#### **Rational Choice**

Paradoxes for the Principle of Expected Utility

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An urn contains 35 blue balls, and 65 other balls, which may be either red ball or green balls. One ball will be drawn from the urn.

	Blue	Red	Green
Ticket 1	QR 300	QRo	QRo
Ticket 2	QRo	QR 300	QRo

An urn contains 35 blue balls, and 65 other balls, which may be either red ball or green balls. One ball will be drawn from the urn.

	Blue	Red	Green
Ticket 3	QRo	QR 300	QR 300
Ticket 4	QR 300	QRO	QR 300



An urn contains 1 blue ball, 10 red balls, and 89 green balls. One ball will be drawn from the urn.

	Blue	Red	Green
Ticket 5	QR 3M	QR 3M	QR 3M
Ticket 6	QRo	QR 15M	QR 3M



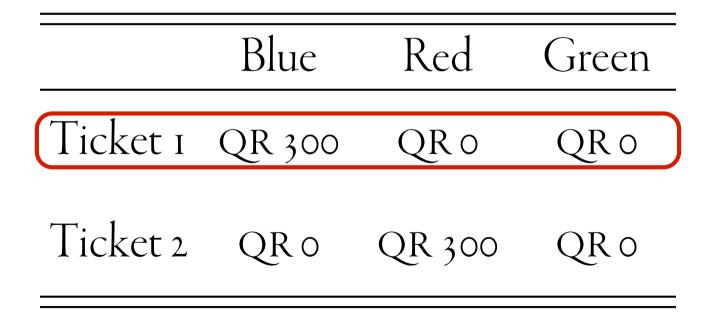
An urn contains 1 blue ball, 10 red balls, and 89 green balls. One ball will be drawn from the urn.

	Blue	Red	Green
Ticket 7	QRoM	QR 15M	QRo
Ticket 8	QR 3M	QR 3M	QRo

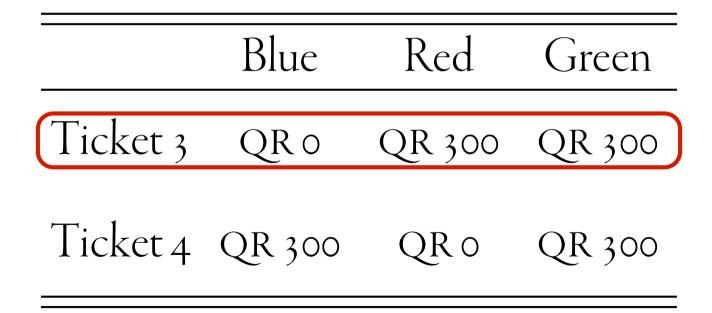
A fair coin is flipped until it shows heads. n is the total number of flips. How much would you pay to buy the following ticket?

	$\mathcal{N} = I$	$\mathcal{N}=2$	$\ldots \mathcal{N} = IO \ldots$	N
Ticket 9	QR 2	QR4	QR 1024	QR 2 <sup>n</sup>

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The typical decisions made in choices 1 and 2 illustrate a problem noticed by Daniel Ellsberg. It is *impossible* for you to consistently maximize expected utility and also choose in these ways.

Let *r* be the number of red balls. So there are 65 - r green balls. Also assume that u(qr o) = o. Calculate the expected utility for each of the tickets from choices 1 and 2:

$$v(T_{I}) = 0.35 \times u(QR 300) + r/_{100} \times u(QR 0) + \frac{(65 - r)}{100} \times u(QR 0)$$
  
= 0.35 × u(QR 300).

$$v(T_{2}) = 0.35 \times u(QR \circ) + r/_{100} \times u(QR 300) + \frac{(65 - r)}{100} \times u(QR \circ)$$
  
=  $r/_{100} \times u(QR 300).$ 

 $v(T_3) = 0.35 \times u(QR \circ) + \frac{r}{100} \times u(QR 300) + \frac{(65 - r)}{100} \times u(QR 300)$ =  $\frac{r}{100} \times u(QR 300) + \frac{(65 - r)}{100} \times u(QR 300).$ 

$$v(T_4) = 0.35 \times u(QR 300) + \frac{r}{100} \times u(QR 0) + \frac{(65 - r)}{100} \times u(QR 300)$$
  
= 0.35 × u(QR 300) +  $\frac{(65 - r)}{100} \times u(QR 300).$ 

Since ticket 1 is typically thought better than ticket 2, expected utility claims that the following must hold:

 $v(T_1) > v(T_2)$ , and so

 $0.35 \times u(QR 300) > r/_{100} \times u(QR 300).$ 

Since ticket 3 is typically thought better than ticket 4, expected utility claims that the following must also hold:

 $v(T_3) > v(T_4)$ , and so  $r_{100} \times u(QR 300) + \frac{(65 - r)}{100} \times u(QR 300) > 0.35 \times u(QR 300) + \frac{(65 - r)}{100} \times u(QR 300)$ , and so  $r_{100} \times u(QR 300) > 0.35 \times u(QR 300)$ .

Since ticket 1 is typically thought better than ticket 2, expected utility claims that the following must hold: But these cannot

 $v(T_1) > v(T_2)$ , and so  $0.35 \times u(QR 300) > r/_{100} \times u(QR 300)$ . both be true! It is a contradiction!!

Since ticket 3 is typically thought better than ticket 4, expected utility claims that the following must also hold:

 $v(T_3) > v(T_4)$ , and so

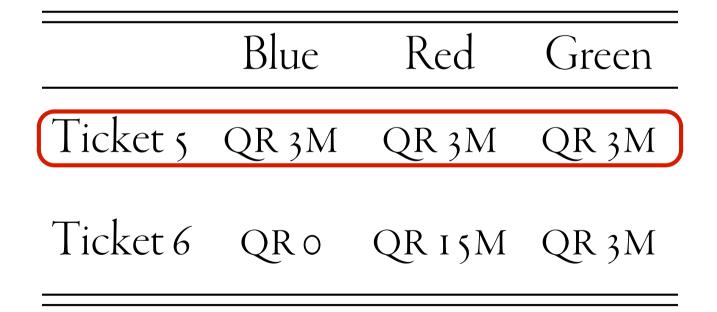
 $r_{100} \times u(QR 300) + \frac{(65 - r)}{100} \times u(QR 300) > 0.35 \times u(QR 300) + \frac{(65 - r)}{100} \times u(QR 300)$ , and so

 $r_{100} \times u(QR 300) > 0.35 \times u(QR 300).$ 

So there is *no* possible value function that maximizes expected utility that can have you pick ticket 1 in choice 1 and pick ticket 3 in choice 2.

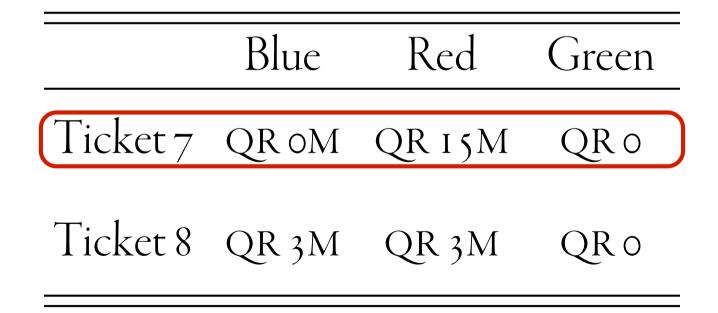


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The standard decisions made in choices 3 and 4 illustrate a problem noticed by Maurice Allias. It is *impossible* for you to consistently maximize expected utility and also choose in these ways.

Once again, assume that  $u(qr \circ) = o$ . Calculate the expected utility for each of the tickets from choices 3 and 4:

$$v(T_5) = 0.01 \times u(QR 3M) + 0.10 \times u(QR 3M) + 0.89 \times u(QR 3M)$$
  
=  $u(QR 3M)$ .

 $v(T_6) = 0.01 \times u(QR 0) + 0.10 \times u(QR 15M) + 0.89 \times u(QR 3M)$  $= 0.10 \times u(QR 15M) + 0.89 \times u(QR 3M).$ 

$$v(T_7) = 0.01 \times u(QR \circ) + 0.10 \times u(QR 15M) + 0.89 \times u(QR \circ)$$
  
= 0.10 × u(QR 15M).

$$v(T_8) = 0.01 \times u(QR 3M) + 0.10 \times u(QR 3M) + 0.89 \times u(QR 0)$$
  
= 0.11 ×  $u(QR 3M)$ .

Since ticket 5 is typically thought better than ticket 6, expected utility claims that the following must hold:

 $v(T_5) > v(T_6)$ , and so  $u(QR 3M) > 0.10 \times u(QR 15M) + 0.89 \times u(QR 3M)$ , and so  $0.11 \times u(QR 3M) > 0.10 \times u(QR 15M)$ .

Since ticket 7 is typically thought better than ticket 8, expected utility claims that the following must also hold:

 $v(T_7) > v(T_8)$ , and so

 $0.10 \times u(QR I 5M) > 0.11 \times u(QR 3M).$ 

Since ticket 5 is typically thought better than ticket 6, expected utility claims that the following must hold: But these cannot

 $v(T_5) > v(T_6)$ , and so

 $u(QR 3M) > 0.10 \times u(QR 15M) + 0.89 \times u(QR 3M)$ , and so

 $0.11 \times u(QR 3M) > 0.10 \times u(QR 15M).$ 

Since ticket 7 is typically thought better than ticket 8, expected utility claims that the following must also hold:

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v(T_7) > v(T_8), and so
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 $0.10 \times u(QR 15M) > 0.11 \times u(QR 3M).$ 

both be true! It is

a contradiction!!

## Alias' Paradox

So there is *no* possible value function that maximizes expected utility that can have you pick ticket 5 in choice 3 and pick ticket 7 in choice 4.

## Paradoxical Choices

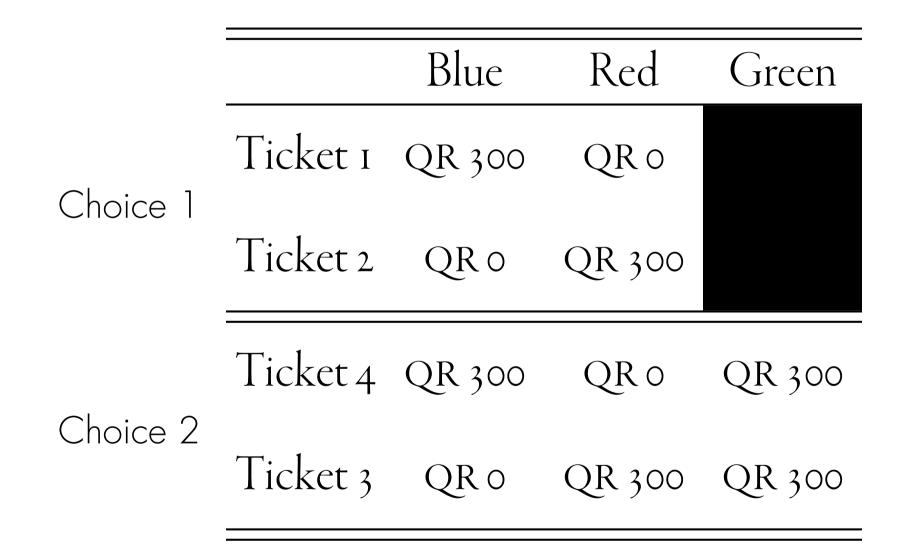
What error are people making when they succumb to either paradox? That is, what axiom of Von Neumann-Morgenstern Theory are they violating?

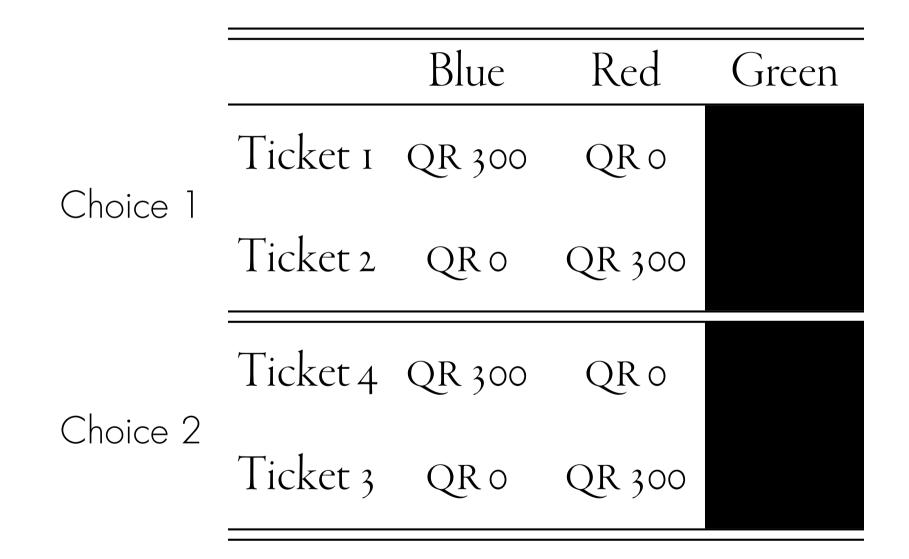
Axiom 1: Ordering.

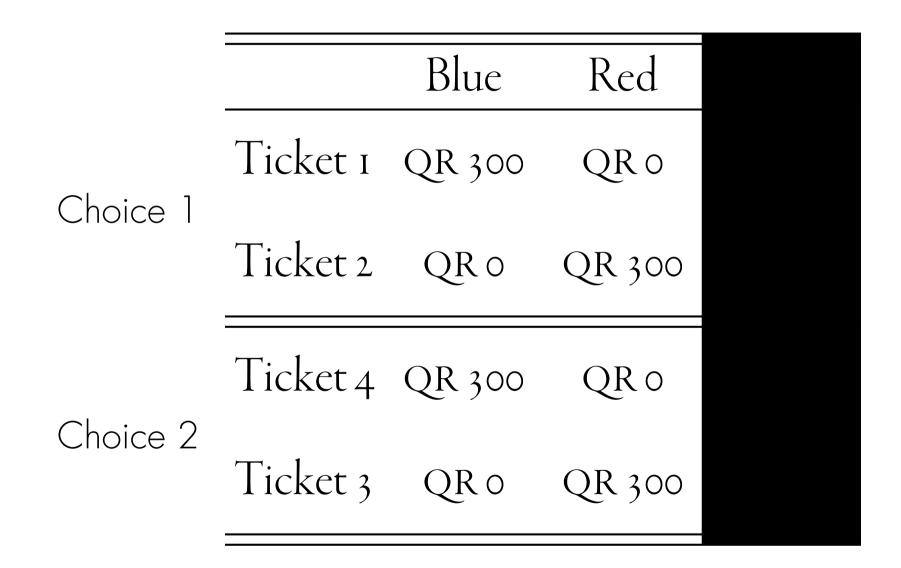
Axiom 2: Independence.

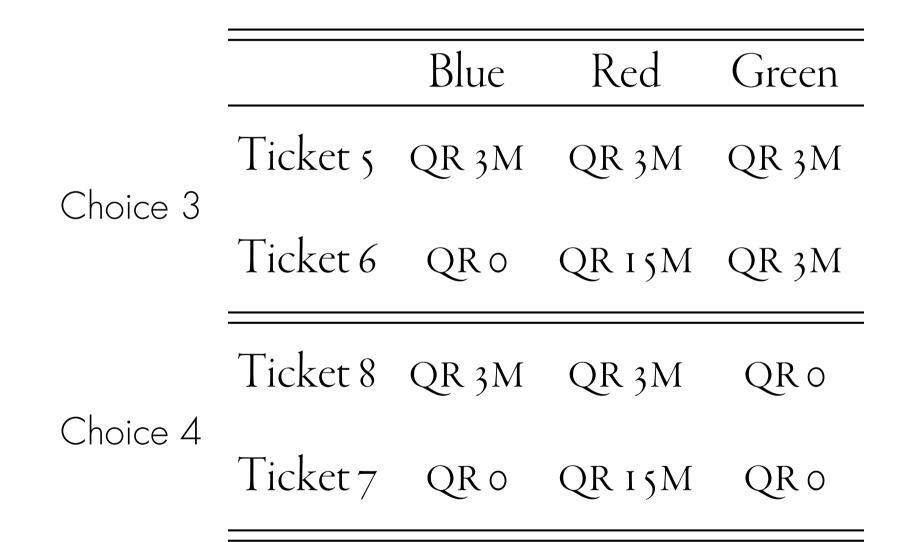
Axiom 3: Archimedean or Continuity Condition.

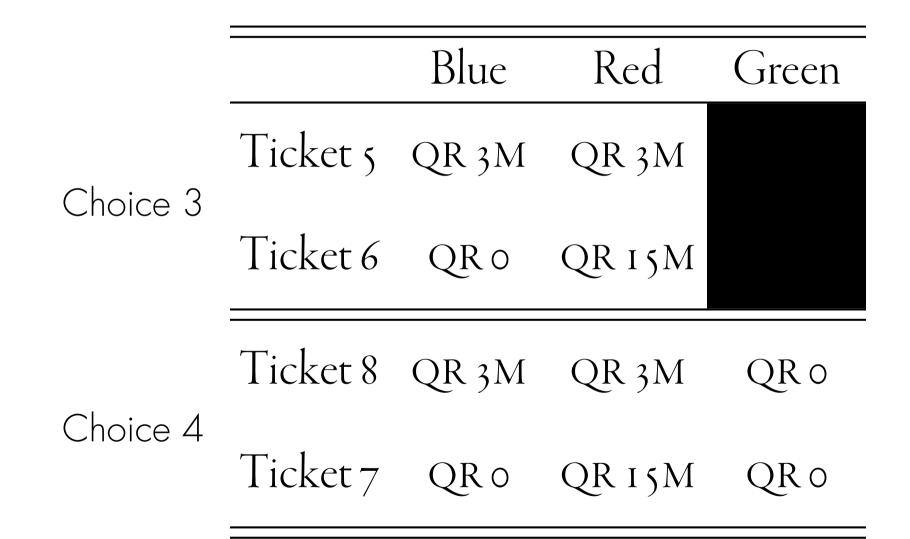
		Blue	Red	Green
Choice 1	Ticket 1	QR 300	QRO	QRo
	Ticket 2	QRo	QR 300	QRO
Choice 2	Ticket 4	QR 300	QRo	QR 300
	Ticket 3	QRo	QR 300	QR 300

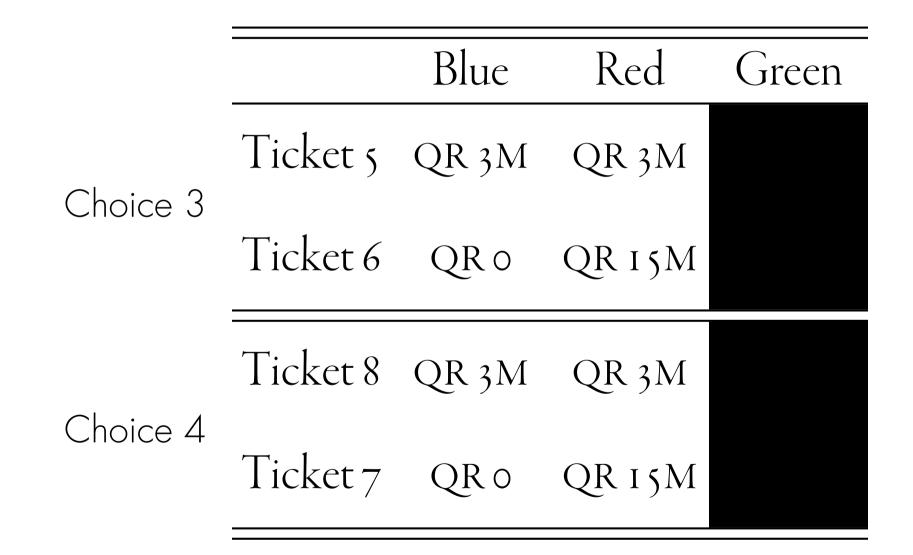


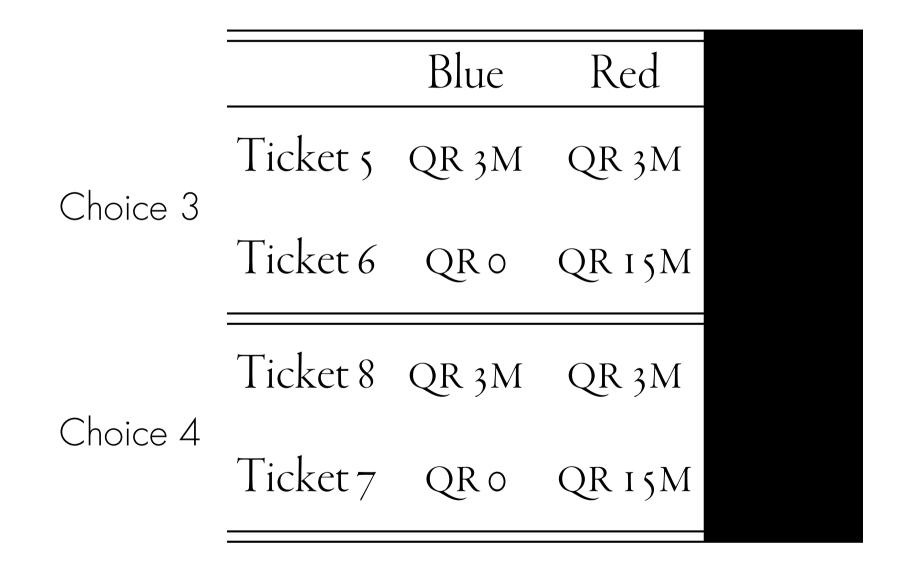












#### Paradoxical Choices

Why might people succumb to these paradoxes, sometimes even after they are told of them? What might be their reasoning? Keep in mind that the reasoning involved in the two paradoxes is probably different, given that their respective choices involve important differences.

A fair coin is flipped until it shows heads. n is the total number of flips. How much would you pay to buy the following ticket?

	$\mathcal{N} = I$	$\mathcal{N}=2$	$\ldots \mathcal{N} = IO \ldots$	N
Ticket 9	QR 2	QR4	QR 1024	QR 2 <sup>n</sup>

#### People tend to give it a value of around QR 15.

## St. Petersburg Paradox

Daniel Bernoulli discovered (while working in St. Petersburg) that the expected monetary gain from this ticket in infinite. So you should be willing to pay *any* finite amount of money to play it, e.g., even QR 1,000,000!

## St. Petersburg Paradox

In this case, assume that the utility for money is linear, i.e., u(QR x) = x.

$$v(T_{9}) = P(n = I) \times u(QR 2) + P(n = 2) \times u(QR 4) + P(n = 3) \times u(QR 8) + \dots + P(n = I0) \times u(QR I024) + \dots = (I_{2}') \times 2 + (I_{4}') \times 4 + (I_{8}') \times 8 + \dots + (I_{1024}') \times I024 + \dots = I + I + I + \dots + I + \dots = \infty.$$

Clearly when infinities are involved, strange things start to happen. (So is this a problem for Pascal's wager when he assigns infinite utility to heaven?)

## St. Petersburg Paradox

This might be avoided by assuming that utility for money has decreasing marginal utility, e.g., u(QR x) = log(x + 1).

$$v(T_9) = P(n = 1) \times u(QR 2) + P(n = 2) \times u(QR 4) + P(n = 3) \times u(QR 8) + \dots + P(n = 10) \times u(QR 1024) + \dots = (\frac{1}{2}) \times \log 3 + (\frac{1}{4}) \times \log 5 + (\frac{1}{8}) \times \log 9 + \dots + (\frac{1}{1024}) \times 1025 + \dots = b$$
, where *b* is some finite number.

The problem with this response is that we can simply reformulate ticket 1 so that the payoffs it gives are in terms of utility and not in terms of dollars. Then you should be willing to spend infinite *utility* to "purchase" the ticket.



We will explore how an account of risk aversion might explain how people make decisions in Ellsbergand Allais-type decision situations.