Introduction to Logical Reasoning

Longer Proofs by Natural Deduction

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The Nine Rules of Inference

1. *Modus Ponens* (M.P.)

   1. $p \rightarrow q$.
   2. p.

   ∴ q.

2. *Modus Tollens* (M.T.)

   1. $p \rightarrow q$.
   2. $\sim q$.

   ∴ $\sim p$.

3. Hypothetical Syllogism (H.S.)

   1. $p \rightarrow q$.
   2. $q \rightarrow r$.

   ∴ $p \rightarrow r$.

4. Disjunctive Syllogism (D.S.)

   1. $p \vee q$.
   2. $\sim p$.

   ∴ q.

5. Constructive Dilemma (C.D.)

   1. $(p \rightarrow q)$ & $(\mathrm{r} \rightarrow \mathrm{s})$.
   2. $p \vee r$.

   ∴ $q \vee s$.

6. Absorption (Abs.)

   1. $p \rightarrow q$.

   ∴ $p \rightarrow (p \,\&\, q)$.

7. Simplification (Simp.)

   1. $p \,\&\, q$.

   ∴ p.

8. Conjunction (Conj.)

   1. p.
   2. q.

   ∴ $p \,\&\, q$.

9. Addition (Add.)

   1. p.

   ∴ $p \vee q$.

Natural Deduction

We have been constructing proofs where the steps are not given to us. To continue this process, today we now look at arguments whose proofs can be done in just three steps. Once you can do this, you will have the skills to tackle most proofs by natural deduction.

Argument 1

1. $(A \vee B) \rightarrow {\sim}C$.
2. $C \vee D$.
3. A.

$\therefore D$.

Argument 1

1. $(A \vee B) \rightarrow \sim C$.
2. $C \vee D$.
3. A.

∴ D.

4. $A \vee B$.	3; Add.
5. $\sim C$.	1, 4; M.P.
6. D.	2, 5; D.S.

Argument 2

1. $(P \to Q) \,\&\, (Q \to P)$.
2. $R \to S$.
3. $P \lor R$.

$\therefore$ $Q \lor S$.

Argument 2

1. $(P \rightarrow Q) \,\&\, (Q \rightarrow P)$.
2. $R \rightarrow S$.
3. $P \vee R$.

$\therefore\ Q \vee S$.

4. $P \rightarrow Q$. 1; Simp.
5. $(P \rightarrow Q) \,\&\, (R \rightarrow S)$. 4, 2; Conj.
6. $Q \vee S$. 5, 3; C.D.

Natural Deduction

As I've said before, the real goal with natural deduction to be able to take arguments in English and verify their validity. So the following argument must first be translated into the language of logic, and then verified with natural deduction. The proof can be done in just three steps.

Argument 3

If Ayah is present, then Bilal is present. If Bilal is present, then either Cala or Dirran will be elected. Cala is not elected. Anisa is present. Therefore, Dirran is elected.

Argument 3

1. $A \rightarrow B$.
2. $B \rightarrow (C \vee D)$.
3. $\sim C$.
4. A.

$\therefore$ D.

Argument 3

1. $A \rightarrow B$.
2. $B \rightarrow (C \vee D)$.
3. $\sim C$.
4. A.

$\therefore$ D.

5. B.	1, 4; M.P.
6. $C \vee D$.	2, 5; M.P.
7. D.	6, 3; D.S.

Learning Natural Deduction

There are only three ways to learn natural deduction:

1. Practice,
2. Practice, and
3. Practice.

If you do not practice this, then you will not be able to do it. I trust you now understand *modus ponens* and *modus tollens*, so you can follow the implications here.

Next Class…

We will do a workshop on creating formal proofs of validity that can be done in either two or three steps.