Rational Choice

Ordinal Utility

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Choice Under Certainty

States of Affairs (Ω)

Acts (A)	ω_1
a_1	o_1
a_2	o_2
$\dots$	
a_i	o_i
$\dots$	
a_m	o_m

Notice that choosing an *action* in this situation is identical with choosing an *outcome*. That is, choosing act a_i is equivalent to choosing outcome o_i.

Choice Under Certainty

Recall that the challenge of rational choice is to generate a ranking of *acts* given a ranking of *outcomes*.

Not too challenging in choice under certainty:

$a_i \succcurlyeq a_j$ if and only if $o_i \succcurlyeq o_j$.

Even so, we can go further, by deriving an underlying numerical value function v evaluating each action.

Representation Theorems

In general, a **representation theorem** states the properties that the judgments in $\succ$ over the actions must possess in order to be represented by a value function v assigning numeric values to those actions, such that the following holds:

$$a_i \succcurlyeq a_j \text{ if and only if } v(a_i) \geq v(a_j).$$

Representational theorems vary by the context of choice, such as under certainty or under risk.

Value Vs. Utility Functions

A **value function** v: $A \to \mathbf{R}$ measures the value associated with each *action*. So $v(a_i)$ is the numerical representation of the value of act a_i.

A **utility function** u: $O \to \mathbf{R}$ measures the value associated with each *outcome*. So $u(o_i)$ is the numerical representation of the "utility" or value of outcome o_i.

(Recall that **R** is the set of real numbers.)

Value Vs. Utility Functions

As a result, responding to the challenge of rational choice involves generating a value function v over *acts* given a utility function u over *outcomes*.

Choice Under Certainty

Since in choice under certainty, action a_i leads to outcome o_i for sure, the value function may be regarded as identical to the utility function:

$$v(a_i) = u(o_i).$$

Therefore, a representational theorem for rational choice under certainty can

Choice Under Certainty

Putting this all together, in the case of choice under certainty, a representation theorem reveals the conditions that the judgments in $\succ$ (over *acts*) must possess in order to be represented by a utility function u (over *outcomes*), such that the following holds:

$$a_i \succcurlyeq a_j \text{ if and only if } u(o_i) \geq u(o_j).$$

The Ordinal Utility Theorem

Ordinal Utility Theorem: $\succcurlyeq$ is a preference relation over actions if and only if there exists some utility function u such that following holds:

$$a_i \succcurlyeq a_j \text{ if and only if } u(o_i) \geq u(o_j).$$

Furthermore: utility function u is an *ordinal* scale. That is, u is unique under positive monotone transformations: any utility function u'—where $u'(x) \geq u'(y)$ if and only if $u(x) \geq u(y)$—generates the same judgments over acts as utility function u.

The Ordinal Utility Theorem

Notice this theorem makes two claims:

1. If you can measure the value of the outcomes according to an ordinal utility function *u*, then your judgements in $\succ$ will be a preference relation.

2. If your judgments in $\succ$ are a preference relation, then you can construct an ordinal utility function *u* measuring the value of the outcomes. The textbook gives a nice algorithm for doing this.

Constructing Ordinal Utilities

Step 1: Take all the outcomes and "index" them by assigning each outcome a unique number.

Sometimes this may have already been done for you, for instance, when the outcomes are given as o_1, o_2, and o_3. So the options are trivially o_1 indexed as follows:

$o_1 = o_1$,

$o_2 = o_2$, and

$o_3 = o_3$.

Constructing Ordinal Utilities

Step 1: Take all the outcomes and "index" them by assigning each outcome a unique number.

Other times you may have to do index them yourself, for instance, when the outcomes are x, y, and z. One way to index these outcomes would be as follows:

$x = x_1$,

$y = y_2$, and

$z = z_3$.

Constructing Ordinal Utilities

Step 1: Take all the outcomes and "index" them by assigning each outcome a unique number.

Other times you may have to do be more clever, for instance, when the outcomes are m_1, m_2, and a_1. One way to index these outcomes would be as follows:

$m_1 = m_1$,

$m_2 = m_2$, and

$a_1 = a_3$.

Constructing Ordinal Utilities

Step 1: Take all the outcomes and "index" them by assigning each outcome a unique number.

No matter what, just be sure to give a unique subscript for indexing each outcome!

Constructing Ordinal Utilities

Step 2: For each outcome x, keep track of the indices of the outcomes strictly worse than x.

That is, we create the following set for each outcome:

$$\mathrm{N}(O, \succ, x) = \{n \mid y_n \in O \text{ and } x \succ y_n\}.$$

(Keep in mind the subscript for y refers to the index that you assigned to y in step 1.)

Constructing Ordinal Utilities

Step 3: Define a utility function u: $O \rightarrow \mathbf{R}$ as follows:

$$u(x) = \sum_{n \in \mathrm{N}(O, >, x)} \left(\tfrac{1}{2}\right)^{n}.$$

The resulting utility function will preserve the order of the outcomes, such that $x \succcurlyeq y$ if and only if $u(x) \geq u(y)$, as one would expect.

Example

Suppose the decision looks like this:

		States of Affairs (Ω)
		ω_1
Acts (A)	a_1	x
	a_2	y
	a_3	z

Example

Suppose I have the following judgments about the outcomes (making up a preference relation):

$z \succ x$, $z \succ y$, and $x \succ y$, or, in other words,

$z \succ x \succ y$.

We can use the method for constructing an ordinal utility function as follows.

Example

Step 1: Take all the outcomes and "index" them by assigning each outcome a unique number.

In this case, we can index them as follows:

$x = x_1$,

$y = y_2$, and

$z = z_3$.

Example

Step 2: For each outcome x, keep track of the indices of the outcomes strictly worse than x.

That is, we create the following set for each outcome:

$$\mathrm{N}(O, \succ, x) = \{n \mid y_n \in O \text{ and } x \succ y_n\}.$$

Here, we get these three sets (one for each outcome):

$\mathrm{N}(O, \succ, x_1) = \{2\}$ (only y_2 is worse than x_1),
$\mathrm{N}(O, \succ, y_2) = \{\} = \varnothing$ (nothing is worse than y_1), and
$\mathrm{N}(O, \succ, z_3) = \{1, 2\}$ (both x_1 and y_2 are worse than z_3).

Example

Step 3: Define a utility function u: $O \to \mathbf{R}$ as follows:

$$u(x) = \sum_{n \in \mathrm{N}(O,\, >,\, x)} \left(\tfrac{1}{2}\right)^n.$$

In this case:

$$u(x_1) = \sum_{n \in \{2\}} \left(\tfrac{1}{2}\right)^n = \left(\tfrac{1}{2}\right)^2 = \tfrac{1}{2} \times \tfrac{1}{2} = \tfrac{1}{4} = 0.25.$$

$$u(y_2) = \sum_{n \in \varnothing} \left(\tfrac{1}{2}\right)^n = 0, \text{ and}$$

$$u(z_3) = \sum_{n \in \{1,2\}} \left(\tfrac{1}{2}\right)^n = \left(\tfrac{1}{2}\right)^1 + \left(\tfrac{1}{2}\right)^2 = \tfrac{1}{2} + \tfrac{1}{4} = \tfrac{3}{4} = 0.75.$$

Keep in mind that u is only an ordinal scale.

Example

Notice that with these utilities are indeed ordinal, just expressing the ranking of the outcomes. That is,

> $z \succ x \succ y$, and indeed, we get a utility function with $u(z) > u(x) > u(y)$.

And given that $v(a_i) = u(o_i)$, we get the following:

> $v(a_3) > v(a_1) > v(a_1)$, implying that $a_3 \succ a_1 \succ a_2$.

The problem of rational choice under certainty has been formally solved!

Constructing Ordinal Utilities

Yes, this feels like a bit of overkill for such a simple scenario. Even so, once the pattern for solving the problem of rational choice under certainty is understood, it can be applied in other situations.

Next Class...

We will discuss how choice under certainty gets more complicated when each outcome has "plural" utility values associated with it. That is, the decision maker has *multiple* utility functions (u_1, u_2, u_3, etc.) evaluating the outcomes instead of just *one* (u).