Rational Choice

Paradoxes for the Principle of Expected Utility

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Choice 1

An urn contains 35 blue balls, and 65 other balls, which may be either red ball or green balls. One ball will be drawn from the urn.

	Blue	Red	Green
Ticket 1	QR 300	QR 0	QR 0
Ticket 2	QR 0	QR 300	QR 0

Choice 2

An urn contains 35 blue balls, and 65 other balls, which may be either red ball or green balls. One ball will be drawn from the urn.

	Blue	Red	Green
Ticket 3	QR 0	QR 300	QR 300
Ticket 4	QR 300	QR 0	QR 300

Choice 3

An urn contains 1 blue ball, 10 red balls, and 89 green balls. One ball will be drawn from the urn.

	Blue	Red	Green
Ticket 5	QR 3M	QR 3M	QR 3M
Ticket 6	QR 0	QR 15M	QR 3M

Choice 4

An urn contains 1 blue ball, 10 red balls, and 89 green balls. One ball will be drawn from the urn.

	Blue	Red	Green
Ticket 7	QR 0M	QR 15M	QR 0
Ticket 8	QR 3M	QR 3M	QR 0

Choice 5

A fair coin is flipped until it shows heads. n is the total number of flips. How much would you pay to buy the following ticket?

	$n = 1$	$n = 2$	…	$n = 10$	…	n
Ticket 9	QR 2	QR 4		QR 1024		QR 2^n

Choice 1

An urn contains 35 blue balls, and 65 other balls, which may be either red ball or green balls. One ball will be drawn from the urn.

	Blue	Red	Green
Ticket 1	QR 300	QR 0	QR 0
Ticket 2	QR 0	QR 300	QR 0

Choice 2

An urn contains 35 blue balls, and 65 other balls, which may be either red ball or green balls. One ball will be drawn from the urn.

	Blue	Red	Green
Ticket 3	QR 0	QR 300	QR 300
Ticket 4	QR 300	QR 0	QR 300

Ellsberg's Paradox

The typical decisions made in choices 1 and 2 illustrate a problem noticed by Daniel Ellsberg. It is *impossible* for you to consistently maximize expected utility and also choose in these ways.

Ellsberg's Paradox

Let r be the number of red balls. So there are $65 - r$ green balls. Also assume that $u(\text{QR } 0) = 0$. Calculate the expected utility for each of the tickets from choices 1 and 2:

$$\begin{aligned} v(\text{T}_1) &= 0.35 \times u(\text{QR } 300) + {}^{r}\!/_{100} \times u(\text{QR } 0) + {}^{(65-r)}\!/_{100} \times u(\text{QR } 0) \\ &= 0.35 \times u(\text{QR } 300). \end{aligned}$$

$$\begin{aligned} v(\text{T}_2) &= 0.35 \times u(\text{QR } 0) + {}^{r}\!/_{100} \times u(\text{QR } 300) + {}^{(65-r)}\!/_{100} \times u(\text{QR } 0) \\ &= {}^{r}\!/_{100} \times u(\text{QR } 300). \end{aligned}$$

$$\begin{aligned} v(\text{T}_3) &= 0.35 \times u(\text{QR } 0) + {}^{r}\!/_{100} \times u(\text{QR } 300) + {}^{(65-r)}\!/_{100} \times u(\text{QR } 300) \\ &= {}^{r}\!/_{100} \times u(\text{QR } 300) + {}^{(65-r)}\!/_{100} \times u(\text{QR } 300). \end{aligned}$$

$$\begin{aligned} v(\text{T}_4) &= 0.35 \times u(\text{QR } 300) + {}^{r}\!/_{100} \times u(\text{QR } 0) + {}^{(65-r)}\!/_{100} \times u(\text{QR } 300) \\ &= 0.35 \times u(\text{QR } 300) + {}^{(65-r)}\!/_{100} \times u(\text{QR } 300). \end{aligned}$$

Ellsberg's Paradox

Since ticket 1 is typically thought better than ticket 2, expected utility claims that the following must hold:

$$v(T_1) > v(T_2), \text{ and so}$$

$$0.35 \times u(\text{QR } 300) > {}^{r}\!/_{100} \times u(\text{QR } 300).$$

Since ticket 3 is typically thought better than ticket 4, expected utility claims that the following must also hold:

$$v(T_3) > v(T_4), \text{ and so}$$

$$ {}^{r}\!/_{100} \times u(\text{QR } 300) + {}^{(65 - r)}\!/_{100} \times u(\text{QR } 300) > 0.35 \times u(\text{QR } 300) + {}^{(65 - r)}\!/_{100} \times u(\text{QR } 300), \text{ and so}$$

$$ {}^{r}\!/_{100} \times u(\text{QR } 300) > 0.35 \times u(\text{QR } 300).$$

Ellsberg's Paradox

Since ticket 1 is typically thought better than ticket 2, expected utility claims that the following must hold:

$v(T_1) > v(T_2)$, and so

$$0.35 \times u(\text{QR } 300) > {}^{r}/_{100} \times u(\text{QR } 300).$$

But these cannot both be true! It is a contradiction!!

Since ticket 3 is typically thought better than ticket 4, expected utility claims that the following must also hold:

$v(T_3) > v(T_4)$, and so

$${}^{r}/_{100} \times u(\text{QR } 300) + {}^{(65 - r)}/_{100} \times u(\text{QR } 300) > 0.35 \times u(\text{QR } 300) + {}^{(65 - r)}/_{100} \times u(\text{QR } 300), \text{ and so}$$

$${}^{r}/_{100} \times u(\text{QR } 300) > 0.35 \times u(\text{QR } 300).$$

Ellsberg's Paradox

So there is *no* possible value function that maximizes expected utility that can have you pick ticket 1 in choice 1 and pick ticket 3 in choice 2.

Choice 3

An urn contains 1 blue ball, 10 red balls, and 89 green balls. One ball will be drawn from the urn.

	Blue	Red	Green
Ticket 5	QR 3M	QR 3M	QR 3M
Ticket 6	QR 0	QR 15M	QR 3M

Choice 4

An urn contains 1 blue ball, 10 red balls, and 89 green balls. One ball will be drawn from the urn.

	Blue	Red	Green
Ticket 7	QR 0M	QR 15M	QR 0
Ticket 8	QR 3M	QR 3M	QR 0

Allais' Paradox

The standard decisions made in choices 3 and 4 illustrate a problem noticed by Maurice Allias. It is *impossible* for you to consistently maximize expected utility and also choose in these ways.

Allais' Paradox

Once again, assume that $u(\text{qr } 0) = 0$. Calculate the expected utility for each of the tickets from choices 3 and 4:

$$\begin{aligned} v(\text{T}_5) &= 0.01 \times u(\text{QR 3M}) + 0.10 \times u(\text{QR 3M}) + 0.89 \times u(\text{QR 3M}) \\ &= u(\text{QR 3M}). \end{aligned}$$

$$\begin{aligned} v(\text{T}_6) &= 0.01 \times u(\text{QR 0}) + 0.10 \times u(\text{QR 15M}) + 0.89 \times u(\text{QR 3M}) \\ &= 0.10 \times u(\text{QR 15M}) + 0.89 \times u(\text{QR 3M}). \end{aligned}$$

$$\begin{aligned} v(\text{T}_7) &= 0.01 \times u(\text{QR 0}) + 0.10 \times u(\text{QR 15M}) + 0.89 \times u(\text{QR 0}) \\ &= 0.10 \times u(\text{QR 15M}). \end{aligned}$$

$$\begin{aligned} v(\text{T}_8) &= 0.01 \times u(\text{QR 3M}) + 0.10 \times u(\text{QR 3M}) + 0.89 \times u(\text{QR 0}) \\ &= 0.11 \times u(\text{QR 3M}). \end{aligned}$$

Allais' Paradox

Since ticket 5 is typically thought better than ticket 6, expected utility claims that the following must hold:

$$v(T_5) > v(T_6), \text{ and so}$$

$$u(\text{QR } 3\text{M}) > 0.10 \times u(\text{QR } 15\text{M}) + 0.89 \times u(\text{QR } 3\text{M}), \text{ and so}$$

$$0.11 \times u(\text{QR } 3\text{M}) > 0.10 \times u(\text{QR } 15\text{M}).$$

Since ticket 7 is typically thought better than ticket 8, expected utility claims that the following must also hold:

$$v(T_7) > v(T_8), \text{ and so}$$

$$0.10 \times u(\text{QR } 15\text{M}) > 0.11 \times u(\text{QR } 3\text{M}).$$

Allais' Paradox

Since ticket 5 is typically thought better than ticket 6, expected utility claims that the following must hold:

$v(T_5) > v(T_6)$, and so

$u(\text{QR 3M}) > 0.10 \times u(\text{QR 15M}) + 0.89 \times u(\text{QR 3M})$, and so

$0.11 \times u(\text{QR 3M}) > 0.10 \times u(\text{QR 15M})$.

But these cannot both be true! It is a contradiction!!

Since ticket 7 is typically thought better than ticket 8, expected utility claims that the following must also hold:

$v(T_7) > v(T_8)$, and so

$0.10 \times u(\text{QR 15M}) > 0.11 \times u(\text{QR 3M})$.

Alias' Paradox

So there is *no* possible value function that maximizes expected utility that can have you pick ticket 5 in choice 3 and pick ticket 7 in choice 4.

Paradoxical Choices

What error are people making when they succumb to either paradox? That is, what axiom of Von Neumann-Morgenstern Theory are they violating?

Axiom 1: Ordering.

Axiom 2: Independence.

Axiom 3: Archimedean or Continuity Condition.

Ellsberg's Paradox

		Blue	Red	Green
Choice 1	Ticket 1	QR 300	QR 0	QR 0
	Ticket 2	QR 0	QR 300	QR 0
Choice 2	Ticket 4	QR 300	QR 0	QR 300
	Ticket 3	QR 0	QR 300	QR 300

Ellsberg's Paradox

		Blue	Red	Green
Choice 1	Ticket 1	QR 300	QR 0	[redacted]
	Ticket 2	QR 0	QR 300	[redacted]
Choice 2	Ticket 4	QR 300	QR 0	QR 300
	Ticket 3	QR 0	QR 300	QR 300

Ellsberg's Paradox

		Blue	Red	Green
Choice 1	Ticket 1	QR 300	QR 0	[redacted]
	Ticket 2	QR 0	QR 300	[redacted]
Choice 2	Ticket 4	QR 300	QR 0	[redacted]
	Ticket 3	QR 0	QR 300	[redacted]

Ellsberg's Paradox

		Blue	Red	[redacted]
Choice 1	Ticket 1	QR 300	QR 0	
	Ticket 2	QR 0	QR 300	
Choice 2	Ticket 4	QR 300	QR 0	
	Ticket 3	QR 0	QR 300	

Allais' Paradox

		Blue	Red	Green
Choice 3	Ticket 5	QR 3M	QR 3M	QR 3M
	Ticket 6	QR 0	QR 15M	QR 3M
Choice 4	Ticket 8	QR 3M	QR 3M	QR 0
	Ticket 7	QR 0	QR 15M	QR 0

Allais' Paradox

		Blue	Red	Green
Choice 3	Ticket 5	QR 3M	QR 3M	[redacted]
	Ticket 6	QR 0	QR 15M	[redacted]
Choice 4	Ticket 8	QR 3M	QR 3M	QR 0
	Ticket 7	QR 0	QR 15M	QR 0

Allais' Paradox

		Blue	Red	Green
Choice 3	Ticket 5	QR 3M	QR 3M	[redacted]
	Ticket 6	QR 0	QR 15M	[redacted]
Choice 4	Ticket 8	QR 3M	QR 3M	[redacted]
	Ticket 7	QR 0	QR 15M	[redacted]

Allais' Paradox

		Blue	Red	[redacted]
Choice 3	Ticket 5	QR 3M	QR 3M	
	Ticket 6	QR 0	QR 15M	
Choice 4	Ticket 8	QR 3M	QR 3M	
	Ticket 7	QR 0	QR 15M	

Paradoxical Choices

Why might people succumb to these paradoxes, sometimes even after they are told of them? What might be their reasoning? Keep in mind that the reasoning involved in the two paradoxes is probably different, given that their respective choices involve important differences.

Choice 5

A fair coin is flipped until it shows heads. n is the total number of flips. How much would you pay to buy the following ticket?

	$n = 1$	$n = 2$	…	$n = 10$	…	n
Ticket 9	QR 2	QR 4		QR 1024		QR 2^n

People tend to give it a value of around QR 15.

St. Petersburg Paradox

Daniel Bernoulli discovered (while working in St. Petersburg) that the expected monetary gain from this ticket in infinite. So you should be willing to pay *any* finite amount of money to play it, e.g., even QR 1,000,000!

St. Petersburg Paradox

In this case, assume that the utility for money is linear, i.e., $u(\text{QR } x) = x$.

$$
\begin{aligned}
v(T_9) &= P(n = 1) \times u(\text{QR } 2) + P(n = 2) \times u(\text{QR } 4) + P(n = 3) \times u(\text{QR } 8) + \ldots + \\
&\quad P(n = 10) \times u(\text{QR } 1024) + \ldots \\
&= (1/2) \times 2 + (1/4) \times 4 + (1/8) \times 8 + \ldots + (1/1024) \times 1024 + \ldots \\
&= 1 + 1 + 1 + \ldots + 1 + \ldots \\
&= \infty.
\end{aligned}
$$

Clearly when infinities are involved, strange things start to happen. (So is this a problem for Pascal's wager when he assigns infinite utility to heaven?)

St. Petersburg Paradox

This might be avoided by assuming that utility for money has decreasing marginal utility, e.g., $u(\text{QR } x) = \log(x + 1)$.

$$v(T_9) = P(n = 1) \times u(\text{QR } 2) + P(n = 2) \times u(\text{QR } 4) + P(n = 3) \times u(\text{QR } 8) + \ldots + P(n = 10) \times u(\text{QR } 1024) + \ldots$$
$$= (1/2) \times \log 3 + (1/4) \times \log 5 + (1/8) \times \log 9 + \ldots + (1/1024) \times 1025 + \ldots$$
$$= b, \text{ where } b \text{ is some finite number.}$$

The problem with this response is that we can simply reformulate ticket 1 so that the payoffs it gives are in terms of utility and not in terms of dollars. Then you should be willing to spend infinite *utility* to "purchase" the ticket.

Next Class

We will explore how an account of risk aversion might explain how people make decisions in Ellsberg- and Allais-type decision situations.