q_{\text{max}}^2, \lambda_s)$, where $\lambda_s < -q_{\text{min}}^2$. Then the number of effective solitons is given by

$$N_s \sim \left\lfloor \frac{S_1(\lambda_s)}{\pi\epsilon} + \frac{1}{2} \right\rfloor \qquad \text{as} \qquad \epsilon \downarrow 0\,. \tag{3.3}$$

Importantly, N_s is independent of κ, $\lambda_s \to -q_{\text{min}}^2$ as $\epsilon \downarrow 0$, and all effective solitons are on the imaginary axis (zero velocity). *Thus, the semiclassical limit of the focusing NLS equation with single-lobe periodic potential is characterized by a semiclassical soliton condensate.*

Lax spectrum of the focusing Zakharov-Shabat spectral problem. In [2] the spectrum of the non-self-adjoint ZS spectral problem (2.3) with a periodic potential was investigated.

Motivation. In [1] it was conjectured that the Lax spectrum of the non-self-adjoint ZS spectral problem (2.3) localizes to the real and imaginary z-axes in the semiclassical limit. In [2] this conjecture was proved. Importantly, rigorous justification of the WKB analysis used in [1] was established. Further, a description of the Lax spectrum was given as $z \to \infty$. These results are important to inverse spectral theory as well as to the theory of non-self-adjoint operators.

Results. In [2], a rigorous asymptotic characterization of the Lax spectrum in the space of periodic essentially bounded potentials was established. Additionally, Floquet theory was used to derive rigorous bounds on the Lax spectrum. Importantly, in the semiclassical limit the spectral bands cluster on the real and imaginary axes. This is relevant to the analysis of solutions to (2.1) with periodic initial data in the semiclassical limit. Further, if the Lax spectrum is confined to the real and imaginary axes only then $q = q(x)$ is a *finite-gap* potential which has important implications to the inverse scattering method in the case of periodic initial data. This work was done in collaboration with Prof. Alexander Tovbis at Univ. Central Florida. Next, I present some of the main results.

Theorem 3.1. *Let $q \in H^1_{loc}(\mathbb{R})$ be L-periodic, and $q_x \in L^\infty(\mathbb{R})$. Then $\Sigma_{Lax} \subset \Lambda^\epsilon(q)$, where*

$$\Lambda^\epsilon(q) := \left\{ z \in \mathbb{C} : |\operatorname{Im} z| \le \|q\|_\infty \right\} \cap \left\{ z \in \mathbb{C} : |\operatorname{Re} z||\operatorname{Im} z| \le \frac{\epsilon}{2}\|q_x\|_\infty \right\}. \tag{3.4}$$

That is, the Lax spectrum is contained in the set $\Lambda^\epsilon(q)$ depending explicitly on the semiclassical parameter. Importantly, Theorem (3.1) shows that $\operatorname{Im} z = O(1/\operatorname{Re} z)$ as $|\operatorname{Re} z| \to \infty$.

Theorem 3.2. *Let $q \in H^1_{loc}(\mathbb{R})$ be L-periodic, and $q_x \in L^\infty(\mathbb{R})$. Define*

$$\Sigma_\infty := \mathbb{R} \cup i[-\|q\|_\infty, \|q\|_\infty]\,. \tag{3.5}$$

Moreover, let $N_\delta(\Sigma_\infty)$ be a δ-neighborhood of Σ_∞. Then for any $\delta > 0$,

$$\Sigma_{Lax} \cap (\mathbb{C} \setminus N_\delta(\Sigma_\infty)) = \emptyset \tag{3.6}$$

for all sufficiently small values of ϵ. That is, for any $\delta > 0$ there exists an $\epsilon^* > 0$ such that (3.6) holds for all $0 < \epsilon \le \epsilon^*$.

Next consider the asymptotic distribution of bands as $z \to \infty$. For this purpose, following [22], the term *spine*, defined as a spectral band that intersects the real axis transversely and does not intersect any other band is introduced. Let $R_{N,q}$ denote the rectangle with vertices $\pm N \pm i\|q\|_\infty$, $N \in \mathbb{N}$.

Theorem 3.3. *For any L-periodic $q \in L^\infty(\mathbb{R})$ there exists some $N = N(\epsilon; q) \in \mathbb{N}$, such that all but finitely many bands of the Lax spectrum are spines located outside $R_{N,q}$. Moreover: i) for each of these spines there exists $n \in \mathbb{Z}$ such that the intersection point between the spine and the real axis is $o(1)$-close to the point $n\pi\epsilon/L$ as $n \to \pm\infty$; ii) only one spine can be $o(1)$-close to $n\pi\epsilon/L$ as $n \to \pm\infty$. Further, if $q \in H^1_{loc}(\mathbb{R})$, the intersection point is $o(1/z)$-close.*

Theorem 3.4. *Let $q \in L^\infty(\mathbb{R})$ be L-periodic. Then q is a finite-gap potential if and only if $\exists N = N(\epsilon; q) \in \mathbb{N}$ such that $(\Sigma_{Lax} \setminus \mathbb{R}) \subset R_{N,q}$.*

Theorem 3.5. *Let the scattering potential q satisfy at least one of the following conditions: a) it is real-valued; b) it is even; c) it is odd. If the periodic and anti-periodic Floquet spectra are real and purely imaginary only, then the entire Lax spectrum is contained within the real and imaginary axes, that is, $\Sigma_{\mathrm{Lax}} \subset \Sigma_\infty$.*

The periodic and anti-periodic spectra correspond to the band edges in the Lax spectrum. Finally, a sharper bound on the Lax spectrum was found for positive real-valued potentials.

Theorem 3.6. *Let $q \in H^1_{loc}(\mathbb{R})$ be real-valued, positive, and L-periodic. Suppose $q_x \in L^\infty(\mathbb{R})$. Take $z \in \Sigma_{Lax}$ with $\operatorname{Re} z > 0$. Then*

$$|\operatorname{Im} z| \le \frac{\epsilon}{2}\|(\ln q)_x\|_\infty\,, \tag{3.7}$$

and so $\operatorname{Im} z = O(\epsilon)$ as $\epsilon \downarrow 0$ from $|\operatorname{Re} z| > 0$.

This shows that for real-valued positive potentials the Lax spectrum is localized along the imaginary z-axis for ϵ small, but not necessarily zero. Further it gives a rigorous justification of the formal WKB approach used in [1].

Current work

- I am currently working on the coexistence problem for the ZS system (2.3) with a one-parameter family of elliptic potentials. A main result of this work is the existence of a sequence of parameter values such that one gets a *finite-gap* potential. This work is in collaboration with A. Tovbis and X.-D. Luo.
- Recently, using the Liouville transformation, I have been able to obtain rigorous error estimates of my WKB approximations in [1] with no turning points. A similar approach should work in the turning point case as well and is currently under construction.

Future work

- A future project of interest is on a Riemann-Hilbert problem approach to the inverse spectral theory of Dirac operators. The current methodology involves complicated reconstruction formulas which makes rigorous asymptotic analysis of solutions with periodic boundary conditions out of reach. Results of this type were recently developed for Hill operators [38].
- To grow as a mathematician, I am also interested in exploring new areas of research as well as in broadening my knowledge of nonlinear waves, integrable systems, and spectral theory.

References

[1] G. Biondini and J. Oregero "Semiclassical dynamics in self-focusing nonlinear media with periodic initial conditions", Stud. Appl. Math. **145** (3): 325–356 (2020)

[2] G. Biondini, J. Oregero and A. Tovbis, "On the spectrum of focusing Zakharov-Shabat operators with periodic potentials" [submitted to J. Spectral Theory (2020)] arXiv:2010.04263

[3] G. B. Whitham, "Linear and Nonlinear Waves", Wiley, New York, 1974

[4] J. Yang, "Nonlinear Waves in Integrable and Nonintegrable Systems", SIAM, 2010

[5] N.J. Zabusky, M.D. Kruskal, "Interaction of 'solitons' in a collisionless plasma and the recurrence of initial states", Phys. Rev. Lett. **15** (6): 240 – 243 (1965)

[6] M. J. Ablowitz and H. Segur, "Solitons and the inverse scattering transform", SIAM, Philadelphia, 1981

[7] C.S. Gardener, J.M. Greene, M.D. Kruskal, and R.M. Miura, "Method for solving the Korteweg de Vries equation", Phys. Rev. Lett. **19**, 1095 – 1097 (1967)

[8] P. Lax, "Integrals of nonlinear equations of evolution and solitary waves", Comm. Pure and Applied Math., **21** (5): 467 – 490 (1968)

[9] V.E. Zakharov and A.B Shabat, "Exact theory of two-dimensional self-focusing and one-dimensional self-modulation of waves in nonlinear media", Soviet Journal of Experimental and Theoretical Physics, **34** (1): 62 – 69 (1972)

[10] M.J. Ablowitz, D.J. Kaup, A.C. Newell, and H. Segur, "The inverse scattering transform – Fourier analysis of nonlinear problems" Stud. in Applied Math., **53** (4): 249 – 315 (1974)

[11] S. P. Novikov, "The periodic problem for the Korteweg–de Vries equation", Funct. Anal. Appl., **8**:3, 236–246 (1974)

[12] A.R. Its and V.B. Matveev, "Schrödinger operators with finite-gap spectrum and soliton solutions of the Korteweg-de Vries equation", Theor. Math. Phys., **23** 51 – 67 (1975)

[13] B. A. Dubrovin, "Periodic problems for the Korteweg–de Vries equation in the class of finite band potentials", Funct. Anal. Appl., **9**:3, 215–223 (1975)

[14] H. P. McKean and P. van Moerbeke, "The spectrum of Hill's equation", Invent. Math. **30**, 217–274 (1975)

[15] H. P. McKean and E. Trubowitz, "Hill's operator and hyperelliptic function theory in the presence of infinitely many branch points", Commun. Pure Appl. Math. **29**, 143–226 (1976)

[16] E. Date and S. Tanaka, "Periodic multi-soliton solutions of Korteweg-de Vries equation and Toda lattice", Progress of Theoretical Physics Supplement, **59**, 107–125 (1976)

[17] P. Deift and X. Zhou, "A steepest descent method for oscillatory Riemann–Hilbert problems. Asymptotics for the MKdV equation", Annals of Math., **137**, 295–368 (1993)

[18] F. S Rofe-Beketov, A. M. Khol'kin, "Spectral Analysis of Differential Operators: Interplay Between Spectral and Oscillatory Properties", World Scientific Monograph Series in Mathematics, 2005

[19] W. Magnus and S. Winkler, "Hill's equation", Dover, 1966

[20] M. S. P. Eastham, "The spectral theory of periodic differential equations", Scottish Academic Press, 1973

[21] J. Satsuma and N. Yajima, "B. Initial value problems of one-dimensional self-modulation of nonlinear waves in dispersive media", Progress of Theoretical Physics Supplement, **55**, 284–306 (1974)

[22] D. W. McLaughlin and E. A. Overman II, "Whiskered Tori for Integrable PDE's: Chaotic Behavior in Near Integrable Pde's", in Surveys in Applied Mathematics, Volume I, Ed. J. P. Keller, D. W. McLaughlin and G. P. Papanicolaou, Plenum, 1995

[23] J. C. Bronski, "Semiclassical eigenvalue distribution of the Zakharov-Shabat eigenvalue problem", Phys. D **97**, 376–397 (1996)

[24] B. Grébert, T. Kappeler, B. Mityagin, "Gap estimates of the spectrum of the Zakharov-Shabat system" Appl. Math. Lett. **11**:4 95–97 (1998)

[25] A. Tovbis and S. Venakides, "The eigenvalue problem for the focusing nonlinear Schrödinger equation: new solvable cases", Phys. D **146**, 150–164 (2000)

[26] P. D. Miller, "Some remarks on a WKB method for the nonselfadjoint Zakharov-Shabat eigenvalue problem with analytic potential and fast phase", Phys. D **152-53**, 145–162 (2001)

[27] M. Klaus and J. K. Shaw, "Purely imaginary eigenvalues of Zakharov-Shabat systems", Phys. Rev. E **65**, 036607 (2002)

[28] M. Klaus and J. K. Shaw, "On the eigenvalues of Zakharov-Shabat systems", SIAM J. Math. Anal. **34**, 759–773 (2003)

[29] V. E. Zakharov, "Turbulence in integrable systems", Stud. Appl. Math. **122**, 219–234 (2009)

[30] J. M. Dudley and J. R. Taylor, "Supercontinuum generation in optical fibers" Cambridge University Press, 2010

[31] G. A. El and M. A. Hoefer "Dispersive shock waves and modulation theory", Phys. D **333**, 11–65 (2016)

[32] S. Trillo and A. Valiani, "Hydrodynamic instability of multiple four-wave mixing", Opt. Lett. **35**, 3967 (2010)

[33] G. Deng, S. Li, G. Biondini and S. Trillo, "Recurrence due to periodic multi-soliton fission in the defocusing nonlinear Schrödinger equation", Phys. Rev. E **96**, 052213 (2017)

[34] S. Kamvissis, K. D. T-R. McLaughlin and P. D. Miller, "Semiclassical soliton ensembles for the focusing nonlinear Schrödinger equation", Princeton (2003)

[35] A. Tovbis, S. Venakides and X. Zhou, "On semiclassical (zero dispersion limit) solutions of the focusing nonlinear Schrödinger equation", Commun. Pure Appl. Math., **57**, 877–985 (2004)

[36] B. Deconinck and J. N. Kutz, "Computing spectra of linear operators using the Floquet-Fourier-Hill method", J. Comput. Phys. **219**, 296–321 (2006)

[37] G. Deng, G. Biondini and S. Trillo, "Small dispersion limit of the Kortweg-de Vries equation with periodic initial condition and analytical description of the Zabusky-Kruskal experiment", Physica D **333**, 137 (2016)

[38] K. T–R McLaughlin and P. V. Nabelek, "A Riemann-Hilbert problem approach to infinite gap Hill's operators and the Korteweg-de Vries equation", Int. Math. Res. Notices, rnz156 (2019)