Rational Choice

Formal Probability Theory

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The Decision Matrix

States of Affairs (Ω)

		ω_1	ω_2	$\dots$	ω_j	$\dots$	ω_n
Acts ($\mathcal{A}$)	a_1	$o_{1,1}$	$o_{1,2}$		$o_{1,j}$		$o_{1,n}$
	a_2	$o_{2,1}$	$o_{2,2}$		$o_{2,j}$		$o_{2,n}$
	$\dots$						
	a_i	$o_{i,1}$	$o_{i,2}$		$o_{i,j}$		$o_{i,n}$
	$\dots$						
	a_m	$o_{m,1}$	$o_{m,2}$		$o_{m,j}$		$o_{m,n}$

Choice Under Risk

In choice under ignorance, the following all hold:

1. There are different outcomes for different states of affairs relevant to the decision,

2. For each combination of action and state of affairs, you know the outcome, and

3. You *do* know how probable each state of affairs is. Let $P = \{p_1, p_2, \ldots, p_j\}$, where $\mathrm{P}(\omega_j) = p_j$ represents the probability that state ω_j occurs.

The Monte Haul Problem

		States of Affairs (Ω)		
		Car behind door #1	Car behind door #2	Car behind door #3
Acts (A)	Pick door #1	Car	Goat	Goat
	Pick door #2	Goat	Car	Goat
	Pick door #3	Goat	Goat	Car

Which door should you pick?

The Monte Haul Problem

Now suppose after you have chosen, the host then says he'll show you a goat behind a door you did not pick. Now there are only two closed doors, your door and the one other, and the host says you can switch to the other door if you want. Are you more likely to win a car by switching or staying with your current door? Or is winning a car equally likely under either option?

Simple Example

Suppose I have an urn with 20 balls in it:

10 are green (G), 5 are red (R), and 5 are blue (B).

I am going to pull one ball (B1) out of the urn, what are the following probabilities?

P(B1 = G) =

P(B1 = R) =

P(B1 = B) =

Solution

$P(B1 = G) = 10 / 20 = 1/2 = 0.50.$

(Since 10 of the 20 balls are green.)

$P(B1 = R) = 5/20 = 1/4 = 0.25.$

(Since 5 of the 20 balls are red.)

$P(B1 = B) = 5/20 = 1/4 = 0.25.$

(Since 5 of the 20 balls are blue.)

Basic Terminology

The **sample space** S is the set of all possible outcomes. It is identical with the universal set $\mathcal{U}$ of outcomes in a given situation. An **event** E, on the other hand, is any subset of S. The idea is that an event may involve more than just a single outcome but multiple outcomes.

Simple Example

In the case of the example of the urn with 20 balls (10 green (G), 5 red (R), and 5 blue(B)), where one ball is drawn (B1) …

What is the sample space S?

What are some possible events?

Probability Theory

One natural way of understanding probability, is by using a **probability measure** P, which assigns numerical values ranging from 0 to 1 on events representing the likelihood that each event will occur, where $\mathrm{P}(E) = 1$ means that event E must occur while and $\mathrm{P}(E) = 0$ represents that the event cannot occur.

The Kolmogorov Axioms

There are three axioms for a probability measure P:

K1: For any event E, $0 \leq \mathrm{P}(E) \leq 1$.

K2: For sample space S, $\mathrm{P}(S) = 1$.

K3: For any events A and B, if $A \cap B = \varnothing$, then $\mathrm{P}(A \cup B) = \mathrm{P}(A) + \mathrm{P}(B)$.

The Kolmogorov Axioms

These axioms can also be formulated so that instead of talking about *events*, they talk about *propositions*, or claims about the word:

K1*: For any proposition E, $0 \leq \mathrm{P}(E) \leq 1$.

K2*: If T is a logical truth, then $\mathrm{P}(T) = 1$.

K3*: For any propositions A and B, if A and B cannot both be true, then $\mathrm{P}(A \text{ or } B) = \mathrm{P}(A) + \mathrm{P}(B)$.

Simple Example

In the case of the example of the urn with 20 balls (10 green (G), 5 red (R), and 5 blue(B)), where one balls is drawn (B1), what is the probability that the ball is either red or green?

$$P(B1 = R \text{ or } B1 = G) =$$

Solution

$$P(B1 = R \text{ or } B1 = G) = P(B1 = R) + P(B1 = G)$$

(by K3* since B1 = R and B1 = G cannot both be true)

$$= 0.25 + 0.50$$

$$= 0.75.$$

Theorems of Probability

Theorem 6.1: $P(A) + P(\overline{A}) = 1$.

The sum of the probabilities of an event and its compliment is 1.

Proof: By definition, $A \cap \overline{A} = \varnothing$. So K3 says that $P(A) + P(\overline{A}) = P(A \cup \overline{A})$. Furthermore, by definition, $A \cup \overline{A} = \mathcal{U} = S$. So $P(A) + P(\overline{A}) = P(S)$. Finally, K1 says that $P(S) = 1$.

Lemma 6.1: $P(\overline{A}) = 1 - P(A)$.

Simple Example

In the case of the example of the urn with 20 balls (10 green (G), 5 red (R), and 5 blue(B)), where one balls is drawn (B1), what is the probability that the ball is not red?

$$P(B1 \neq R) =$$

Solution

$$P(B1 \neq R) = 1 - P(B1 = R)$$

(by Lemma 6.1 since $B1 = R$ and $B1 \neq R$ are compliments)

$$= 1 - 0.25$$

$$= 0.75.$$

Conditional Probability

We are often interested in situations where we assume that one event happens (or one proposition is true) and determine how that influences the probability of another event.

This is represented by **conditional probability**, where $\mathrm{P}(A \mid B)$ stands for the probability that A occurs (or is true) given that B occurs (or is true).

Definition: $\mathrm{P}(A \mid B) = \dfrac{\mathrm{P}(A \text{ and } B)}{\mathrm{P}(B)}$, where $\mathrm{P}(B) \neq 0$.

Conditional Probability

In certain cases, the probability of one event is not influenced by the occurrence of another event.

Definition 6.2: A and B are **independent** if and only if $\mathrm{P}(A) = \mathrm{P}(A \mid B)$.

The Gambler's Fallacy denies this important insight about independence.

Theorems of Probability

Theorem 6.2: If A and B are logically equivalent, then $\mathrm{P}(A) = \mathrm{P}(B)$.

Theorem 6.3: $\mathrm{P}(A \text{ or } B) = \mathrm{P}(A) + \mathrm{P}(B) - \mathrm{P}(A \text{ and } B)$.

Theorem 6.4: If A is independent of B, then $\mathrm{P}(A \text{ and } B) = \mathrm{P}(A) \times \mathrm{P}(B)$.

Complex Example

Suppose I have the urn with 20 balls in it:

10 are green (G), 5 are red (R), and 5 are blue (B).

I am going to pull one ball (B1) out of the urn, note its color, put it back, and then pull one ball (B2).

What is the sample space *S*?

What are some possible events?

Complex Example

In the case of the example of the urn with 20 balls (10 green (G), 5 red (R), and 5 blue(B)), where two balls are drawn (B1 and B2) with replacement, what is the probability that the second ball is green given that the first ball was green?

$$P(B2 = G \mid B1 = G) =$$

Solution

$$P(B2 = G \mid B1 = G) = P(B2 = G)$$

(from Definition 6.2 since $B2 = G$ and $B1 = G$ are independent)

$$= 0.50$$

(fill in the values for $P(B1 = G)$).

Complex Example

In the case of the example of the urn with 20 balls (10 green (G), 5 red (R), and 5 blue(B)), where two balls are drawn (B1 and B2) with replacement, what is the probability that the first ball is red and the second ball is green?

$$P(B1 = R \text{ and } B2 = G) =$$

Solution

$$
\begin{aligned}
P(B1 = R \text{ and } B2 = G) &= P(B1 = R) \times P(B2 = G) \\
&\quad \text{(by Theorem 6.4 since } B1 = R \text{ and} \\
&\quad B2 = G \text{ are independent)} \\
&= 0.25 \times 0.50 \\
&= 0.125.
\end{aligned}
$$

Complex Example

In the case of the example of the urn with 20 balls (10 green (G), 5 red (R), and 5 blue(B)), where two balls are drawn (B1 and B2) with replacement, what is the probability that neither ball is blue?

$$P(B1 \neq B \text{ and } B2 \neq B) =$$

Solution

$$
\begin{aligned}
P(B1 \neq B \text{ and } B2 \neq B) &= P(B1 \neq B) \times P(B2 \neq B) \\
&\qquad \text{(by Theorem 6.4 since } B1 \neq B \text{ and} \\
&\qquad B2 \neq B \text{ are independent)} \\
&= [1 - P(B1 = B)] \times [1 - P(B2 = B)] \\
&\qquad \text{(by Lemma 6.1)} \\
&= [1 - 0.25] \times [1 - 0.25] \\
&= 0.5625.
\end{aligned}
$$

Complex Example

In the case of the example of the urn with 20 balls (10 green (G), 5 red (R), and 5 blue(B)), where two balls are drawn (B1 and B2) with replacement, what is the probability that one ball is red and one ball is blue?

$$P((B1 = R \text{ and } B2 = B) \text{ or } (B1 = B \text{ and } B2 = R)) =$$

Solution

$$
\begin{aligned}
P((B1 = R \text{ and } B2 = B) \text{ or } (B1 = B \text{ and } B2 = R)) = \\
& P(B1 = R \text{ and } B2 = B) + P(B1 = B \text{ and } B2 = R) \\
& \qquad \text{(by K3)} \\
& = [P(B1 = R) \times P(B2 = B)] + [P(B1 = R) \times P(B2 = B)] \\
& \qquad \text{(by Theorem 6.4)} \\
& = [0.25 \times 0.25] + [0.25 \times 0.25] \\
& = 0.125.
\end{aligned}
$$

Next Class…

We will look at the von Neumann-Morgenstern framework for dealing with options involving risk.