Rational Choice

Difficulties With Subjective Probability

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Bank Robbery

You and I are at the bank when there is a robbery. You say to me, “This guy is pathetic. I don’t think he knows how to use that ak-47. I could take him down before he could shoot and kill me.”

As a good Bayesian, I respond:
“How much you wanna bet?”

Betting Your Life

	E	$\bar{E}$
Bet on E	QR 100	QR 0
Bet against E	QR 0	QR 100

State of affairs E is the state where you are not killed;
$\bar{E}$ is the state where you are killed.

The Problem

Most people would prefer to bet on E than to bet against it. Why is that?

This preference means that, according to Bayesians, most people must believe that $P(E) > P(\bar{E})$. Do you think that most people really believe that?

State-Dependent Utilities

The problem here is one concerning **state-dependent utilities**. That is, the utility (or value) of a reward or outcome depends upon which state holds when it is given or when it occurs. In this example, QR 100 is worthless to you when you are dead!

Bayesians, like Leonard Savage, believe the problem is resolved by re-describing the rewards or outcomes so that their utilities are not state-dependent.

Betting Your Life

	E	$\bar{E}$
Bet on E	QR 100 plus fame and glory.	QR 0 and you are dead.
Bet against E	QR 0, but fame and glory.	QR 100 goes to your next of kin.

State of affairs E is the state where you are not killed; $\bar{E}$ is the state where you are killed.

The Deeper Problem

Mark Schervish, Teddy Seidenfeld, and Joseph Kadane (**Carnegie Mellon** alert!) show that the problem remains: in certain cases, it may be impossible to find rewards or outcomes whose utilities never change from state to state.

Rewards in Dollars

	ω_1	ω_2	ω_3
Lottery 1	$100	$0	$0
Lottery 2	$0	$100	$0
Lottery 3	$0	$0	$100

Rewards in Dollars

Suppose I am indifferent between these three lotteries and that my utility for money is linear. According to Bayesian theory, you can then infer that $P(\omega_1) = P(\omega_2) = P(\omega_3) = 1/3$.

Rewards in Yen

	ω_1	ω_2	ω_3
Lottery 4	¥100	¥0	¥0
Lottery 5	¥0	¥125	¥0
Lottery 6	¥0	¥0	¥150

Rewards in Yen

Suppose that I am also indifferent between these three lotteries and that my utility is linear in yen, as it was in dollars. According to Bayesianism, you can then infer that $P(\omega_1) = 1.25 \times P(\omega_2) = 1.5 \times P(\omega_3)$. And so $P(\omega_1) = {}^{15}\!/_{37}$, $P(\omega_2) = {}^{12}\!/_{37}$, and $P(\omega_3) = {}^{10}\!/_{37}$.

But the states are the same! You now think that I am committed to two different probability distributions over these same three states.

Sidebar: The Math

Given $P(\omega_1) = 1.25 \times P(\omega_2) = 1.5 \times P(\omega_3)$, solve for $P(\omega_1)$, $P(\omega_2)$, and $P(\omega_3)$.

We know that $P(\omega_1) + P(\omega_2) + P(\omega_3) = 1$. So $P(\omega_3) = 1 - P(\omega_1) - P(\omega_2)$. Therefore, $1.25 \times P(\omega_2) = 1.5 \times (1 - P(\omega_1) - P(\omega_2))$.

Since $P(\omega_1) = 1.25 \times P(\omega_2)$, we can substitute this for $P(\omega_1)$, to get that $1.25 \times P(\omega_2) = 1.5 \times (1 - (1.25 \times P(\omega_2)) - P(\omega_2))$. Solving for $P(\omega_2)$, we get that $P(\omega_2) = {}^{12}/_{37}$.

Since $P(\omega_1) = 1.25 \times P(\omega_2)$ and $P(\omega_2) = {}^{12}/_{37}$, we can substitute this for $P(\omega_2)$, to get that $P(\omega_1) = 1.25 \times {}^{12}/_{37}$. Solving for $P(\omega_1)$, we get that $P(\omega_1) = {}^{15}/_{37}$.

Finally, since $P(\omega_3) = 1 - P(\omega_1) - P(\omega_2)$, $P(\omega_1) = {}^{15}/_{37}$, and $P(\omega_2) = {}^{12}/_{37}$, we can solve for $P(\omega_3)$ to get that $P(\omega_3) = {}^{10}/_{37}$.

Therefore, $P(\omega_1) = {}^{15}/_{37}$, $P(\omega_2) = {}^{12}/_{37}$, and $P(\omega_3) = {}^{10}/_{37}$.

Exchange Rates

The problem is that I may be perfectly coherent. For instance, the three states could represent three different exchange rates between dollars and yen:

$$\omega_1 = \{\$100 \text{ is worth } ¥100\},$$

$$\omega_2 = \{\$100 \text{ is worth } ¥125\}, \text{ and}$$

$$\omega_3 = \{\$100 \text{ is worth } ¥150\}.$$

Supposing I can exchange different currencies without penalty, then am indifferent between all the lotteries.

Problem with Exchanges

The two probability distributions were constructed under conflicting assumptions. The first distribution was derived under the assumption that $u(\$100)$ is the same in all three states. The second distribution was derived under the assumption that $u(¥100)$ is the same in all three states. Given the definitions of these three states, both assumptions cannot be true.

State-independent utilities may hold, but it is unclear under which description of the rewards or outcomes.

Problem for Elicitation

This means problems for a Bayesian trying to determine my probabilities. Suppose I use state-independent utility for dollars and I have the first probability distribution. But then you, as the Bayesian, offer me the lotteries in yen. Thus you are going to "discover" that I hold the second distribution, which is false, even though I am perfectly coherent and satisfy all of Savage's axioms!

State-Dependent Utilities

So state-dependent utilities cause a problem for Bayesians: what may appear to be a constant prize or outcome may not actually have the same value to an agent in all the states of nature.

This is a huge problem for theories of probability and statistics because they want to focus solely on probabilities and deal not with utilities. This concern is, however, that probabilities and utilities may be intimately linked in unanticipated ways.

The Wise Predictor

Suppose that Professor G is very good at predicting other people's choices (he is a rational choice professor, after all). You believe that 99% of all his predictions come out correctly. Now he offers you a choice concerning boxes B_1 and B_2. Box B_1 for sure contains QR 3,000. Box B_2 either contains nothing or QR 3,000,000 (you don't know which). Professor G says you have the choice between (choice 1) taking both B_1 and B_2 or (choice 2) taking only B_2.

But here's the trick: Professor G has *already* made a prediction about what you will choose, and if he has predicted you will pick both boxes (choice 1), he has put nothing in B_2. But if he has predicted you will pick only B_2 (choice 2), he has put QR 3,000,000 in it.

You may now make your choice and claim your rewards!

The Choices

	Nothing in B_2	QR 3,000,000 in B_2
Take both B_1 and B_2	QR 3,000	QR 3,003,000
Take only B_2	QR 0	QR 3,000,000

Making the Choice

One way to reason is by **weak Pareto** (or weak dominance). Under each state of affairs, taken separately, it is always better to take both the boxes. So you should take both boxes (choice 1).

Making the Choice

But this ignores the fact that Professor G is a great predictor of decisions. You might want to take that into account. If you take both boxes, he has predicted it with 99% accuracy, so it is 99% likely you get only QR 3,000. Whereas if you take only B_2, it is 99% likely you get QR 3,000,000. It is therefore pretty clear that the **principle of maximizing expected utility** says to take only the second box (choice 2).

Sidebar: The Math

$$v(\text{Choice 1}) = 0.99 \times u(\text{QR } 3{,}000) + 0.01 \times u(\text{QR } 3{,}003{,}000)$$

$$= 3{,}3000 \text{ (assuming } u(\text{QR } x) = x\text{), and}$$

$$v(\text{Choice 2}) = 0.01 \times u(\text{QR } 0) + 0.99 \times u(\text{QR } 3{,}000{,}000)$$

$$= 2{,}970{,}000 \text{ (assuming } u(\text{QR } x) = x\text{).}$$

Therefore, $v(\text{Choice 2}) > v(\text{Choice 1})$.

The Problem

In this case, two principles (usually in unison) have come apart. Weak Pareto and the principle of maximizing expected utility make alternative recommendations. What went wrong here? Which one should you go with?

Act/State Dependence

The root of this conflict lies in the fact that there is **act/state dependence** going on here: your actions influence the probability of which state of affair holds. That is, if you pick both boxes, this makes the state of affairs where there is no money in B_2 more likely; and if you pick only B_2, this makes the state of affairs where there is no money in B_2 less likely.

Act/State Dependence

Act/state dependence also cause a problem for Bayesians: if you believe that your actions influence the probabilities for the states, then you may make decisions that the Bayesians think are irrational (e.g., by violating dominance).

Next Class…

We will look at the Dutch book theorem, which Bayesians use to justify the fact that even though probability is subjective, this does not mean that a person should violate *coherence*, i.e., by not satisfying all three of the Kolmogorov axioms for probability.