Rational Choice

Utility and Its Maximization

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The Decision Matrix

$$
\begin{array}{cc}
 & \text{States of Affairs } (\Omega) \\
\text{Acts } (A) &
\begin{array}{c|cccccc|}
 & \omega_1 & \omega_2 & \ldots & \omega_j & \ldots & \omega_n \\
\hline
a_1 & o_{1,1} & o_{1,2} & & o_{1,j} & & o_{1,n} \\
a_2 & o_{2,1} & o_{2,2} & & o_{2,j} & & o_{2,n} \\
\ldots & & & & & & \\
a_i & o_{i,1} & o_{i,2} & & o_{i,j} & & o_{i,n} \\
\ldots & & & & & & \\
a_m & o_{m,1} & o_{m,2} & & o_{m,j} & & o_{m,n} \\
\hline
\end{array}
\end{array}
$$

Choice Under Risk

In choice under ignorance, the following all hold:

1. There are different outcomes for different states of affairs relevant to the decision,

2. For each combination of action and state of affairs, you know the outcome, and

3. You *do* know how probable each state of affairs is. Let $P = \{p_1, p_2, \ldots, p_j\}$, where $\mathrm{P}(\omega_j) = p_j$ represents the probability that state ω_j occurs.

The Challenge of Rational Choice

How can a ranking of the *outcomes* be used to generate a ranking of the *acts*?

In choice under risk, the most common answer is to rank the acts based their expected utility:

$$v(a_i) = \sum_{j=1}^{n} [p_j \times u(o_{i,j})].$$

Utility

But what is meant by "utility"? For instance, is "expected utility" identical with "expected monetary outcome"?

Utility

Expected utility and expected monetary outcome are different because …

1. People care about things other than money, and
2. The value of money tends to diminish as it accumulates (this is usually referred to as the diminishing marginal utility of money).

Example

Suppose I have an urn with three green balls (G) and one blue ball (B). I will pull out one ball (B1) from it.

Acts (A)	States of Affairs (Ω): B1 = B	B1 = G
t_1	QR 0	QR 10
t_2	QR 0	QR 100

I give you t_1. How much would you pay to swap for t_2?

Example

The expected monetary gain from each ticket is:

$$v(t_1) = (0.25 \times \text{QR } 0) + (0.75 \times \text{QR } 10) = \text{QR } 7.50\text{, and}$$

$$v(t_2) = (0.25 \times \text{QR } 0) + (0.75 \times \text{QR } 100) = \text{QR } 75.00.$$

So the expected gain from t_1 to t_2 is QR 67.50. So you should be willing to pay around this amount.

Example

Suppose I have an urn with three green balls (G) and one blue ball (B). I will pull out one ball (B1) from it.

		States of Affairs (Ω): B1 = B	B1 = G
Acts (A)	t_3	QR 0	QR 1,000,010
	t_4	QR 0	QR 1,000,100

I give you t_3. How much would you pay to swap for t_4?

Example

The expected monetary gain from each ticket is:

$$v(t_3) = (0.25 \times \text{QR } 0) + (0.75 \times \text{QR } 1{,}000{,}010) = \text{QR } 750{,}007.50, \text{ and}$$

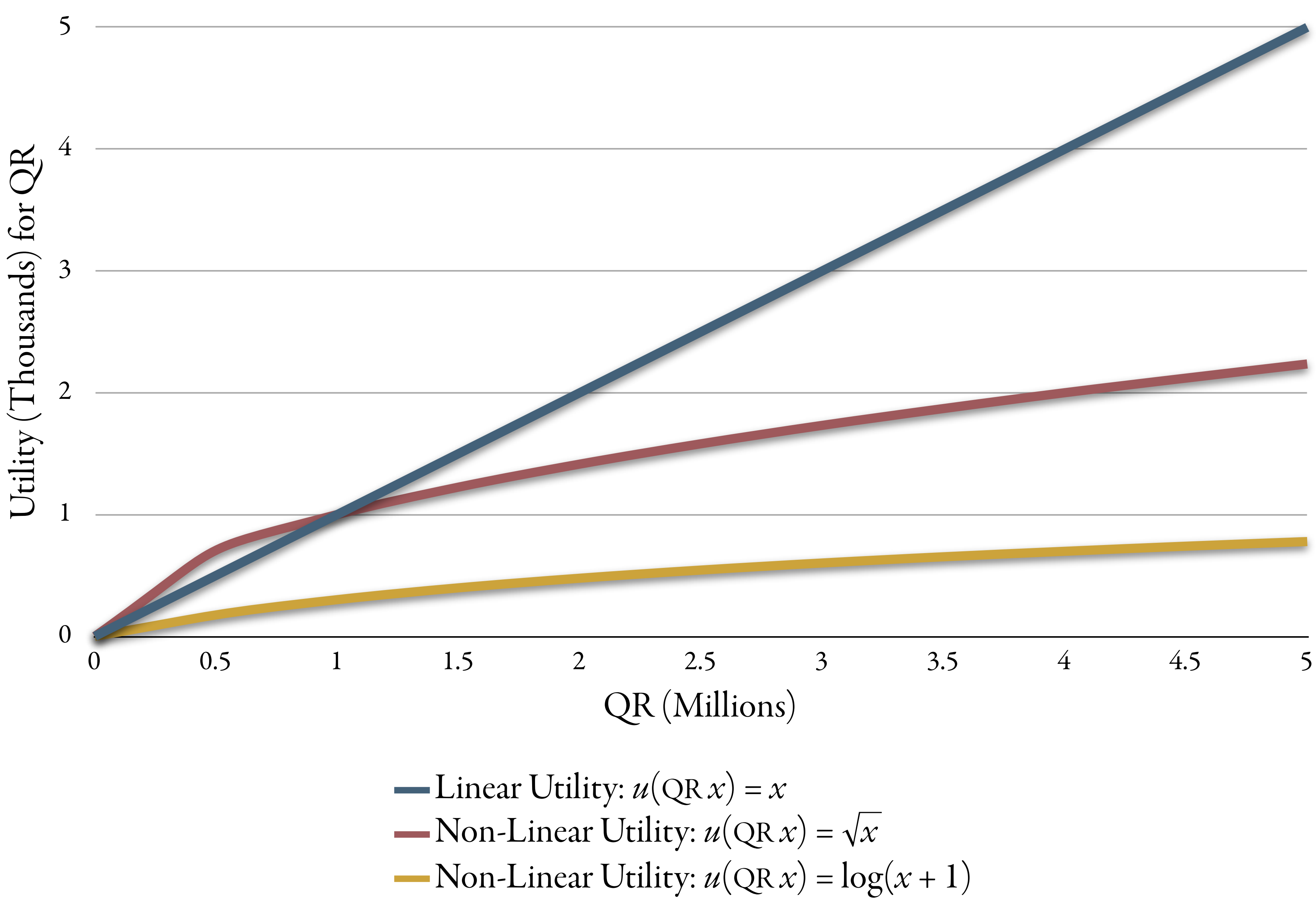
$$v(t_4) = (0.25 \times \text{QR } 0) + (0.75 \times \text{QR } 1{,}000{,}100) = \text{QR } 750{,}075.00.$$

So the expected gain from t_3 to t_4 is QR 67.50, the same as from t_1 to t_2. Yet most people would pay much less in this case.

Diminishing Marginal Utility

This example illustrates how the marginal value of money decreases as you get more of it. A QR 90 bonus is worth a lot more when you only have QR 10. When you already have QR 1,000,000, that same QR 90 bonus no longer seems as valuable. As a result, the utility people get from money is often treated as non-linear.

Utility for Money



Example

Suppose I have an urn with ten green balls (G), five red balls (R), and five blue balls (B). I will pull out one ball (B1) from it.

		States of Affairs (Ω)		
		B1 = G	B1 = R	B1 = B
Acts ($\mathcal{A}$)	t_1	QR 64	QR 36	QR 36
	t_2	QR 49	QR 121	QR 1
	t_3	QR 100	QR 1	QR 4

Which ticket is best if your utility for money is linear, where $u(\text{QR}\,x) = x$?

Example

$$v(t_1) = \left(\tfrac{1}{2} \times u(\text{QR } 64)\right) + \left(\tfrac{1}{4} \times u(\text{QR } 36)\right) + \left(\tfrac{1}{4} \times u(\text{QR } 36)\right)$$
$$= \left(\tfrac{1}{2} \times 64\right) + \left(\tfrac{1}{4} \times 36\right) + \left(\tfrac{1}{4} \times 36\right) = 50.00.$$

$$v(t_2) = \left(\tfrac{1}{2} \times u(\text{QR } 49)\right) + \left(\tfrac{1}{4} \times u(\text{QR } 121)\right) + \left(\tfrac{1}{4} \times u(\text{QR } 1)\right)$$
$$= \left(\tfrac{1}{2} \times 49\right) + \left(\tfrac{1}{4} \times 121\right) + \left(\tfrac{1}{4} \times 1\right) = 55.00.$$

$$v(t_3) = \left(\tfrac{1}{2} \times u(\text{QR } 100)\right) + \left(\tfrac{1}{4} \times u(\text{QR } 1)\right) + \left(\tfrac{1}{4} \times u(\text{QR } 4)\right)$$
$$= \left(\tfrac{1}{2} \times 100\right) + \left(\tfrac{1}{4} \times 1\right) + \left(\tfrac{1}{4} \times 4\right) = 51.25.$$

So ticket 2 (t_2) is best if your utility for money is linear, where $u(\text{QR } x) = x$.

Example

Suppose I have an urn with ten green balls (G), five red balls (R), and five blue balls (B). I will pull out one ball (B1) from it.

		States of Affairs (Ω)		
		B1 = G	B1 = R	B1 = B
Acts ($\mathcal{A}$)	t_1	QR 64	QR 36	QR 36
	t_2	QR 49	QR 121	QR 1
	t_3	QR 100	QR 1	QR 4

Which ticket is best if your utility for money is non-linear, where $u(\text{QR}\,x) = \sqrt{x}$?

Example

$$v(t_1) = \left(\tfrac{1}{2} \times u(\text{QR }64)\right) + \left(\tfrac{1}{4} \times u(\text{QR }36)\right) + \left(\tfrac{1}{4} \times u(\text{QR }36)\right)$$
$$= \left(\tfrac{1}{2} \times 8\right) + \left(\tfrac{1}{4} \times 6\right) + \left(\tfrac{1}{4} \times 6\right) = 7.00.$$

$$v(t_2) = \left(\tfrac{1}{2} \times u(\text{QR }49)\right) + \left(\tfrac{1}{4} \times u(\text{QR }121)\right) + \left(\tfrac{1}{4} \times u(\text{QR }1)\right)$$
$$= \left(\tfrac{1}{2} \times 7\right) + \left(\tfrac{1}{4} \times 11\right) + \left(\tfrac{1}{4} \times 1\right) = 6.50.$$

$$v(t_3) = \left(\tfrac{1}{2} \times u(\text{QR }100)\right) + \left(\tfrac{1}{4} \times u(\text{QR }1)\right) + \left(\tfrac{1}{4} \times u(\text{QR }4)\right)$$
$$= \left(\tfrac{1}{2} \times 10\right) + \left(\tfrac{1}{4} \times 1\right) + \left(\tfrac{1}{4} \times 2\right) = 5.75.$$

Now ticket 1 (t_1) is best if your utility for money is non-linear, where $u(\text{QR }x) = \sqrt{x}$.

Expected Utility Maximization

There are two sorts of arguments defending rational choice under risk as maximizing expected utility:

1. The appeal to the law of large numbers, and
2. An axiomatic approach.

The Law of Large Numbers

According to the **law of large numbers**, maximizing expected utility makes the decision maker better off (by getting him or her more utility) in the long run. That is, if the same problem appears many, many times, the *expected* utility of an action becomes the average *actual* utility of repeating that action.

Example

Suppose I have an urn with one green ball (G) and ninety-nine blue balls (B). I will pull out one ball (B1) from it.

	States of Affairs (Ω): B1 = B	States of Affairs (Ω): B1 = G
Acts (A): t_1	QR 1	QR 1
Acts (A): t_2	QR 0	QR 90

In the long run, what is the average payoff of each ticket?

The Law of Large Numbers

The obvious problem with appealing to the law of large numbers is that many (if not most) important decisions are made only once and not repeatedly.

The Axiomatic Approach

Axiomatic justifications of maximizing expected utility come in two different forms, indirect and direct. According to **indirect** approaches, as long as your decisions obey certain axioms concerning judgments $\succ$, then I can model your decisions "as if" you had a utility function u over outcomes whose expected utility you were maximizing. We saw this last time with the Von Neumann-Morgenstern theory of cardinal utility.

The Direct Axiomatic Approach

According to **direct** approaches, however, you *do* have a utility function u over outcomes. These approaches then stipulate some axioms that you should obey when using that utility function to rank actions. Finally, they show that if you accept these axioms, then you must use your utility function u and ultimately rank all the actions based on maximizing their expected utility. One common direct approache appeals to four axioms.

Axiom 1

Constant Utility: Given action a_i, if for all states of affairs ω_j, $u(o_{i,j}) = x$, then $v(a_i) = x$.

In other words, if the outcome of an action has the same utility value—call it x—no matter which state of affairs occurs, then the value of that action is also x.

Example

States of Affairs (Ω)

Acts (A)	ω_1	ω_2	ω_3	ω_4	ω_5	ω_6	ω_7
a_1	6	6	6	6	6	6	6
a_2	7	8	10	7	8	10	11

If you satisfy axiom 1, what is $v(t_1)$ and what is $v(t_2)$?

Axiom 2

Weak Pareto (Under Risk): Given two actions a_1 and a_2, if for all states of affairs ω_j, $u(o_{1,j}) > u(o_{2,j})$, then $v(a_1) > v(a_2)$. However, if instead $u(o_{1,j}) = u(o_{2,j})$, then $v(a_1) = v(a_2)$.

So if one action has more utility than a second action for each state of affairs, then the first action has a higher value than the second action. However, if both actions have the same utility for each state of affairs, then both actions have the same value.

Example

		States of Affairs (Ω)						
		ω_1	ω_2	ω_3	ω_4	ω_5	ω_6	ω_7
Acts (A)	a_1	6	6	6	6	6	6	6
	a_2	7	8	10	7	8	10	11

If you satisfy axiom 2, what is the relationship between $v(a_1)$ and $v(a_2)$?

Axiom 3

Splitting into Equiprobable States: Given a decision problem for actions $a_1, a_2, \ldots, a_m$, convert this into a new decision problem for actions $a'_1, a'_2, \ldots, a'_m$ by splitting the states of affairs in the original problem so that all the states have the same probability in the new decision problem. Do this and then $v(a_i) = v(a'_i)$.

Example

		States of Affairs (Ω)	
		ω_1	ω_2
		$p_1 = 0.20$	$p_2 = 0.80$
Acts ($\mathcal{A}$)	a_1	1	3
	a_2	6	2

Example

		States of Affairs (Ω)	
		ω_1	ω_2
		$p_1 = 0.20$	$p_2 = 0.80$
Acts ($\mathcal{A}$)	a_1	1	3
	a_2	6	2

		States of Affairs (Ω')				
		ω'_1	ω'_2	ω'_3	ω'_4	ω'_5
		$p_1 = 0.20$	$p_2 = 0.20$	$p_2 = 0.20$	$p_2 = 0.20$	$p_2 = 0.20$
Acts ($\mathcal{A}'$)	a'_1	1	3	3	3	3
	a'_2	6	2	2	2	2

Example

		States of Affairs (Ω)	
		ω_1	ω_2
		$p_1 = 0.20$	$p_2 = 0.80$
Acts ($\mathcal{A}$)	a_1	1	3
	a_2	6	2

		States of Affairs (Ω')				
		ω'_1	ω'_2	ω'_3	ω'_4	ω'_5
		$p_1 = 0.20$	$p_2 = 0.20$	$p_2 = 0.20$	$p_2 = 0.20$	$p_2 = 0.20$
Acts ($\mathcal{A}'$)	a'_1	1	3	3	3	3
	a'_2	6	2	2	2	2

Example

	States of Affairs (Ω)	
	ω_1	ω_2
Acts ($\mathcal{A}$)	$p_1 = 0.20$	$p_2 = 0.80$
a_1	1	3
a_2	6	2

If you satisfy axiom 3, then $v(a_1) = v(a'_1)$ and $v(a_2) = v(a'_2)$.

	States of Affairs (Ω')				
	ω'_1	ω'_2	ω'_3	ω'_4	ω'_5
Acts ($\mathcal{A}'$)	$p_1 = 0.20$	$p_2 = 0.20$	$p_2 = 0.20$	$p_2 = 0.20$	$p_2 = 0.20$
a'_1	1	3	3	3	3
a'_2	6	2	2	2	2

Axiom 4

Trade Off Condition: Suppose that (1) two outcomes of an action a are equiprobable, and that (2) a new action a' is created by making the better of a's two equiprobable outcomes slightly worse by subtracting some fixed utility ε_1, from it. If (1) and (2) hold, then there exists some other value ε_2 that can be added to the other outcome to compensate for that loss such that $v(a) = v(a')$.

Example

		States of Affairs (Ω)	
		ω_1	ω_2
		$p_1 = 0.50$	$p_2 = 0.50$
Acts ($\mathcal{A}$)	a_1	5	5
	a_2	2	10

Example

		States of Affairs (Ω)	
		ω_1	ω_2
		$p_1 = 0.50$	$p_2 = 0.50$
Acts ($\mathcal{A}$)	a_1	5	5
	a_2	2	10

		States of Affairs (Ω)	
		ω_1	ω_2
		$p_1 = 0.50$	$p_2 = 0.50$
Acts ($\mathcal{A}$)	a_1	5	5
	a'_2	$2 + \varepsilon_2$	$10 - \varepsilon_1$

Example

		States of Affairs (Ω)	
		ω_1	ω_2
		$p_1 = 0.50$	$p_2 = 0.50$
Acts ($\mathcal{A}$)	a_1	5	5
	a_2	2	10

		States of Affairs (Ω)	
		ω_1	ω_2
		$p_1 = 0.50$	$p_2 = 0.50$
Acts ($\mathcal{A}$)	a_1	5	5
	a'_2	$2 + \varepsilon_2$	$10 - \varepsilon_1$

If you satisfy axiom 4, then for *any* ε_1 there exists *some* ε_2 such that $v(a_2) = v(a'_2)$.

Direct Theorem

Direct Theorem for Maximizing Expected Utility: If a value function v over actions satisfies axioms 1, 2, 3, and 4, then the following holds:

$$v(a_i) = \sum_{j=1}^{n} [p_j \times u(o_{i,j})].$$

Direct Vs. Indirect Approaches

Direct Approaches . . .

1. Have axioms setting constraints on how utilities u over outcomes translate into values v over acts.

2. Presuppose that the decision maker has a utility function u, and use the axioms to justify maximizing expected utility using u.

Indirect Approaches . . .

1. Have axioms setting constraints on how certain judgments $\succ$ over acts translate into other judgments $\succ$ over acts.

2. Do not presuppose that the decision maker has a utility function u. Instead, the axioms show that your judgments work "as if" you are maximizing expected utility with such a function u.

Next Class…

We will look at some results from psychology showing how the decisions of real people fail to maximize expected utility.

Until then, have a good spring break! I know y'all are working hard, so you deserve it!