Introduction to Logical Reasoning

Introduction to Natural Deduction

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A Long Argument

1. $A \rightarrow B$.
2. $B \rightarrow C$.
3. $C \rightarrow D$.
4. $\sim D$.
5. $A \vee E$.

$\therefore$ E.

Constructing a truth table for this would be tedious! Since there are five letters involved, there would then be $2^5 = 32$ rows!! Yuck!

The Truth Table Monster!

A	B	C	D	Conclusion E	Premise 4 ~D	Premise 1 A → B	Premise 2 B → C	Premise 3 C → D	Premise 5 A ∨ E
T	T	T	T	T	F	T	T	T	T
T	T	T	T	F	F	T	T	T	T
T	T	T	F	T	T	T	T	F	T
T	T	T	F	F	T	T	T	F	T
T	T	F	T	T	F	T	F	T	T
T	T	F	T	F	F	T	F	T	T
T	T	F	F	T	T	T	F	T	T
T	T	F	F	F	T	T	F	T	T
T	F	T	T	T	F	F	T	T	T
T	F	T	T	F	F	F	T	T	T
T	F	T	F	T	T	F	T	F	T
T	F	T	F	F	T	F	T	F	T
T	F	F	T	T	F	F	T	T	T
T	F	F	T	F	F	F	T	T	T
T	F	F	F	T	T	F	T	T	T
T	F	F	F	F	T	F	T	T	T
F	T	T	T	T	F	T	T	T	T
F	T	T	T	F	F	T	T	T	F
F	T	T	F	T	T	T	T	F	T
F	T	T	F	F	T	T	T	F	F
F	T	F	T	T	F	T	F	T	T
F	T	F	T	F	F	T	F	T	F
F	T	F	F	T	T	T	F	T	T
F	T	F	F	F	T	T	F	T	F
F	F	T	T	T	F	T	T	T	T
F	F	T	T	F	F	T	T	T	F
F	F	T	F	T	T	T	T	F	T
F	F	T	F	F	T	T	T	F	F
F	F	F	T	T	F	T	T	T	T
F	F	F	T	F	F	T	T	T	F
F	F	F	F	T	T	T	T	T	T
F	F	F	F	F	T	T	T	T	F

A Shorter Form of Assessment

1. $A \rightarrow B$.
2. $B \rightarrow C$.
3. $C \rightarrow D$.
4. $\sim D$.
5. $A \vee E$.

$\therefore$ E.

6. $A \rightarrow C$. 1, 2; Hypothetical Syllogism.
7. $A \rightarrow D$. 6, 3; Hypothetical Syllogism.
8. $\sim A$. 7, 4; *Modus Tollens.*
9. E. 5, 8; Disjunctive Syllogism.

But there is a more "natural" way to show that it is a deductively valid argument.

Natural Deduction

Natural deduction is a method of deriving the conclusion of a deductive argument by using rules of inference. This allows us to construct a formal proof of validity for any deductively valid argument. Once mastered, it is more efficient, elegant, and more illuminating than checking validity with a truth table.

There are nine important rules of inference that we will focus on for this course.

Modus Ponens (M.P.)

1. $p \rightarrow q$.

2. p.

$\therefore q$.

Recall that the pattern for M.P. says that affirming both (1) a hypothetical and (2) its antecedent allows you to also (∴) affirm its consequent.

Examples of Using M.P.

Prove that the following argument is valid:

1. $A \rightarrow B$.
2. A.

$\therefore$ B.

Examples of Using M.P.

Prove that the following argument is valid:

1. $A \rightarrow B$.

2. A.

$\therefore$ B.

3. B. 1, 2; M.P.

We just add a new line, putting a new number for it. Then state the inference rule used to get it along with the number of the premises used with that rule. In this case, we get the argument's conclusion right away just by using M.P.

Examples of Using M.P.

Prove that the following argument is valid:

1. $A \rightarrow B$.

2. A.

∴ B.

Just put A in for *p*, and put B in for *q*, and this then has the same pattern as M.P. You saw this before, when we covered argument patterns.

3. B. 1, 2; M.P.

Examples of Using M.P.

Prove that the following argument is valid:

1. $A \rightarrow B$.

2. A.

∴ B.

3. B. 1, 2; M.P.

The first number tells us which line in the proof is acting as the first line for M.P. (the line affirming the hypothetical), while the second number tell us which line is acting as the second line for M.P. (the line affirming the antecedent).

Examples of Using M.P.

Prove that the following argument is valid:

1. $A \rightarrow B$.

2. A.

$\therefore B$.

So this completes the proof, explaining how the conclusion follows from the premises.

3. B. 1, 2; M.P.

Examples of Using M.P.

Prove that the following argument is valid:

1. C.
2. $C \rightarrow F$.

$\therefore F$.

Examples of Using M.P.

Prove that the following argument is valid:

1. C.
2. $C \rightarrow F$.

$\therefore F$.

We just put C in for *p*, and put F in for *q*, and this then has the same pattern as M.P.

3. F. 2, 1; M.P.

Examples of Using M.P.

Prove that the following argument is valid:

1. C.
2. $C \rightarrow F$.

$\therefore F$.

3. F. 2, 1; M.P.

Even if the order of the premises is reversed, the rule still applies. Just put the number labels in correct order for the step in the proof. For M.P. the first number is the line where the hypothetical is affirmed, and the second number is the line where the antecedent is affirmed.

Examples of Using M.P.

Prove that the following argument is valid:

1. C.

2. $C \rightarrow F$.

$\therefore$ F.

So this completes the proof!

3. F. 2, 1; M.P.

Examples of Using M.P.

Prove that the following argument is valid:

1. $\sim(D \,\&\, Z) \rightarrow (A \rightarrow D)$.
2. $\sim(D \,\&\, Z)$.

$\therefore\ A \rightarrow D$.

Examples of Using M.P.

Prove that the following argument is valid:

1. $\sim(D \& Z) \rightarrow (A \rightarrow D)$.
2. $\sim(D \& Z)$.

$\therefore\ A \rightarrow D$.

Just put $\sim(D \& Z)$ in for *p*, and $A \rightarrow D$ in for *q*, and this then has the same pattern as M.P.

3. $A \rightarrow D$. 1, 2; M.P.

Examples of Using M.P.

Prove that the following argument is valid:

1. $\sim(D \,\&\, Z) \rightarrow (A \rightarrow D)$.
2. $\sim(D \,\&\, Z)$.

$\therefore\ A \rightarrow D$.

So even if the statements are more complex, the rule still applies as long as the general pattern conforms to the rule of inference.

3. $A \rightarrow D$. 1, 2; M.P.

Familiar Rules of Inference

1. *Modus Ponens* (M.P.)

   1. $p \rightarrow q.$
   2. $p.$

   $\therefore\ q.$

2. *Modus Tollens* (M.T.)

   1. $p \rightarrow q.$
   2. $\sim q.$

   $\therefore\ \sim p.$

3. Hypothetical Syllogism (H.S.)

   1. $p \rightarrow q.$
   2. $q \rightarrow r.$

   $\therefore\ p \rightarrow r.$

4. Disjunctive Syllogism (D.S.)

   1. $p \vee q.$
   2. $\sim p.$

   $\therefore\ q.$

New Rules of Inference

5. Constructive Dilemma (C.D.)

   1. $(p \rightarrow q) \ \& \ (r \rightarrow s)$.
   2. $p \vee r$.

   $\therefore \ q \vee s$.

6. Absorption (Abs.)

   1. $p \rightarrow q$.

   $\therefore \ p \rightarrow (\mathrm{p} \ \& \ q)$.

7. Simplification (Simp.)

   1. $p \ \& \ q$.

   $\therefore \ p$.

8. Conjunction (Conj.)

   1. p.
   2. q.

   $\therefore \ p \ \& \ q$.

9. Addition (Add.)

   1. p.

   $\therefore \ p \vee q$.

Constructive Dilemma (C.D.)

1. $(p \rightarrow q) \,\&\, (r \rightarrow s)$.

2. $p \vee r$.

$\therefore\ q \vee s$.

Absorption (Abs.)

1. $p \rightarrow q$.

$\therefore\ p \rightarrow (p \,\&\, q)$.

Simplification (Simp.)

$$
\begin{array}{l}
1.\ p \,\&\, q. \\
\hline
\therefore\ p.
\end{array}
$$

Conjunction (Conj.)

1. p.

2. q.

$\therefore\ p \,\&\, q$.

Addition (Add.)

1. p.

$$\therefore p \vee q.$$

The Nine Rules of Inference

1. Modus Ponens (M.P.)

1. $p \rightarrow q$.
2. p.

$\therefore q$.

2. Modus Tollens (M.T.)

1. $p \rightarrow q$.
2. $\sim q$.

$\therefore \sim q$.

3. Hypothetical Syllogism (H.S.)

1. $p \rightarrow q$.
2. $q \rightarrow r$.

$\therefore p \rightarrow r$.

4. Disjunctive Syllogism (D.S.)

1. $p \vee q$.
2. $\sim p$.

$\therefore q$.

5. Constructive Dilemma (C.D.)

1. $(p \rightarrow \mathrm{q}) \;\&\; (\mathrm{r} \rightarrow \mathrm{s})$.
2. $p \vee \mathrm{r}$.

$\therefore \mathrm{q} \vee \mathrm{s}$.

6. Absorption (Abs.)

1. $p \rightarrow q$.

$\therefore p \rightarrow (p \;\&\; q)$.

7. Simplification (Simp.)

1. $p \;\&\; q$.

$\therefore p$.

8. Conjunction (Conj.)

1. p.
2. q.

$\therefore p \;\&\; q$.

9. Addition (Add.)

1. p.

$\therefore p \vee q$.

Pattern Matching

Given all these rules, the first thing to practice is recognizing patterns in arguments. That is, when given an argument, can you see how the rules of inference might be applied.

Argument 1

Can you figure out the pattern here?

1. $(A \& B) \rightarrow C.$

$\therefore\ (A \& B) \rightarrow [(A \& B) \& C].$

Understanding Formal Proofs

In starting to practice natural deduction, it is useful to begin by looking at correct formal proofs of validity, but with the explanation of each step left blank. We then fill in these blanks in the proof by trying to recognize which rule of inference can be used to get us to that step.

Proof 1

Fill in the blanks for the following proof of validity:

1. $A \& B$.
2. $(A \lor C) \rightarrow D$.

$\therefore A \& D$.

3. A. ____________________
4. $A \lor C$. ____________________
5. D. ____________________
6. $A \& D$. ____________________

Proof 1

Fill in the blanks for the following proof of validity:

1. A & B.
2. (A ∨ C) → D.

∴ A & D.

3. A. _____1; Simp._____
4. A ∨ C. _____3; Add._____
5. D. _____2, 4; M.P._____
6. A & D. _____3; 5; Add._____

Learning Natural Deduction

There are only three ways to learn natural deduction:

1. Practice,
2. Practice, and
3. Practice.

If you do not practice this, then you will not be able to do it. I trust you now understand *modus ponens* and *modus tollens*, so you can follow the implications here.

Lecture Survey

You should have a short survey about these types of problems attached to your lecture slides. Please gently detach it and fill it out.

Do not put your name on it, but put your clicker ID on it instead.

Please hand it in when you drop off your clicker at the front of the room as you are leaving.

Next Class…

We will do a workshop on doing simple formal proofs of validity using natural deduction.