# Testing a discrete model for quantum spin with two sequential Stern-Gerlach detectors and photon Fock states 

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#### Abstract

Despite its unparalleled success, quantum mechanics (QM) is an incomplete theory of nature. Longstanding concerns regarding its mathematical foundations and physical interpretation persist, a full century beyond its conception. Limited by these issues, efforts to move beyond QM have struggled to gain traction within the broader physics community. One approach to progress in this direction, which is deeply rooted in the tradition of physics, is the development of new models for physical systems otherwise treated by QM. One such model is presented here, which concerns the interaction of a spin system with sequences of two Stern-Gerlach detectors that may be independently rotated. Rather than employing the traditional formalism of QM, the proposed model is supported by tools from discrete mathematics, such as finite groups, set theory, and combinatorics. Equipped with this novel toolkit, an analog of Wigner's d-matrix formula is derived and shown to deviate slightly from QM. With these results, the proposed model is extended to an optical system involving photon number states passing through a beam splitter. Leveraging recent advancements in high precision experiments on these systems, we then propose a means of testing the new model using a tabletop experiment. Hence, the proposed model not only makes clear testable predictions, but also provides valuable insight into the essential principles of quantum theory.


## 1 Introduction

In 1922, physicists Otto Stern and Walther Gerlach reported their experimental results concerning the discrete nature of angular momentum [1]. The experiment they performed was proposed as a test of the Bohr-Sommerfeld model of the atom, in which electrons were restricted to fixed orbits around the nucleus, a feature known then as "space quantization" [2]. While the Bohr-Sommerfeld model was later supplanted, the experimental results of Stern and Gerlach (SG), which showed that the projection of angular momentum was indeed quantized within the spatial frame of their choice, continued to play an important role in the development of modern quantum mechanics (QM). Today, sequences of SG detectors are of significant pedagogical importance, often being introduced in the early chapters of undergraduate texts on QM [3].

Wrapped up in the treatment of sequences of SG detectors are many foundational questions in physics. There are the familiar ones surrounding QM, such as non-determinism, non-commutativity, and the measurement problem. There are also questions about the nature of spin, the intrinsic angular momentum carried by fundamental particles. In particular, what is the origin of this degree of freedom, why is it quantized, and what is its relationship to space and time? With varying degrees of success, each of these questions has been addressed over the last century.

We raise these questions to highlight the non-trivial physics involved in experiments comprised of sequences of SG detectors. In this paper, our focus will be a new model for experiments involving two SG detectors, which may differ in spatial orientation. A depiction of this experimental setup is provided in Figure 1 , where the rotations being modeled are limited to the angle $\theta_{a b}$. Of course, QM has provided a satisfactory treatment of this physical system for nearly 100 years. So, what motivation could there be for a new model? First, we note that by studying alternative models or formulations, we may improve our understanding of QM in the process. This has been a guiding motivation for a great deal of important research [4-11]. In this


Figure 1: Within each SG detector is a magnetic field gradient, the spatial orientation of which is indicated by a red arrow for Alice's detector and a blue arrow for Bob's detector. Incoming particles are deflected by these field gradients at an angle which depends on their spin projection quantum number $m$ within the chosen spatial frame. The rotations modeled here are limited to $\theta_{a b}$, which is defined in the plane perpendicular to the dashed gray line.
case, however, our motivation arises primarily from the intriguing nature of the formalism upon which the model is built, which deviates significantly from QM [12].

This alternative formalism begins with the $Z_{2}$ group members 0 and 1 , equipped with the addition modulo two operation $(\oplus)$. We also introduce a direct product $(\times)$, which is used to construct higher order structures such as base-2 sequences. Additional formalism is then introduced as a means of studying these higher order structures, which includes the natural numbers, sets of sequences, Cartesian products ( $\otimes$ ), and probabilities.

The discrete nature of this formalism distinguishes the proposed model from large portions of the literature concerning the foundations of physics. This is due primarily to the ubiquitous assumption of continuity in theoretical physics, which can not be empirically confirmed. Over the past six decades, there have been several notable attempts to challenge various manifestations of this important assumption. The most prominent of these attempts being Roger Penrose's work on spin networks and the subsequent development of Loop Quantum Gravity [13, 14]. More recently, the Cellular Automata program has gained some attention through the work of Gerard 't Hooft [15]. Also of relevance is the work of David Finkelstein on the issue of a Space-Time Code [16], as well as the work by Chang et al. on their Galois Field Quantum Mechanics [1720]. Though not directly related to the issue of continuity, the Amplituhedron research program initiated by Arkani-Hamed et al. is relevant here due its use of combinatorial techniques in the calculation of probabilities [21, 22]. Collectively, these research efforts have contributed significantly to the literature upon which the present work is based.

To motivate our specific approach to modeling the physical system of interest, we will begin with a topdown analysis of a result provided by QM. The expression which generates the probability of observing a particular spin projection $m^{\prime}$, given a system with total spin $j$, initial spin projection $m$, and relative rotation of spatial frames $\theta$, is as follows:

$$
\begin{align*}
&\left(d_{m^{\prime}, m}^{j}(\theta)\right)^{2}=\sum_{q^{a}}(-1)^{m^{\prime}-m+q^{a}} \frac{(j+m)!(j-m)!}{\left(j+m-q^{a}\right)!q^{a}!\left(m^{\prime}-m+q^{a}\right)!\left(j-m^{\prime}-q^{a}\right)!} \\
& \times\left(\cos \left(\frac{\theta}{2}\right)\right)^{2 j+m-m^{\prime}-2 q^{a}}\left(\sin \left(\frac{\theta}{2}\right)\right)^{m^{\prime}-m+2 q^{a}} \\
& \times \sum_{q^{b}}(-1)^{m^{\prime}-m+q^{b}} \frac{\left(j+m^{\prime}\right)!\left(j-m^{\prime}\right)!}{\left(j+m-q^{b}\right)!q^{b}!\left(m^{\prime}-m+q^{b}\right)!\left(j-m^{\prime}-q^{b}\right)!}  \tag{1}\\
& \times\left(\cos \left(\frac{\theta}{2}\right)\right)^{2 j+m-m^{\prime}-2 q^{b}}\left(\sin \left(\frac{\theta}{2}\right)\right)^{m^{\prime}-m+2 q^{b}}
\end{align*}
$$

The expression given in equation (1) can be obtained in a variety of ways within the formalism of QM. It was first proposed by Eugene Wigner in 1927, who relied on group theoretic arguments [23, 24]. Decades later, Julian Schwinger provided an alternative derivation which employed the operator algebra of simple harmonic oscillators in QM [25]. The latter derivation provides the conceptual starting point for the top-down analysis in section 2.

Beginning with section 3, we use the conclusions of section 2 to guide the pedagogical development of a new mathematical model for the system of interest. In particular, we focus on the spacial case of a spin $\frac{1}{2}$ particle. This development process culminates in section 6 , in which a general expression for calculating probabilities is proposed and compared to QM. In section 7, we show how the new model can be applied to an optical system, which provides an experimental advantage over spin systems. We also discuss several ways to test the proposed model. In section 8 , we provide a brief summary of the formalism which supports the proposed model, arguing that it may be of interest beyond the specific model presented here. Finally, in section 9, we offer a short discussion regarding the position of this new model within the broader landscape of beyond QM models, as well as plans for future work.

## 2 Changing variables

In Schwinger's oscillator model for angular momentum, he begins with two uncoupled simple harmonic oscillators which may be called "plus type" and "minus type" [26]. Ladder operators associated with a particular spatial frame are then introduced for each oscillator, such that operators acting on different oscillators commute. The total angular momentum $j$ of a physical system is then built up from the vacuum state by successive applications of creation operators, which are denoted as $a_{+}^{\dagger}$ and $a_{-}^{\dagger}$ for the plus and minus type oscillator, respectively. The projection of angular momentum $m$, within the chosen spatial frame, is then defined as the difference between the number of $a_{+}^{\dagger}$ 's and $a_{-}^{\dagger}$ 's which are used to lift the state from vacuum. Both $j$ and $m$ can be expressed in terms of the eigenvalue of the number operators for each oscillator type:

$$
\begin{array}{r}
j \equiv \frac{n_{+}+n_{-}}{2} \\
m \equiv \frac{n_{+}-n_{-}}{2} \tag{3}
\end{array}
$$

A second set of ladder operators can then be defined as rotated versions of the originals. In this case, the $a_{+}^{\dagger}$ operator becomes $a_{+}^{\dagger} \cos \left(\frac{\theta}{2}\right)+a_{-}^{\dagger} \sin \left(\frac{\theta}{2}\right)$ and the $a_{-}^{\dagger}$ operator becomes $a_{-}^{\dagger} \cos \left(\frac{\theta}{2}\right)-a_{+}^{\dagger} \sin \left(\frac{\theta}{2}\right)$. This implies that the projection of angular momentum in the rotated spatial frame may be different than in the unrotated frame, which can be defined in terms of the rotated number operator eigenvalues:

$$
\begin{equation*}
m^{\prime} \equiv \frac{n_{+}^{\prime}-n_{-}^{\prime}}{2} \tag{4}
\end{equation*}
$$

The expressions given in equations (2-4) will be structurally identical to those used in the new model. However, we will no longer interpret the positive integers $n_{+}, n_{-}, n_{+}^{\prime}$, and $n_{-}^{\prime}$ as the eigenvalues of number operators for simple harmonic oscillators. Rather, we will think of them as the number of times an abstract symbol appears within a sequence. This interpretation is motivated by the combinatorial terms in equation (1), which are restated here:

$$
\begin{align*}
& \frac{(j+m)!(j-m)!}{\left(j+m-q^{a}\right)!q^{a}!\left(m^{\prime}-m+q^{a}\right)!\left(j-m^{\prime}-q^{a}\right)!}  \tag{5}\\
& \frac{\left(j+m^{\prime}\right)!\left(j-m^{\prime}\right)!}{\left(j+m-q^{b}\right)!q^{b}!\left(m^{\prime}-m+q^{b}\right)!\left(j-m^{\prime}-q^{b}\right)!} \tag{6}
\end{align*}
$$

Each of these expressions can be thought of as counting permutations of sequences comprised of four symbols, which we shall call base- 4 sequences. There is, however, an additional constraint imposed by the two factorials in the numerators. This constraint requires us to interpret each base- 4 sequence as being an ordered pair of two base- 2 sequences, which shall be composed of the symbols $C$ and $D$. An example of one
such construction is given below, where the subscripts $a 1$ and $b 2$ shall distinguish the left and right base- 2 sequence, respectively:

$$
\left(\begin{array}{l}
C  \tag{7}\\
D \\
D \\
C
\end{array}\right)_{a 1} \otimes\left(\begin{array}{c}
C \\
D \\
C \\
D
\end{array}\right)_{b 2}=\left(\begin{array}{c}
C C \\
D D \\
D C \\
C D
\end{array}\right)_{a 1 . b 2}
$$

The constraint imposed by the two factorials in the numerator of equations (5) and (6) can now be interpreted as holding one of the base- 2 sequences fixed while counting base- 4 permutations. To make this picture more clear, we must show that each of the arguments in these combinatorial expressions can be interpreted as the number of times a symbol appears in a base- 4 sequence. In general, the number of times a symbol appears in a sequence shall be called a count, which we denote by placing a tilde atop the symbol of interest like so: $\widetilde{C C}, \widetilde{C D}, \widetilde{D C}$, and $\widetilde{D D}$. Following from the construction depicted in equation (7), we can express the base- 2 counts associated with the $a 1$ and $b 2$ sequences in terms of base- 4 counts like so:

$$
\begin{array}{ll}
\tilde{C}_{a 1}=\widetilde{C C}+\widetilde{C D}, & \tilde{D}_{a 1}=\widetilde{D C}+\widetilde{D D} \\
\tilde{C}_{b 2}=\widetilde{C C}+\widetilde{D C}, & \tilde{D}_{b 2}=\widetilde{C D}+\widetilde{D D} \tag{9}
\end{array}
$$

As previously stated, the positive integers $n_{+}, n_{-}, n_{+}^{\prime}$, and $n_{-}^{\prime}$ interpreted by Schwinger as number operator eigenvalues will now be interpreted as counts:

$$
\begin{array}{ll}
n_{+} \rightarrow \tilde{C}_{a 1}, & n_{-} \rightarrow \tilde{D}_{a 1} \\
n_{+}^{\prime} \rightarrow \tilde{C}_{b 2}, & n_{-}^{\prime} \rightarrow \tilde{D}_{b 2} \tag{11}
\end{array}
$$

The quantum numbers $j, m$, and $m^{\prime}$ can now be expressed as follows, where we replace the primed/unprimed notation with the $a 1 / b 2$ notation:

$$
\begin{array}{r}
j \equiv \frac{\tilde{C}_{a 1}+\tilde{D}_{a 1}}{2}=\frac{\tilde{C}_{b 2}+\tilde{D}_{b 2}}{2} \\
m \rightarrow m_{a 1} \equiv \frac{\tilde{C}_{a 1}-\tilde{D}_{a 1}}{2} \\
m^{\prime} \rightarrow m_{b 2} \equiv \frac{\tilde{C}_{b 2}-\tilde{D}_{b 2}}{2} \tag{14}
\end{array}
$$

One last definition is necessary to accomplish our goal of interpreting each factorial argument as a count. This definition is for the summing parameter $q$, which may carry the superscript $a$ or $b$ depending on which sum it is associated with in equation (1):

$$
\begin{equation*}
q^{a} \equiv \widetilde{C D}^{a}, \quad q^{b} \equiv \widetilde{C D}^{b} \tag{15}
\end{equation*}
$$

Using the definitions offered in equations (8-9) and (12-15), we may now execute the following change of variables for the remaining factorial arguments in equations (5) and (6):

$$
\begin{align*}
& j+m_{a 1}=\tilde{C}_{a 1}, \quad j-m_{a 1}=\tilde{D}_{a 1}  \tag{16}\\
& j+m_{b 2}=\tilde{C}_{b 2}, \quad j-m_{b 2}=\tilde{D}_{b 2}  \tag{17}\\
& j+m_{a 1}-q^{a}=\widetilde{C C}^{a}, \quad m_{b 2}-m_{a 1}+q^{a}=\widetilde{D C}^{a}, \quad j-m_{b 2}-q^{a}=\widetilde{D D}^{a}  \tag{18}\\
& j+m_{a 1}-q^{b}=\widetilde{C C}^{b}, \quad m_{b 2}-m_{a 1}+q^{b}=\widetilde{D C}^{b}, \quad j-m_{b 2}-q^{b}=\widetilde{D D}^{b} \tag{19}
\end{align*}
$$

With these results, equations (5) and (6) become the following:

$$
\begin{equation*}
\frac{\tilde{C}_{a 1}!\tilde{D}_{a 1}!}{\widetilde{C C}^{a}!\widetilde{C D}^{a}!\widetilde{D C}^{a}!\widetilde{D D}^{a}!} \tag{20}
\end{equation*}
$$

$$
\begin{equation*}
\frac{\tilde{C}_{b 2}!\tilde{D}_{b 2}!}{\widetilde{C C}^{b}!\widetilde{C D}^{b}!\widetilde{D C}^{b}!\widetilde{D D}^{b}!} \tag{21}
\end{equation*}
$$

Expressed in this form, the proposed interpretation of these combinatorial terms becomes more clear. We may think of them as counting base- 4 sequences, such that the ordering of the symbols in one of the component base-2 sequences remain fixed. In the case of equation (20), it is the base-2 sequence with the subscript $a 1$ which is held fixed, while the base-2 sequence with the subscript $b 2$ is held fixed in equation (21). One example of a set of sequences being counted by equation (20) is as follows, where $\tilde{C}_{a 1}=\tilde{D}_{a 1}=$ $\tilde{C}_{b 2}=\tilde{D}_{b 2}=2$ and $\widetilde{C C}^{a}=\widetilde{C D}^{a}=\widetilde{D C}^{a}=\widetilde{D D}^{a}=1$ :

$$
\left(\begin{array}{l}
C  \tag{22}\\
D \\
D \\
C
\end{array}\right)_{a 1} \otimes\left\{\left(\begin{array}{l}
C \\
D \\
C \\
D
\end{array}\right)_{b 2},\left(\begin{array}{l}
D \\
D \\
C \\
C
\end{array}\right)_{b 2},\left(\begin{array}{l}
C \\
C \\
D \\
D
\end{array}\right)_{b 2},\left(\begin{array}{l}
D \\
C \\
D \\
C
\end{array}\right)_{b 2}\right\}
$$

For the example given above, the combinatorial expression in equation (20) yields:

$$
\frac{\tilde{C}_{a 1}!\tilde{D}_{a 1}!}{\widetilde{\widetilde{C C}}^{a}!\widetilde{C D}^{a}!\widetilde{D C}^{a}!\widetilde{D D}}{ }^{a}!=\frac{2!2!}{1!1!1!1!}=4
$$

Motivated by this combinatorial picture, we propose that sets of sequences like the one depicted in equation (22) be interpreted as ontic state spaces. Here, we use the term "ontic state" to imply that there indeed exists a real fundamental state of the system. As a conceptual aid, we assign an observer named Alice (a) and Bob (b) to each of the SG detectors involved in the experiment. Through the use of their assigned detector, each of these observers has access to one component of the underlying ontic state. Individually, we interpret these components as the events which occur in each observer's detector. In the top-down analysis offered above, these events are modeled by the base- 2 sequences carrying the subscripts $a 1$ (Alice) and $b 2$ (Bob), where the associated counts can be used to define the quantum numbers $j, m_{a 1}$, and $m_{b 2}$. Equations (20) and (21) can then be interpreted as the cardinality of Alice's and Bob's ontic state spaces, which ultimately represent the information each observer has about the experiment.

Among other things, the conceptual picture proposed above does not explain the sine and cosine terms in equation (1). Within this equation, these terms not only account for the relative rotation $\theta$ of the spatial frames, but they also serve to normalize the combinatorial terms. We are thus left with a significant impasse in our top-down approach to deriving a fully combinatorial interpretation of equation (1). In the next section, we will begin a pedagogical introduction of a new model, which will be guided by the conclusions of this section. Throughout the development process, the concepts discussed in the previous paragraph, such as events, observers, and ontic state spaces, will be reintroduced and formally defined. Ultimately, we will obtain an expression which can produce predictions for the physical system of interest that are highly competitive with QM.

## 3 Spin 1/2 $(\mathrm{n}=1)$

In the previous section, we made the suggestion that sequences comprised of the abstract symbols $C$ and $D$ should be interpreted as mathematical models for events. To clarify what we mean by this, we will treat the simplest possible case of an experiment involving two SG detectors, which is for a spin $\frac{1}{2}$ particle. In Figure 2 , we see an example of such an experiment.

The first thing that happens in this experiment is the interaction, or "event", which occurs inside of Alice's detector. As a result of this event, the spin $\frac{1}{2}$ particle is deflected into one of two possible paths, one for each value of $m_{a 1} \in\left\{+\frac{1}{2},-\frac{1}{2}\right\}$, a range that can be inferred from equation (13). For the spin $\frac{1}{2}$ case, the only two sequences that could be associated with this event, each of which have length $n=1$, are as follows:

$$
\begin{equation*}
\left\{(C)_{a 1},(D)_{a 1}\right\} \tag{23}
\end{equation*}
$$

Collectively, these sequences form the "ontic state space" of Alice's event. That is, we assume that the event in Alice's detector is associated with a definite state of reality, which we model using a single sequence of abstract symbols. Continuing to follow the spin $\frac{1}{2}$ particle's path in Figure 2, we see that there will also


Figure 2: An event occurs at Alice's detector which deflects the $j=\frac{1}{2}$ particle into one of two paths (red), one for each possible value of the quantum number $m_{a 1} \in\left\{+\frac{1}{2},-\frac{1}{2}\right\}$. Depending on which value of $m_{a 1}$ is of interest, Bob's detector is then placed along one of these paths. Bob then rotates his detector with respect to Alice's by the angle $\theta_{a b}$. Finally, an event occurs at Bob's detector which again deflects the $j=\frac{1}{2}$ particle into one of two paths (blue), one for each possible value of the quantum number $m_{b 2} \in\left\{+\frac{1}{2},-\frac{1}{2}\right\}$.
be an event in Bob's detector. Like before, this event will cause the particle to be deflected into one of two possible paths. As we did with Alice's event, we can write down the ontic state space for Bob's event like so:

$$
\begin{equation*}
\left\{(C)_{b 2},(D)_{b 2}\right\} \tag{24}
\end{equation*}
$$

For any given experiment, an event occurs at Alice's and Bob's detector. Thus, the ontic state space for the full experiment should consist of all ordered pairs of sequences in the ontic state spaces for each event. That's just the Cartesian product of equations (23) and (24):

$$
\begin{align*}
& (C)_{a 1} \otimes(C)_{b 2}=(C C)_{a 1, b 2}  \tag{25}\\
& (C)_{a 1} \otimes(D)_{b 2}=(C D)_{a 1, b 2}  \tag{26}\\
& (D)_{a 1} \otimes(C)_{b 2}=(D C)_{a 1, b 2}  \tag{27}\\
& (D)_{a 1} \otimes(D)_{b 2}=(D D)_{a 1, b 2} \tag{28}
\end{align*}
$$

We are now ready to begin discussing rotation, which we may think of as an abstract operation which maps Alice's event to Bob's. The extreme cases of rotation are $\theta_{a b}=0$ and $\theta_{a b}=\pi$, which should correspond to $m_{a 1}=m_{b 2}$ and $m_{a 1}=-m_{b 2}$, respectively. Returning to the ontic state space of the experiment given in equations (25-28), we see that equations (25) and (28) are associated with the $\theta_{a b}=0$ case ( $m_{a 1}=m_{b 2}$ ), while equations (26) and (27) are associated with the $\theta_{a b}=\pi$ case ( $m_{a 1}=-m_{b 2}$ ).

Since we want to think of rotation as an operation, we need to assign the symbols $C$ and $D$ some algebraic property which allows them to be mapped into one another. There are many ways to approach this, but in the end, making the symbols $C$ and $D$ elements of the finite group $Z_{2} \times Z_{2}$ will prove most fruitful. Because there are four elements in this group, which is known as the Klein four-group, we need to add the symbols $A$ and $B$ to our emerging alphabet. What remains is to decide which elements of this group to associate with each of the four symbols. Of the 24 possibilities, we make the following choice:

$$
\begin{equation*}
A \equiv 0 \times 0, B \equiv 1 \times 1, C \equiv 1 \times 0, D \equiv 0 \times 1 \tag{29}
\end{equation*}
$$

Note that an important consequence of using the finite group $Z_{2} \times Z_{2}$ is that we may actually regard base- 4 sequences composed of the symbols $A, B, C$, and $D$ as ordered pairs of base- 2 sequences composed of the symbols 0 and 1 . This is the origin of the subscript notation we have been using thus far, where the indices $a$ and 1, for example, distinguish the base-2 sequences within the ordered pair. An example of this construction is given here:

$$
\left(\begin{array}{l}
0  \tag{30}\\
1 \\
1 \\
0 \\
1 \\
0
\end{array}\right)_{a} \otimes\left(\begin{array}{l}
0 \\
1 \\
0 \\
1 \\
1 \\
0
\end{array}\right)_{1}=\left(\begin{array}{l}
A \\
B \\
C \\
D \\
B \\
A
\end{array}\right)_{a 1}
$$

The group operation of interest for rotations is addition modulo two $(\bmod 2)$, which we denote using the symbol $\oplus$. In general, the operators which map ontic states for Alice's event into Bob's are sequences comprised of the symbols $A, B, C$, and $D$, along with the $\oplus$ operation. Though, for rotations, we are only concerned with sequences filled with $A$ 's and $B$ 's. This restriction is motivated in part by the need for Alice's and Bob's events to share a common value of $j$. The full addition table for these symbols is as follows:

$$
\begin{align*}
& A \oplus A=B \oplus B=C \oplus C=D \oplus D=A \\
& A \oplus B=B \oplus A=C \oplus D=D \oplus C=B \\
& A \oplus C=C \oplus A=B \oplus D=D \oplus B=C  \tag{31}\\
& A \oplus D=D \oplus A=B \oplus C=C \oplus B=D
\end{align*}
$$

With our brief interlude into finite group theory complete, we can return to the physical system we are attempting to model. The goal now is to show that the ontic state space for the experiment, which was defined in equations (25-28), can be generated by applying maps to the ontic state space of Alice's event, for example. For the case we have been considering, there are two ontic states associated with Alice's event and two possible maps for each, yielding four possible scenarios:

$$
\begin{align*}
& (C)_{a 1} \oplus(A)_{m a p}=(C)_{b 2}  \tag{32}\\
& (C)_{a 1} \oplus(B)_{m a p}=(D)_{b 2}  \tag{33}\\
& (D)_{a 1} \oplus(B)_{m a p}=(C)_{b 2}  \tag{34}\\
& (D)_{a 1} \oplus(A)_{m a p}=(D)_{b 2} \tag{35}
\end{align*}
$$

Now that we have an understanding of how maps work, we need to make a choice regarding the definition of $\theta_{a b}$ in terms of these maps. Taking a look at equations (32-35), we see that the $m_{a 1}=m_{b 2}$ case is associated the symbol $A\left(\theta_{a b}=0\right)$, while the $m_{a 1}=-m_{b 2}$ case is associated with the symbol $B\left(\theta_{a b}=\pi\right)$. Motivated by this observation, we propose the following provisional definition of $\theta_{a b}$, where $\tilde{B}_{\text {map }}$ is the number of $B$ 's that appear in a map of length $n$ :

$$
\begin{equation*}
\theta_{a b} \equiv \frac{\tilde{B}_{m a p}}{n} \pi \tag{36}
\end{equation*}
$$

We have finally developed enough machinery to check if the proposed model indeed predicts the correct behavior of a spin $\frac{1}{2}$ particle within an experiment involving two SG detectors. Though, we are currently limited to relative rotations of $\theta_{a b}=0$ and $\theta_{a b}=\pi$ between Alice's and Bob's detectors. In the case of $\theta_{a b}=0$, if Alice observes $m_{a 1}=+\frac{1}{2}\left(m_{a 1}=-\frac{1}{2}\right)$ then Bob must observe $m_{b 2}=+\frac{1}{2}\left(m_{b 2}=-\frac{1}{2}\right)$, as seen in equations (32) and (35). Likewise for $\theta_{a b}=\pi$, if Alice observes $m_{a 1}=+\frac{1}{2}\left(m_{a 1}=-\frac{1}{2}\right)$ then Bob must observe $m_{b 2}=-\frac{1}{2}\left(m_{b 2}=+\frac{1}{2}\right)$, as seen in equations (33) and (34). Of course, these results are unsurprising given the approach we took when selecting a definition for $\theta_{a b}$. The issue we must now face is that of arbitrary rotations.

## $4 \quad$ Spin $1 / 2(n=2)$

Continuing our incremental approach to developing this new model, we would now like to consider the case in which Alice's and Bob's detector have a relative rotation of $\theta_{a b}=\frac{\pi}{2}$. From our definition of $\theta_{a b}$ given in equation (36), the smallest value of $n$ that would support such a rotation is 2 . Given that we are modeling a spin $\frac{1}{2}$ particle, along with the definition of $j$ given in equation (12), the total number of $C$ 's and $D$ 's that can appear in ontic states for events is still limited to one. This leaves us with only two options for lengthening our sequences, which is adding an $A$ or a $B$. This modification leaves us with the following ontic state spaces for Alice's and Bob's event:

$$
\begin{align*}
& \left\{\binom{A}{C}_{a 1},\binom{C}{A}_{a 1},\binom{B}{C}_{a 1},\binom{C}{B}_{a 1},\binom{A}{D}_{a 1},\binom{D}{A}_{a 1},\binom{B}{D}_{a 1},\binom{D}{B}_{a 1}\right\}  \tag{37}\\
& \left\{\binom{A}{C}_{b 2},\binom{C}{A}_{b 2},\binom{B}{C}_{b 2},\binom{C}{B}_{b 2},\binom{A}{D}_{b 2},\binom{D}{A}_{b 2},\binom{B}{D}_{b 2},\binom{D}{B}_{b 2}\right\} \tag{38}
\end{align*}
$$

The inclusion of the symbols $A$ and $B$ in ontic states for events has several important implications, but the first issue that must be addressed is the need to define two new quantum numbers, which we may think of as analogues of $j$ and $m$ (other definitions are possible):

$$
\begin{array}{ll}
g_{a 1} \equiv \frac{\tilde{A}_{a 1}+\tilde{B}_{a 1}}{2}, & g_{b 2} \equiv \frac{\tilde{A}_{b 2}+\tilde{B}_{b 2}}{2} \\
l_{a 1} \equiv \frac{\tilde{A}_{a 1}-\tilde{B}_{a 1}}{2}, & l_{b 2} \equiv \frac{\tilde{A}_{b 2}-\tilde{B}_{b 2}}{2} \tag{40}
\end{array}
$$

We now have four quantum numbers associated with events, which are $j, m, g$, and $l$. However, we will often replace $g$ with $n$ when listing the degrees of freedom for events, where $n=2 j+2 g$. For the time being, we remain agnostic with respect to the physical interpretation of these new quantum numbers. In most cases, we can simply sum over this degree of freedom, though in section 5 we will see that the quantum number $l$ has an interesting effect on the predictions generated by the new model.

Beyond the new quantum numbers, the inclusion of $A$ 's and $B$ 's also lifts the one-to-one correspondence between quantum states and ontic states for spin $\frac{1}{2}$ particles. For example, there are now four ontic states in equations (37) and (38) associated with each possible value of $m_{a 1}$ and $m_{b 2}$, respectively. This change from one-to-one to one-to-many leads directly to non-determinism within this model.

As an example of how non-determinism arises within this model, consider the case in which the event at Alice's detector results in $m_{a 1}=+\frac{1}{2}$ and $l_{a 1}=+\frac{1}{2}$, with $\theta_{a b}=\frac{\pi}{2}$. In equation (37), there are two ontic states associated with the combination of $m_{a 1}=+\frac{1}{2}$ and $l_{a 1}=+\frac{1}{2}$, of which we will select one to model the event that occurred in Alice's detector. To this ontic state we then apply all possible maps associated with the quantum numbers $n=2$ and $\theta_{a b}=\frac{\pi}{2}$ :

$$
\begin{equation*}
\left\{\binom{C}{A}_{a 1}\right\} \oplus\left\{\binom{A}{B}_{\text {map }},\binom{B}{A}_{\text {map }}\right\}=\left\{\binom{C}{B}_{b 2},\binom{D}{A}_{b 2}\right\} \tag{41}
\end{equation*}
$$

The right side of equation (41) can be interpreted as the ontic state space for Bob's event given $m_{a 1}=+\frac{1}{2}$, $l_{a 1}=+\frac{1}{2}$, and $\theta_{a b}=\frac{\pi}{2}$, along with a particular choice of ontic state for Alice's event. The key observation being the presence of two possibilities, each associated with a different value of the quantum number $m_{b 2}$. Thus, the non-determinism observed in experiments involving sequences of SG detectors is modeled here by considering all possible maps. Or equivalently, by hiding the information stored in the ordering of the $A$ 's and $B$ 's within maps.

The example illustrated in equation (41) correctly predicts the relative frequency of the two possible values of $m_{b 2}$. That is, each occur an equal number of times in the resulting ontic state space. As previously stated, we will typically sum over the quantum numbers $l_{a 1}$ and $l_{b 2}$. For the example considered in equation (41), that amounts to choosing a sequence from the ontic state space for Alice's event for each possible value of $l_{a 1}$ and then applying the maps to each. In this case, such an operation will yield the same relative frequency for the two possible values of $m_{b 2}$. However, as $n$ becomes larger, this relative frequency will generally depend on $l_{a 1}$. This dependence will be discussed further in section 5 .

$$
\begin{array}{ll}
\tilde{A}_{a 1}=\frac{n}{2}-j+l_{a 1} & \tilde{A}_{b 2}=\frac{n}{2}-j+l_{b 2} \\
\tilde{B}_{a 1}=\frac{n}{2}-j-l_{a 1} & \tilde{B}_{b 2}=\frac{n}{2}-j-l_{b 2} \\
\tilde{C}_{a 1}=j+m_{a 1} & \tilde{C}_{b 2}=j+m_{b 2} \\
\tilde{D}_{a 1}=j-m_{a 1} & \tilde{D}_{b 2}=j-m_{b 2} \\
\tilde{A}_{m a p}=n\left(1-\frac{\theta_{a b}}{\pi}\right) & \tilde{B}_{m a p}=n \frac{\theta_{a b}}{\pi}
\end{array}
$$

Table 1: Base-4 counts as functions of quantum numbers

## 5 Spin 1/2 (arbitrary n)

The next logical step is to consider arbitrary values of $\theta_{a b}$, which in turn requires $n$ to become arbitrarily large. As we saw in the $n=2$ example, it is essential that we be able to count the number of ontic states associated with a given set of quantum numbers. For the $n=2$ case, this can be done explicitly due to the relatively small number of ontic states. Unfortunately, as $n$ increases, this approach quickly becomes infeasible. To overcome this, we must borrow some technology from combinatorics. This will enable us to efficiently count the number of sequences associated with each unique set of quantum numbers. However, to make use of this technology, we need to take another step in our formalism.

At the moment, we can use the quantum numbers $n, j, m$, and $l$ to count the number ontic states associated with a given event. We just need a map which converts these quantum numbers into the counts $\tilde{A}, \tilde{B}, \tilde{C}$, and $\tilde{D}$, which is a straight forward task in linear algebra (Table 1). These counts can then be used in the following combinatorial expression to determine the number of sequences associated with the given quantum numbers, where $n=\tilde{A}+\tilde{B}+\tilde{C}+\tilde{D}$ :

$$
\begin{equation*}
\frac{n!}{\tilde{A}!\tilde{B}!\tilde{C}!\tilde{D}!} \tag{42}
\end{equation*}
$$

Provided we know $n$ and $\theta_{a b}$, we can also use this approach to count the number of maps. The only difference in that case is that the counts $\tilde{C}$ and $\tilde{D}$ are always 0 . Thus, given particular choices for the quantum numbers $n, j, m_{a 1}, l_{a 1}, m_{b 2}, l_{b 2}$, and $\theta_{a b}$, we can determine the cardinality of the ontic state spaces associated with Alice's and Bob's events, as well as the total number of maps. Unfortunately, this combination of information is useless to us.

To generalize our approach to calculating relative frequencies for large $n$, which we will refer to as probabilities herein, we need to learn how to count the ontic states of full experiments. In particular, we need a way to count the ontic states of experiments that allow us to control not only $m_{a 1}$ and $m_{b 2}$, but also $\theta_{a b}$. To accomplish this task, we will need to once again expand our alphabet.

Because a single experiment consists of an event at Alice's detector and an event Bob's detector, we can model the ontic states of the full experiment as ordered pairs of base- 4 sequences, as depicted in equations (25-28). In general, we may treat ordered pairs of base- 4 sequences as base- 16 sequences comprised of the following symbols:

$$
\begin{equation*}
\{A A, A B, A C, A D, B A, B B, B C, B D, C A, C B, C C, C D, D A, D B, D C, D D\} \tag{43}
\end{equation*}
$$

For each symbol in this base-16 alphabet, the base- 4 symbol on the left is associated with Alice's event and the base- 4 symbol on the right is associated with Bob's. Because we are only interested in maps containing the symbols $A$ and $B$, we can actually ignore eight of the symbols introduced in equation (43). After accounting for this restriction, we are left with the following set of eight symbols, which will serve as the basis for ontic states of experiments:

$$
\begin{equation*}
\{A A, A B, B A, B B, C C, C D, D C, D D\} \tag{44}
\end{equation*}
$$

We can now express the base- 4 counts associated with Alice's and Bob's events in terms of base- 8 counts like so:

$$
\begin{array}{ll}
\tilde{C}_{a 1}=\widetilde{C C}+\widetilde{C D}, & \tilde{D}_{a 1}=\widetilde{D C}+\widetilde{D D} \\
\tilde{C}_{b 2}=\widetilde{C C}+\widetilde{D C}, & \tilde{D}_{b 2}=\widetilde{C D}+\widetilde{D D} \tag{46}
\end{array}
$$

$$
\begin{aligned}
& \widetilde{A A}=\frac{1}{4} n\left(1-\frac{\theta_{a b}}{\pi}\right)-\frac{1}{2} j+\frac{1}{2} l_{a 1}+\frac{1}{2} l_{b 2}+\frac{1}{2} \mu_{a 1, b 2} \\
& \widetilde{A B}=\frac{1}{4} n\left(1+\frac{\theta_{a b}}{\pi}\right)-\frac{1}{2} j+\frac{1}{2} l_{a 1}-\frac{1}{2} l_{b 2}-\frac{1}{2} \mu_{a 1, b 2} \\
& \widetilde{B A}=\frac{1}{4} n\left(1+\frac{\theta_{a b}}{\pi}\right)-\frac{1}{2} j-\frac{1}{2} l_{a 1}+\frac{1}{2} l_{b 2}-\frac{1}{2} \mu_{a 1, b 2} \\
& \widetilde{B B}=\frac{1}{4} n\left(1-\frac{\theta_{a b}}{\pi}\right)-\frac{1}{2} j-\frac{1}{2} l_{a 1}-\frac{1}{2} l_{b 2}+\frac{1}{2} \mu_{a 1, b 2} \\
& \widetilde{C C}=\frac{1}{4} n\left(1-\frac{\theta_{a b}}{\pi}\right)+\frac{1}{2} j+\frac{1}{2} m_{a 1}+\frac{1}{2} m_{b 2}-\frac{1}{2} \mu_{a 1, b 2} \\
& \widetilde{C D}=\frac{1}{4} n\left(\frac{\theta_{a b}}{\pi}-1\right)+\frac{1}{2} j+\frac{1}{2} m_{a 1}-\frac{1}{2} m_{b 2}+\frac{1}{2} \mu_{a 1, b 2} \\
& \widetilde{D C}=\frac{1}{4} n\left(\frac{\theta_{a b}}{\pi}-1\right)+\frac{1}{2} j-\frac{1}{2} m_{a 1}+\frac{1}{2} m_{b 2}+\frac{1}{2} \mu_{a 1, b 2} \\
& \widetilde{D D}=\frac{1}{4} n\left(1-\frac{\theta_{a b}}{\pi}\right)+\frac{1}{2} j-\frac{1}{2} m_{a 1}-\frac{1}{2} m_{b 2}-\frac{1}{2} \mu_{a 1, b 2}
\end{aligned}
$$

Table 2: Base-8 counts as functions of quantum numbers

$$
\begin{array}{rlr}
\tilde{A}_{a 1} & =\widetilde{A A}+\widetilde{A B}, & \tilde{B}_{a 1}=\widetilde{B A}+\widetilde{B B} \\
\tilde{A}_{b 2} & =\widetilde{A A}+\widetilde{B A}, & \tilde{B}_{b 2}=\widetilde{A B}+\widetilde{B B} \tag{48}
\end{array}
$$

With these identities, the definitions given in equations (12-14), (36), and (40) become the following, where we also define $n$ rather than $g_{a 1}$ or $g_{b 2}$ :

$$
\begin{gather*}
n=\widetilde{A A}+\widetilde{A B}+\widetilde{B A}+\widetilde{B B}+\widetilde{C C}+\widetilde{C D}+\widetilde{D C}+\widetilde{D D}  \tag{49}\\
j=\frac{\widetilde{C C}+\widetilde{C D}+\widetilde{D C}+\widetilde{D D}}{2}  \tag{50}\\
m_{a 1}=\frac{\widetilde{C C}+\widetilde{C D}-\widetilde{D C}-\widetilde{D D}}{2}, \quad m_{b 2}=\frac{\widetilde{C C}+\widetilde{D C}-\widetilde{C D}-\widetilde{D D}}{2}  \tag{51}\\
l_{a 1}=\frac{\widetilde{A A}+\widetilde{A B}-\widetilde{B A}-\widetilde{B B}}{2}, \quad l_{b 2}=\frac{\widetilde{A A}+\widetilde{B A}-\widetilde{A B}-\widetilde{B B}}{2}  \tag{52}\\
\theta_{a b}=\frac{\pi}{n}(\widetilde{A B}+\widetilde{B A}+\widetilde{C D}+\widetilde{D C}) \tag{53}
\end{gather*}
$$

So far, we have defined seven quantum numbers in terms of base- 8 counts, each having been previously introduced. However, to count base- 8 sequences, we will need to define an eighth quantum number. This new degree of freedom, which is closely related to the summing parameter $q$ in equation (1), will not be important for the spin $\frac{1}{2}$ case. In section 6 , however, it will be shown that this degree of freedom is central to the "quantum" nature of the proposed model. This owes to that fact that, no matter which definition we choose, this final quantum number is an exclusive property of the full ontic state of the experiment. That is, it cannot be associated with Alice's or Bob's events, nor the map which relates the two. There are actually several equivalent ways to define this base- 8 quantum number, but the following is perhaps the most instructive:

$$
\begin{equation*}
\mu_{a 1, b 2} \equiv \frac{\widetilde{C D}+\widetilde{D C}+\widetilde{A A}+\widetilde{B B}}{2} \tag{54}
\end{equation*}
$$

Given a complete set of eight quantum numbers, along with a map which converts these quantum numbers into base- 8 counts (Table 2), we can count base- 8 sequences using an analog of equation (42):

$$
\begin{equation*}
\frac{n!}{\widetilde{A A}!\widetilde{A B}!\widetilde{B A}!\widetilde{B B}!\widetilde{C C}!\widetilde{C D}!\widetilde{D C}!\widetilde{D D}!} \tag{55}
\end{equation*}
$$

We can now use a combination of the expressions introduced in equations (42) and (55) to calculate probabilities. Though, we will also need to sum over various quantum numbers, such as $m_{b 2}, l_{b 2}, \mu_{a 1, b 2}$, and possibly $l_{a 1}$. To simplify our notation, we introduce an elementary set of ontic states for experiments, which we denote as $\varepsilon^{a}\left(n, j, m_{a 1}, l_{a 1}, m_{b 2}, l_{b 2}, \theta_{a b}, \mu_{a 1, b 2}\right)$. For each unique combination of eight quantum numbers, $\varepsilon^{a}$ represents a unique set of ontic states, where the superscript indicates that Alice's event is held fixed within this set, as it is in equation (41). For a given set of eight quantum numbers, the cardinality of $\varepsilon^{a}$ is given by the following:

$$
\begin{equation*}
\left|\varepsilon^{a}\left(n, j, m_{a 1}, m_{b 2}, l_{a 1}, l_{b 2}, \theta_{a b}, \mu_{a 1, b 2}\right)\right|=\frac{\tilde{A}_{a 1}!\tilde{B}_{a 1}!\tilde{C}_{a 1}!\tilde{D}_{a 1}!}{\widetilde{A A}!\widetilde{A B}!\widetilde{B A}!\widetilde{B B}!\widetilde{C C}!\widetilde{C D}!\widetilde{D C}!\widetilde{D D}!} \tag{56}
\end{equation*}
$$

Before taking the next step in calculating probabilities, we should point out the similarity between equations (20) and (56). Clearly, we have recovered the combinatorial terms associated with the symbols $C$ and $D$, but with the addition of new terms involving the symbols $A$ and $B$. These terms, along with the normalization scheme to be introduced at the end of this section, play an analogous role to the sine and cosine terms in equation (1). There is still the issue of equation (21), however. That is, we have not recovered a combinatorial expression that depends the base- 4 counts associated with Bob's event $\left(\widetilde{A}_{b 2}, \widetilde{B}_{b 2}\right.$, $\left.\widetilde{C}_{b 2}, \widetilde{D}_{b 2}\right)$.

At the end of section 2, we suggested that the combinatorial terms in equations (20) and (21) be interpreted as the cardinalities of Alice's and Bob's ontic state spaces, respectively. In other words, they represent the information each observer has about the experiment being performed. Extending this interpretation to $\varepsilon^{a}$, we may now view this as an elementary set of ontic states of the experiment, according to Alice. That is, it contains the set of all possible ontic states associated with Bob's event, which are simultaneously compatible with the given quantum numbers and a particular fixed event at Alice's detector. Based on the results of the top down analysis of section 2 , we are thus motivated to introduce the elementary set of ontic states $\varepsilon^{b}$, where the superscript indicates that Bob's event is fixed within this ontic state space, rather than Alice's. The cardinality of $\varepsilon^{b}$ is given by the following:

$$
\begin{equation*}
\left|\varepsilon^{b}\left(n, j, m_{a 1}, m_{b 2}, l_{a 1}, l_{b 2}, \theta_{a b}, \mu_{a 1, b 2}\right)\right|=\frac{\tilde{A}_{b 2}!\tilde{B}_{b 2}!\tilde{C}_{b 2}!\tilde{D}_{b 2}!}{\widetilde{A A}!\widetilde{A B}!\widetilde{B A}!\widehat{B B}!\widetilde{C C}!\widetilde{C D}!\widetilde{D C}!\widetilde{D D}!} \tag{57}
\end{equation*}
$$

We now offer an example of $\varepsilon^{a}$ and $\varepsilon^{b}$ for the same complete set of eight quantum numbers $(n=4, j=$ $\left.\frac{1}{2}, m_{a 1}=+\frac{1}{2}, m_{b 2}=+\frac{1}{2}, l_{a 1}=+\frac{1}{2}, l_{b 2}=+\frac{1}{2}, \theta_{a b}=\frac{\pi}{2}, \mu_{a 1, b 2}=\frac{1}{2}\right):$

$$
\varepsilon^{a}=\left(\begin{array}{c}
C  \tag{58}\\
B \\
A \\
A
\end{array}\right)_{a 1} \otimes\left\{\left(\begin{array}{c}
C \\
A \\
A \\
B
\end{array}\right)_{b 2},\left(\begin{array}{c}
C \\
A \\
B \\
A
\end{array}\right)_{b 2}\right\}, \quad \varepsilon^{b}=\left\{\left(\begin{array}{c}
A \\
A \\
C \\
B
\end{array}\right)_{a 1},\left(\begin{array}{c}
A \\
B \\
C \\
A
\end{array}\right)_{a 1}\right\} \otimes\left(\begin{array}{c}
B \\
A \\
C \\
A
\end{array}\right)_{b 2}
$$

The last point we will make before returning to the task of calculating probabilities is regarding the choice of the fixed events in $\varepsilon^{a}$ and $\varepsilon^{b}$. Simply put, this choice will have no influence on the cardinality of $\varepsilon^{a}$ or $\varepsilon^{b}$, which is the relevant quantity when calculating probabilities. Thus, the quantities $\left|\varepsilon^{a}\right|$ and $\left|\varepsilon^{b}\right|$ each have a permutation symmetry relating to this freedom of choice.

We can now apply the various tools developed in this section, as well as the hints provided by the top down analysis of section 2, to begin building our expression for probabilities in the special case of $j=\frac{1}{2}$. The cardinality we are now interested in calculating is that of the Cartesian product space of Alice's and Bob's ontic state spaces, which is given by the following expression:

$$
\begin{equation*}
\left|\varepsilon^{a} \otimes \varepsilon^{b}\right|=\left|\varepsilon^{a}\right|\left|\varepsilon^{b}\right| \tag{59}
\end{equation*}
$$

We can interpret this product space as the set of all ordered pairs of ontic states which are compatible with the information each observer has about the experiment. To calculate the probability of observing a particular quantum number, we must define a sum over equation (59) for all possible combinations of unknown quantum numbers. For the case under consideration in this section, we are interested in the probability of observing a particular value of $m_{b 2}$, given $n, j=\frac{1}{2}, m_{a 1}$, and $\theta_{a b}$. We find this by first defining the following elementary counting expression, where the arguments of $\left|\varepsilon^{a}\right|$ and $\left|\varepsilon^{b}\right|$ not being summed over have been suppressed for notational ease:

$$
\begin{equation*}
v\left(n, j=\frac{1}{2}, m_{a 1}, m_{b 2}, l_{a 1}, l_{b 2}, \theta_{a b}\right)=\sum_{\mu_{a 1, b 2}}\left|\varepsilon^{a}\left(\mu_{a 1, b 2}\right)\right|\left|\varepsilon^{b}\left(\mu_{a 1, b 2}\right)\right| \tag{60}
\end{equation*}
$$

We then sum over equation (60) for all possible combinations of the quantum numbers $l_{a 1}$ and $l_{b 2}$ to obtain the following:


Figure 3: A comparison of models for $n=100, j=\frac{1}{2}, m_{a 1}=+\frac{1}{2}$, and $m_{b 2}=+\frac{1}{2}$, where $|\Delta|$ is the magnitude of the difference between equations (1) and (62). Left: all values of $l_{a 1}$ summed over. Right: $l_{a 1}= \pm \frac{49}{2}$.

$$
\begin{equation*}
\Upsilon\left(n, j=\frac{1}{2}, m_{a 1}, m_{b 2}, \theta_{a b}\right)=\sum_{l_{a 1}, l_{b 2}} v\left(n, j=\frac{1}{2}, m_{a 1}, m_{b 2}, l_{a 1}, l_{b 2}, \theta_{a b}\right) \tag{61}
\end{equation*}
$$

The probability of observing a particular value of $m_{b 2}$ is simply equation (61) normalized by the sum over all possible values of $m_{b 2}$ :

$$
\begin{equation*}
P\left(m_{b 2} \mid n, j=\frac{1}{2}, m_{a 1}, \theta_{a b}\right)=\frac{\Upsilon\left(n, j=\frac{1}{2}, m_{a 1}, m_{b 2}, \theta_{a b}\right)}{\sum_{m_{b 2}} \Upsilon\left(n, j=\frac{1}{2}, m_{a 1}, m_{b 2}, \theta_{a b}\right)} \tag{62}
\end{equation*}
$$

In Figure 3 (left), a comparison between equations (1) and (62) is offered for the case in which $n=100$, $j=\frac{1}{2}, m_{a 1}=+\frac{1}{2}$, and $m_{b 2}=+\frac{1}{2}$, for a full range of $\theta_{a b}$. Though there is a clear functional similarity between QM and the new model, differences on the order of five percent do exist for certain values of $\theta_{a b}$. The extent to which these two models agree can be improved somewhat by increasing $n$, though computational errors associated with large $n$ must also be accounted for. There are additional degrees of freedom we may exploit to further improve agreement. As an example, we may assume $l_{a 1}$ as an additional conditioning variable, rather than summing over all possible values. The value of $l_{a 1}$ can then be used as a tuning parameter, where $l_{a 1} \approx \pm \frac{1}{2}\left(\frac{n}{2}-j\right)$ leads to significant improvement in agreement (see Figure 3 (right)).

Instead of fine tuning $l_{a 1}$, there may be other ways to improve agreement between the model proposed here and QM. This is made possible by the first principles construction of the expression in equation (62). At each step in the development of that expression, certain choices were made that could be modified. For example, we could have chosen a different set of complete quantum numbers. Perhaps some alternative to $l_{a 1}$ and $l_{b 2}$ would produce a better result with a more obvious connection to something physical. Or, we could have defined $\theta_{a b}$ to be something other than a simple linear function of $\tilde{B}_{m a p}$. Modifications of this type are of interest for future work.

## 6 Spin 1 (arbitrary n)

The final issue we must address in the new model is interference, which is driven by the summing parameter $q$ in equation (1). The simplest physical scenario in which this becomes relevant is for a spin 1 particle, which will be the experiment of interest in this section. A visual depiction of this experiment is provided in Figure 4. As mentioned in the previous sections, $q$ is closely related to the quantum number $\mu_{a 1, b 2, ~}^{\text {, }}$ which was defined in equation (54). Recall that $\mu_{a 1, b 2}$ is unique among the other quantum numbers in that it is exclusively a base- 8 quantum number. That is, it is a property of the entire experiment, rather than individual events or maps. To help provide some intuition for this interesting quantity, we offer two examples


Figure 4: An event occurs at Alice's detector which deflects the $j=1$ particle into one of three paths (red), one for each possible value of the quantum number $m_{a 1} \in\{+1,0,-1\}$. Depending on which value of $m_{a 1}$ is of interest, Bob's detector is then placed along one of these paths. Bob then rotates his detector with respect to Alice's by the angle $\theta_{a b}$. Finally, an event occurs at Bob's detector which again deflects the $j=1$ particle into one of three paths (blue), one for each possible value of the quantum number $m_{b 2} \in\{+1,0,-1\}$.
of spin 1 ontic states which share the same seven quantum numbers $\left(n=6, j=1, m_{a 1}=0, m_{b 2}=0, l_{a 1}=0\right.$, $\left.l_{b 2}=1, \theta_{a b}=\frac{\pi}{2}\right)$, but differ in the value of $\mu_{a 1, b 2}$ :

$$
\left(\begin{array}{c}
A A  \tag{63}\\
B B \\
D C \\
C D \\
B A \\
A A
\end{array}\right)_{\mu_{a 1, b 2}=\frac{5}{2}}, \quad\left(\begin{array}{c}
A B \\
B A \\
D D \\
C C \\
B A \\
A A
\end{array}\right)_{\mu_{a 1, b 2}=\frac{1}{2}}
$$

Under the current procedure, when counting pairs of elements from the ontic state spaces $\varepsilon^{a}$ and $\varepsilon^{b}$, we require that each element share the same set of eight quantum numbers. We now lift this restriction, allowing for the possibility that these pairs differ in the value of $\mu_{a 1, b 2}$. The magnitude of this difference is then used to drive interference, where pairs with an odd value of $\left|\Delta \mu_{a 1, b 2}\right| / 2$ annihilate pairs with an even value. We apply this modification to the elementary counting expression defined in equation (60):

$$
\begin{equation*}
v\left(n, j, m_{a 1}, m_{b 2}, l_{a 1}, l_{b 2}, \theta_{a b}\right)=\sum_{\mu_{a 1, b 2}^{a}, \mu_{a 1, b 2}^{b}}(-1)^{\left|\Delta \mu_{a 1, b 2}\right| / 2}\left|\varepsilon^{a}\left(\mu_{a 1, b 2}^{a}\right)\right|\left|\varepsilon^{b}\left(\mu_{a 1, b 2}^{b}\right)\right| \tag{64}
\end{equation*}
$$

Equation (64) can then be summed over all possible combinations of $l_{a 1}$ and $l_{b 2}$ to obtain $\Upsilon$, as in equation (61). We can now write down the general expression for probabilities in the new model, which is the analog of equation (1) for arbitrary $j$ :

$$
\begin{equation*}
P\left(m_{b 2} \mid n, j, m_{a 1}, \theta_{a b}\right)=\frac{\Upsilon\left(n, j, m_{a 1}, m_{b 2}, \theta_{a b}\right)}{\sum_{m_{b 2}} \Upsilon\left(n, j, m_{a 1}, m_{b 2}, \theta_{a b}\right)} \tag{65}
\end{equation*}
$$

In Figure 5, the predictions of the new model and QM are compared for a full range of $\theta_{a b}$ for $j=1$, $m_{a 1}=+1$ (left), and $m_{a 1}=0$ (right). As discussed in section 5 , agreement can be improved by increasing $n$ and fine tuning $l_{a 1}$.


Figure 5: A comparison of models for $n=100, j=1, m_{a 1}=+1$ (left), and $m_{a 1}=0$ (right), where $|\Delta|$ is the magnitude of the difference between equations (1) and (65).

## 7 Testing the model

With just a few modifications, the model developed for sequences of SG detectors can also be used to model the behavior of photon number states through a beam splitter [27]. The photon number state entering the two input ports of a beam splitter are modeled using the base- 4 counts $\tilde{C}_{a 1}$ and $\tilde{D}_{a 1}$, while the photon number states exiting the two output ports are modeled using $\tilde{C}_{b 2}$ and $\tilde{D}_{b 2}$ (see Figure 6 (left)).

The effect of the beam splitter is then modeled using a map composed of $A$ and $B$ symbols only, where the following definition relates the number of $B$ 's within these maps to the transmittance $\left(\tau_{a b}\right)$ of the beam splitter:

$$
\begin{equation*}
\tau_{a b} \equiv \cos ^{2}\left(\frac{\tilde{B}_{m a p}}{n} \frac{\pi}{2}\right) \tag{66}
\end{equation*}
$$

The only remaining modifications necessary are the quantum numbers we defined in section 5 . Instead of using $j, m_{a 1}$, and $m_{b 2}$, we will use the following, where $\tilde{D}_{b 2}=\tilde{C}_{a 1}+\tilde{D}_{a 1}-\tilde{C}_{b 2}$ :

$$
\begin{align*}
\tilde{C}_{a 1} & =\widetilde{C C}+\widetilde{C D}  \tag{67}\\
\tilde{D}_{a 1} & =\widetilde{D D}+\widetilde{D C}  \tag{68}\\
\tilde{C}_{b 2} & =\widetilde{C C}+\widetilde{D C} \tag{69}
\end{align*}
$$

To calculate the probability of observing an output photon number state with quantum numbers $\tilde{C}_{b 2}$ and $\tilde{D}_{b 2}$, given $\tilde{C}_{a 1}, \tilde{D}_{a 1}$, and $\tau_{a b}$, along with a choice of sequence length $n$, we simply apply this change of variables to equation (65) to obtain the following:

$$
\begin{equation*}
P\left(\tilde{C}_{b 2}, \tilde{D}_{b 2} \mid n, \tilde{C}_{a 1}, \tilde{D}_{a 1}, \tau_{a b}\right)=\frac{\sum_{l a 1}, l_{b 2}}{} \Upsilon\left(n, \tilde{C}_{a 1}, \tilde{D}_{a 1}, \tilde{C}_{b 2}, l_{a 1}, l_{b 2}, \tau_{a b}\right) \tag{70}
\end{equation*}
$$

Through a superficial change of variables, we have successfully applied the model for sequences of SG detectors to a new physical system. This particular application is noteworthy because of the experimental advantages we gain by working with optical systems, rather than spin systems. This makes high precision tests of the proposed model more practical and cost effective [28]. In particular, this configuration avoids some experimental pitfalls, such as lossy beam splitters or non-ideal detectors.

In Figure 6 (right), we offer a comparison of equation (70) with QM for two different combinations of $\tilde{C}_{a 1}, \tilde{D}_{a 1}, \tilde{C}_{b 2}$, and $\tilde{D}_{b 2}$, as a function of $\tau_{a b}$, where $n=100$. To mitigate experimental challenges such as



Figure 6: (left) Photon number states modeled by the base-4 counts $\tilde{C}_{a 1}$ and $\tilde{D}_{a 1}$ enter the input ports of a beam splitter with transmittance $\tau_{a b}$. The photon number states exiting the output ports of the beam splitter, which are modeled by the base- 4 counts $\tilde{C}_{b 2}$ and $\tilde{D}_{b 2}$, then interact with a pair of detectors. (right) Blue: Probability of $\tilde{C}_{b 2}=2, \tilde{D}_{b 2}=0$ given $\tilde{C}_{a 1}=2, \tilde{D}_{a 1}=0$. Red: Probability of $\tilde{C}_{b 2}=1, \tilde{D}_{b 2}=1$ given $\tilde{C}_{a 1}=2, \tilde{D}_{a 1}=0$. In all cases, $n=100$ and $|\Delta|$ is the magnitude of the difference between the new model and QM. The green line identifies a value of $\tau_{a b}$ of experimental interest.
determining the beam splitter splitting ratio $\tau_{a b}$ for a wide variety of values, a comparative measurement on the same beam splitter would be preferable. For example, in Figure 6 (right) we used a solid green line to indicate a specific beam splitter value ( $\tau_{a b}=0.4$ ) where the new model agrees with QM for one output state (red line), but disagrees with QM for another (blue line). With this approach, we can use a specific input configuration ( $\left.\widetilde{C}_{a 1}=2, \widetilde{D}_{a 1}=0\right)$ and output configuration $\left(\widetilde{C}_{b 2}=2, \widetilde{D}_{b 2}=0\right)$ as a calibration for this specific beam splitter ratio. For the same input configuration ( $\left.\widetilde{C}_{a 1}=2, \widetilde{D}_{a 1}=0\right)$, but different output configuration ( $\widetilde{C}_{b 2}=1, \widetilde{D}_{b 2}=1$ ), we would be able to identify any experimental discrepancies for the specific beam splitter ratio. Using this comparative approach would practically eliminate any experimentally hard to determine parameters.

As with equation (65), we can improve agreement between the new model and QM by increasing $n$ and fine tuning the additional conditioning variable $l_{a 1}$. Thus, experiments like the one we have suggested provide an opportunity to place constraints on these parameters. However, small deviations from QM are unavoidable, especially for finite $n$. This arises due to the definition of $\tau_{a b}$ (or $\theta_{a b}$ ) within the proposed model. In QM, this quantity is assumed to be an element of the real number line. In the new model, it is an element of the rational number line when $n$ is finite. This granularity implies that for $\theta_{a b}\left(\tau_{a b}\right)$ sufficiently close to $\pi(0)$ or $0(1)$, certain combinations of $m_{a 1}$ and $m_{b 2}\left(\tilde{C}_{a 1}, \tilde{D}_{a 1}, \tilde{C}_{b 2}, \tilde{D}_{b 2}\right)$ will not be possible in the new model, but are possible in QM. The following is a simple example, where $n=6, j=1, m_{a 1}=+1$ and $\theta_{a b}=\frac{\pi}{6}$ :

$$
\begin{align*}
& \left(\begin{array}{l}
A \\
C \\
B \\
C \\
B \\
A
\end{array}\right)_{a 1} \oplus\left(\begin{array}{l}
A \\
A \\
A \\
B \\
A \\
A
\end{array}\right)_{m a p}=\left(\begin{array}{l}
A \\
C \\
B \\
D \\
B \\
A
\end{array}\right)_{b 2} \rightarrow\left(m_{b 2}=0\right)  \tag{71}\\
& \left(\begin{array}{l}
A \\
C \\
B \\
C \\
B \\
A
\end{array}\right)_{a 1}\left(\begin{array}{c}
A \\
A \\
A \\
A \\
B \\
A
\end{array}\right)_{\operatorname{map}}=\left(\begin{array}{l}
A \\
C \\
B \\
C \\
A \\
A
\end{array}\right)_{b 2} \rightarrow\left(m_{b 2}=+1\right) \tag{72}
\end{align*}
$$

The missing spin state in equations (71) and (72) is $m_{b 2}=-1$. Because there is only one $B$ in the map
relating Alice's and Bob's event, there is no way to generate this state. So, the probability of observing $m_{b 2}=-1$ is zero according to the new model, where as it is $\approx 0.5 \%$ in QM . The magnitude of this signal will decrease with increasing $n$, however. If we continue with $\tilde{B}_{\text {map }}=1$, but increase $n$ to 60 , the probability predicted by QM becomes $\approx 0.00005 \%$. Though this signal becomes increasingly difficult to detect for large $n$, the fact that the null condition indicates discovery may provide an experimental advantage.

## 8 Summary

In sections 3-6 we presented a pedagogical development of the proposed model for sequences of two SG detectors. In this section, we summarize this development while making an effort to highlight the underlying formalism, which has applications beyond this specific model. We begin with the mathematical building blocks of the formalism, which are the members of the finite group $Z_{2}$. Beginning with such simple mathematical building blocks forces one to reassess their assumptions regarding the most fundamental physical building blocks. In the proposed model, the physical building block is the event, where an event may also be thought of as an interaction or measurement. Unlike QM and QFT, which take particles or fields to be the essential physical object, we attempt to encode all relevant physics into events and networks of events. Our motivation for choosing the event to be our physical building block is rooted in epistemology. After all, it is events alone that inform our empirical description of the universe.

During an experiment involving two SG detectors, an event occurs at each detector. To these detectors we assign observers named Alice and Bob, as depicted in Figure 1. We assume that the non-determinism observed in this experiment is a consequence of hidden information. This implies that an underlying ontic state space exists for this experiment, which we attempt to model directly.

The first step in the development of this formalism is the introduction of the set $S(n)$, which is the set of all $n^{t h}$ order direct products of the finite group elements of $Z_{2}=\{0,1\}$, otherwise known as base- 2 sequences. We model the ontic state of an event with an ordered pair of base-2 sequences, as seen in equation (30). Thus, the full ontic state space for events is the Cartesian product of two copies of $S(n)$, which is the set of all base-4 sequences:

$$
\begin{equation*}
S^{2}(n)=S(n) \otimes S(n) \tag{73}
\end{equation*}
$$

We use subscripts to distinguish the base- 2 constituents of each base- 4 sequence, one of which we associate with the observer of the event being modeled. In general, we refer to this base- 2 sequence as the reference sequence. The elements comprising base- 4 sequences are members of the Klein four-group $Z_{2} \times Z_{2}$, which are denoted using the symbols $A, B, C$, and $D$, as defined in equation (29). Finally, the full ontic state space for an experiment is the Cartesian product of two copies of $S^{2}(n)$, which is the set of all base- 16 sequences composed of the symbols given in equation (43):

$$
\begin{equation*}
S^{4}(n)=S^{2}(n) \otimes S^{2}(n) \tag{74}
\end{equation*}
$$

Within this formalism, a count is defined as the number of times a symbol appears in a sequence. Formally, counts are measures on individual sequences which enable a partial ordering on sets of sequences. This partial ordering leads directly to non-determinism within the proposed model, due to the one-to-many mapping between counts and sequences. Counts, which form the basis of the quantum numbers defined in sections 2-5, may be thought of as discrete variables which are greater than or equal to 0 and always sum to $n$.

In the proposed model, the information observers are allowed to have about an experiment is limited to counts associated with maps and events. This epistemic constraint is analogous to others employed in hidden information theories, such as the "Knowledge Balance Principle" of the Spekkens toy model [10]. Naively, the probabilities associated with a particular experimental outcome should then be related to the number of ontic states (sequences) associated with that outcome. In that case, modeling physics would require one to encode the relevant degrees of freedom into counts and then calculate the cardinality of the associated partitions of $S^{4}(n)$. While these are indeed necessary skills, nature apparently had something slightly more interesting in mind. Rather than constructing a single ontic state space, we must build one for each observer.

The physical system under study here requires two events, or measurements, to fully characterize the outcome. We may think of these two measurements as being performed by two separate observers, or
the same observer at two separate times. Either way, a full characterization of the system under study requires two sources of information. For this reason, the mathematical structure of interest when calculating probabilities is the product of Alice's and Bob's ontic state spaces. We should note that this procedure is closely connected to the Born rule in QM, in which one multiplies amplitudes by their complex conjugates to obtain a probability. This connection provides an opportunity to establish a correspondence between various elements of QM and the model presented here. Though, such a correspondence is beyond the scope of this paper.

As a consequence of hiding information from observers, their knowledge regarding the ontic state of an experiment is limited to a subset of $S^{4}(n)$, as opposed to a single element of this set. We denote these subsets as $E^{a}$ and $E^{b}$, where the superscript indicates which observer each is associated with. For each unique choice of 16 counts, or 8 in the case of the model presented here, these subsets can be partitioned further into the elementary ontic state spaces $\varepsilon^{a}$ and $\varepsilon^{b}$. The only difference between these is the base- 4 sequence which is held fixed. In $\varepsilon^{a}$ it is the base- 4 sequence associated with Alice's event, while in $\varepsilon^{b}$ it is the base- 4 sequence associated with Bob's event. Fixed base- 4 sequences imply that each event is, or will be, a definite state of reality. To be clear, holding events fixed is not equivalent to knowing the ordering of symbols. This distinction manifests as a permutation symmetry for $\varepsilon^{a}$ and $\varepsilon^{b}$, as briefly discussed in section 5 . Due to this permutation symmetry, the information contained in these ontic state spaces is actually limited to the cardinalities $\left|\varepsilon^{a}\right|$ and $\left|\varepsilon^{b}\right|$.

When calculating probabilities, the mathematical object of interest is a modified Cartesian product of $E^{a}$ and $E^{b}\left(E^{a} \bar{\otimes} E^{b}\right)$. This modification requires that all ordered pairs of ontic states share the same set of "local" quantum numbers, with the only exception being "non-local" quantum numbers, such as $\mu_{a 1, b 2}$ defined in equation (54). These "non-local" quantum numbers may then be used to drive interference, as discussed in section 6. For the physical model of interest in this paper, there is only one such quantum number, where ordered pairs of ontic states with odd values of $\left|\Delta \mu_{a 1, b 2}\right|$ annihilate those with even values. Admittedly, a clear first principles justification for this feature of the calculation is lacking, though further study of this model may clarify the associated conceptual picture. Most importantly, the outcome of this calculation is a prediction which can be made to closely match that of QM, as discussed in sections 5 and 6 .

Perhaps the most notable feature of the formalism highlighted here is its freedom from any assumptions regarding the nature of space and time. Even $\theta_{a b}$, which is a spatial degree of freedom, is an emergent property of the Cartesian product of two copies of $S^{2}(n)$. This implies that models for other spacetime degrees of freedom may be within reach. The development of such models would likely benefit from the clear picture of what it means to be "quantum" within the context of this new formalism. It is also noteworthy that the discrete nature of this formalism automatically protects against the divergences and singularities which plague modern theories.

## 9 Discussion

Though differences remain, the similarity between the predictions generated by the model presented here and QM brings some credence to the proposed conceptual picture, the central figures of which are observers, events, and ontic state spaces. We argue that many of the strangest features of QM , such as non-determinism, non-locality, and contextuality, become more clear in this new view. While a detailed treatment of each of these is beyond the scope of this paper, we offer here a short comment on each.

Non-determinism is an empirical property of nature. Yet, the physical origins of this property remain unclear. A great deal of debate on this and related issues have taken place within the context of QM. In particular, there is the question of whether or not the wavefunction is itself ontic, or if it results from incomplete information [29]. Because a one-to-one correspondence between the wavefunction in QM and the Cartesian product of the ontic state spaces $E^{a}$ and $E^{b}$ discussed in section 8 is lacking, framing the new model within the context of this debate is challenging. With that being said, there is no question that probabilities, as defined in section 6, are epistemic within this model. That is, they arise due to observers' inability to resolve certain details about the ontic state of the physical system under study. For the time being, we remain agnostic with respect to the physical interpretation of this obscurement, though we do argue that the underlying ontic state indeed exists. However, we readily acknowledge the possibility of alternative interpretations of this "ontic" sub-structure of the quantum state. Further study of the model is
necessary to better understand this point.
A wide variety of epistemic models are possible, but three critical features distinguish the one presented here. First, the quantum numbers associated with events are functions of the observer's reference sequence, making them observer dependent (equation (30)). This feature alone negates the applicability of the PBR no-go theorem [30]. It also implies that observables are inherently relational [11]. The second feature of note is the disjointness of quantum states within this model. That is, no single ontic state can ever be associated with more than one unique combination of quantum numbers. This feature distinguishes this model from the well known toy model by Spekkens, for example [10].

The third and final distinguishing feature of this epistemic model is the nature of the hidden information, which is directly related to the issues of non-locality and contextuality. As discussed in section 8, the hidden information is stored in the configuration of symbols within base- 4 sequences, which are in turn used to model events. Experiments are then defined as ordered pairs of events, which require the introduction of "nonlocal" quantum numbers such as $\mu_{a 1, b 2}$. Unlike most hidden variable models, which are broadly excluded by Bell's theorem and the Kochen-Speker theorem [31-33], this model is not an attempt to reformulate QM in classical terms. Rather, it is a first principles construction which is anything but classical. Admittedly, we have not yet presented a model for the Bell test, which is the exemplar of non-locality in modern physics. To model a Bell test, we must combine the results presented in this paper with the those presented in [12]. While this development is certainly necessary to further validate our methodology, the conceptual issues surrounding non-locality and contextuality are essentially resolved by the event centric picture.

Beyond the narrow set of issues we have briefly discussed, there remain many open questions about the proposed model and the underlying formalism. These questions can only be addressed through continued model development and testing. Of particular interest is a model for a Bell test, which holds the promise of revealing a new perspective on the all important property of entanglement. Extending the model developed in this paper to include a third SG detector is also of interest, which would enable us to explore the issues of non-commutativity, Heisenberg's Uncertainty Principle, and even the possibility of modeling additional spatial degrees of freedom. Should the methodology employed here continue to successfully produce models for important physical systems, the implications will be quite significant. This would point to the possible existence of an ontic substructure which underpins the quantum state. Such a state of affairs would likely open new directions of inquiry within a variety of fields.

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