Rational Choice

The Dutch Book Theorem

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World Cup Win

Let event Q be that Qatar wins its first game in the 2022 World Cup, while event O is that Qatar loses or draws its first game in the 2022 World Cup. Notice that $O = \bar{Q}$.

Now suppose that right now you believe that $P(Q) = 0.2$ and $P(O) = P(\bar{Q}) = 0.9$.

Question: Is there anything wrong with this?

World Cup Win

Question: If probability is subjective, as Bayesians claim, what is the problem with claiming that $P(Q) = 0.2$ and $P(O) = 0.9$?

Answer: $P(O) = P(\bar{Q})$, and so $P(Q) + P(\bar{Q}) = 1.1$, violating theorem 6.1, which states that mathematical probabilities must obey $P(Q) + P(\bar{Q}) = 1$.

Question: So? Why do I need mathematical probabilities obeying the Kolmogorov axioms?

World Cup Win

Given that P(Q) = 0.2, Bayesian calibration says that you must believe that the following bet is fair*:

	Q	$\bar{Q}$
Bet 0.2 on Q	0.8	− 0.2

*Recall that a bet is fair when betting ~ abstaining.

Sidebar: The Math

$$v(\text{bet } 0.2 \text{ on } Q) = [P(Q) \times 0.8] + [P(\bar{Q}) \times (-0.2)]$$

$$= [0.2 \times 0.8] + [0.8 \times (-0.2)] = 0.16 - 0.16 = 0, \text{ and}$$

$$v(\text{abstaining}) = 0.$$

Therefore betting 0.2 on Q ~ abstaining, and therefore this bet is fair.

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Given that P(O) = 0.9, Bayesian calibration says that you must believe that the following bet is also fair:

	O	$\bar{O}$
Bet 0.9 on O	0.1	− 0.9

Sidebar: The Math

$$v(\text{bet } 0.9 \text{ on O}) = [P(O) \times 0.1] + [P(\bar{O}) \times (-0.9)]$$

$$= [0.9 \times 0.1] + [0.1 \times (-0.9)] = 0.09 - 0.09 = 0, \text{ and}$$

$$v(\text{abstaining}) = 0.$$

Therefore betting 0.9 on O ~ abstaining, and therefore this bet is fair.

World Cup Win

Since you believe that each bet individually is fair, you should believe that both bets taken together are also fair. Now notice that there only two possible states of affairs that may occur in this situation:

ω_1 = Q occurs (and so $\bar{O}$ must occur), or

ω_2 = O occurs (and so $\bar{Q}$ must occur).

Let's see what happens if you take the fair bet on Q and the fair bet on O.

World Cup Win

If ω_1 holds (so both Q and $\bar{O}$ occur), these are the resulting payoffs:

	Q	$\bar{Q}$
Bet 0.2 on Q	0.8	−0.2

	O	$\bar{O}$
Bet 0.9 on O	0.1	−0.9

So you *win* the bet on Q (and win 0.8 utiles), but you *lose* the bet on O (and lose 0.9 utiles). This is a net loss of 0.1 utiles.

World Cup Win

However, if ω_2 holds (so both $\bar{Q}$ and O occur), these are the resulting payoffs:

	Q	$\bar{Q}$
Bet 0.2 on Q	0.8	−0.2

	O	$\bar{O}$
Bet 0.9 on O	0.1	−0.9

So you *win* the bet on O (and win 0.1 utiles), but you *lose* the bet on Q (and lose 0.2 utiles). This is a net loss of 0.1 utiles.

World Cup Win

The problem here is that, based on your probability distribution, you should believe that taking both bets is fair—but taking both bets makes you a sure loser no matter what happens. There are only those two possible states of affairs (ω_1 and ω_2) and no others.

Therefore, abstaining strictly dominates taking both bets. But then accepting these bets cannot be fair, implying a contradiction. Something has gone seriously wrong.

Dutch Books

A **Dutch book** is a combination of bets, each of which is thought fair, that leads to certain loss. One is better off abstaining.

In economics and finance, this is known as achieving **arbitrage**, where there is a price difference between two or more markets that allow a person to be the "Dutch bookie" and make a sure profit with zero risk.

The Dutch Book Theorem

Frank Ramsey, Bruno de Finetti, and others have proven that subjective probability distributions are not mathematical probabilities (i.e., they do not satisfy Kolmogorov's three axioms) if and only if a Dutch book can be constructed using them. This result is known as the **Dutch book theorem**.

I will not prove this directly (see Box 7.2 in the text), but I will give you some examples of Dutch books which provide a flavor for how this theorem works.

Framework for Dutch Booking

A bet on/against event E is specified by its payoffs:

	E	$\bar{E}$
Bet x on E	$1 - x$	$-x$
Bet x against E	$-(1 - x)$	x
Abstain	0	0

Framework for Dutch Booking

For any E and x, exactly *one* of these must hold:

1. The bet on E is **fair**:

   Bet x on E ~ abstain ~ bet x against E.

2. The bet on E is **favorable**:

   Bet x on E > abstain > bet x against E.

3. The bet on E is **unfavorable**:

   Bet x against E > abstain > bet x on E.

Framework for Dutch Booking

Each event E carries a unique **fair odds** x_E.
That is, for any event E, there is a single number x_E such that betting x_E on E is fair.
This simply means that $P(E) = x_E$.

Constructing Dutch Books

The key to constructing a Dutch book is to recognize which axiom of mathematical probabilities the subjective probability distribution violates. Then use that to set up some bets that lead to a sure loss no matter what state actually occurs. For more complex cases, you should try to arrange the bets so that at least one bet is always a loser and that this loss is sufficient to wipe out any gains.

Violating Axiom K1

Suppose that I hold that P(E) = 1.2. My unique fair odds for E must then be 1.2, making x_E = 1.2. I then must believe that it is fair to bet on E as follows:

	E	$\bar{E}$
Bet 1.2 on E	– 0.2	– 1.2

But no matter what happens this bet is a sure loser!

Sidebar: The Math

Suppose that E happens (and so $\bar{E}$ does not happen). You "win" the bet on E, and so lose 0.2 utiles.

Now suppose that $\bar{E}$ happens (and so E does not happen). You lose the bet on E, and so lose 1.2 utiles.

There are no other possible states of affairs to consider, and so you are a sure loser, even though you should think this bet is fair!

Violating Axiom K2

Suppose that I hold that P(S) = 0.9, where S a sure thing, i.e., it is certain to happen. So $x_S = 0.9$. I then must believe that it is fair to bet *against* S as follows:

	S	$\overline{S}$
Bet 0.9 against S	– 0.1	0.9

But S cannot occur, so this bet is a sure loser!

Sidebar: The Math

Suppose that S happens (and so $\overline{S}$ does not happen). You lose your bet against S, and so lose 0.1 utiles.

Unfortunately, S is a sure thing, and so $\overline{S}$ can never happen. You can therefore never win betting against S.

There are no other possible states of affairs to consider, and so you are a sure loser, even though you should think this bet is fair!

Violating Axiom K3

Suppose that I hold that P(A) = 0.4, P(B) = 0.3, and P(A∪B) = 0.9, where A∩B = ∅. So $x_A = 0.4$, $x_B = 0.3$, and $x_{A \cup B} = 0.9$. I must then believe that each of the following bets is fair:

	A	$\overline{A}$
Bet 0.4 against A	−0.6	0.4

	B	$\overline{B}$
Bet 0.3 against B	−0.7	0.3

	A∪B	$\overline{(A \cup B)}$
Bet 0.9 on A∪B	0.1	−0.9

Violating Axiom K3

Here there are only three possible states of affairs:

ω_1 = A occurs (and so $\overline{B}$ and $A \cup B$ must occur),

ω_2 = B occurs (and so $\overline{A}$ and $A \cup B$ must occur), or

ω_3 = Neither A nor B occurs (and so $\overline{A}$, $\overline{B}$, and $\overline{A \cup B}$ must all occur).

Let's see what happens if you take all three "fair" bets listed on the previous slide.

Violating Axiom K3

If ω_1 holds (A, $\overline{B}$, and A∪B occur), these are the resulting payoffs:

	A	$\overline{A}$
Bet 0.4 against A	−0.6	0.4

	B	$\overline{\mathbf{B}}$
Bet 0.3 against B	−0.7	0.3

	A∪B	$\overline{(A \cup B)}$
Bet 0.9 on A∪B	0.1	−0.9

There is a net loss here because $-0.6 + 0.3 + 0.1 = -0.2$.

Violating Axiom K3

If ω_2 holds ($\overline{A}$, B, and A∪B occur), these are the resulting payoffs:

	A	$\overline{A}$
Bet 0.4 against A	−0.6	0.4

	B	$\overline{B}$
Bet 0.3 against B	−0.7	0.3

	A∪B	$\overline{(A \cup B)}$
Bet 0.9 on A∪B	0.1	−0.9

There is a net loss here because $0.4 - 0.7 + 0.1 = -0.2$.

Violating Axiom K3

If ω_3 holds ($\overline{A}$, $\overline{B}$, and $\overline{A \cup B}$ occur), these are the resulting payoffs:

	A	$\overline{A}$
Bet 0.4 against A	−0.6	0.4

	B	$\overline{B}$
Bet 0.3 against B	−0.7	0.3

	A∪B	$\overline{(A \cup B)}$
Bet 0.9 on A∪B	0.1	−0.9

There is a net loss here because $0.4 + 0.3 - 0.9 = -0.2$.

Violating Axiom K3

So while you think each of the three bets is fair, taking all of them leads to a sure loss no matter what.

	A	$\overline{A}$
Bet 0.4 against A	−0.6	0.4

	B	$\overline{B}$
Bet 0.3 against B	−0.7	0.3

	$A \cup B$	$\overline{(A \cup B)}$
Bet 0.9 on $A \cup B$	0.1	−0.9

The Choices

	Nothing in B_2	QR 3,000,000 in B_2
Take both B_1 and B_2	QR 3,000	QR 3,003,000
Take only B_2	QR 0	QR 3,000,000

Constructing Dutch Books

So keep in mind the pattern for creating Dutch books. The person states their probabilities for some events. This immediately gives you their fair odds for betting on/against those events. Use these fair odds to construct bets that lead to a sure loss. Sometimes you may need to play around before you get the right combination of bets.

Next Class…

We will look at the phenomenon of the "hot hand" in basketball, a discussion that reveals how bad people are at detecting patterns.