Introduction to Logical Reasoning

Logical Analysis via Truth Tables

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Truth and Falsity

Recall that a statement is either true or false. According to classical logic (the logic assumed for this class), a proposition *cannot* be both.

The truth or falsity of a compound statement ultimately depends upon the truth values of the positive statements making it up.

Conjunction

The **conjunctive statement** p & q asserts that *both* statements p and q are true. It is false if and only if *any one* conjunct is false. This meaning of conjunction is neatly expressed with what is called a "truth table":

p	q	p & q
T	T	T
T	F	F
F	T	F
F	F	F

Negation

The **negative statement** $\sim p$ asserts that statement p is false. This negative statement is itself false if and only if p is true. The truth table for negation is here:

p	$\sim p$
T	F
F	T

Disjunction

The **disjunctive statement** $p \vee q$ asserts that *at least one* of statements p and q is true. It is false if and only if *both* disjuncts are false. This is expressed in the truth table for disjunction:

p	q	$p \vee q$
T	T	T
T	F	T
F	T	T
F	F	F

Implication

The **hypothetical statement** $p \rightarrow q$ asserts that whenever p is true, then q must be true as well. It is false if and only if the antecedent is *true* but the consequent is *false*. Here is its truth table:

p	q	$p \rightarrow q$
T	T	T
T	F	F
F	T	T
F	F	T

This is the one that is tricky to remember!

Truth Table Construction

Given any statement, we can construct a truth table to see what possible truth values it can have.

For instance, let's construct the truth table for the following statement:

> If I do not hate logic, then I am smart.

Truth Table Construction

Step 1: Translate the statement.

In this example, there are two positive statements to symbolize:

1. I **h**ate logic. (H)
2. I am **s**mart. (S)

So the entire statement is symbolized as follows:

$$\sim H \rightarrow S$$

Truth Table Construction

Step 2: Construct the columns.

The columns are determined by taking apart the statement until we reach positive statements that cannot be broken down any further (because they are just a single letter).

Truth Table Construction

Start with the original statement:

$\sim H \rightarrow S$

Truth Table Construction

Next, identify the main connective:

$$\sim \mathbf{H} \rightarrow \mathbf{S}$$

Truth Table Construction

Now identify the main parts it connects.

$$\sim H \rightarrow S$$

Truth Table Construction

Next add one column for each part.

S	~H	~H → S

To make things easier later on, always put any single letters to the far left (because they cannot be taken further apart).

Truth Table Construction

Repeat the process with the parts just found.

S	~H	~H → S

S cannot be broken down any further, but ~H can.

Truth Table Construction

Keep putting anything that cannot be broken down to the far left.

H	S	~H	~H → S

Truth Table Construction

And since H cannot be broken down any further, there is nothing more to do.

H	S	~H	~H → S

Truth Table Construction

Step 3: Construct the rows.

If there are an *n* number of single letters, then there will be 2^n rows.

In this example, there is only H and S, so $n = 2$. As a result, there are 2^2 or 4 rows.

Truth Table Construction

For the first single letter (in this case H), set the first half of the rows to T and the second half to F.

H	S	~H	~H → S
T			
T			
F			
F			

Truth Table Construction

For each of these halves, repeat this process for the next single letter (in this case S).

H	S	~H	~H → S
T	T		
T	F		
F			
F			

Truth Table Construction

For each of these halves, repeat this process for the next single letter (in this case S).

H	S	~H	~H → S
T	T		
T	F		
F	T		
F	F		

Truth Table Construction

Repeat this process for all the single letters (in this case, there are no more to do).

H	S	~H	~H → S
T	T		
T	F		
F	T		
F	F		

Notice that these are ALL the possible truth value combinations for these two statements.

Step 4: Fill out the remaining columns.

Work across each column from left to right, calculating the truth value for each column based on the truth values of statements to the left and the connective used in that column.

Truth Table Construction

Staring with the leftmost column that is not filled in, the main connective is ~ and the main part is H.

H	**S**	**~H**	**~H → S**
T	T		
T	F		
F	T		
F	F		

Truth Table Construction

Use the truth table for negation, given the values from H's column, to fill in ~H's column.

H	S	~H	~H → S
T	T	F	
T	F	F	
F	T	T	
F	F	T	

Truth Table Construction

Repeat this for the next column; here the main connective is → and the main parts are ~H and S.

H	S	~H	~H → S
T	T	F	
T	F	F	
F	T	T	
F	F	T	

Truth Table Construction

Use implication's truth table, given the values from ~H's and S's columns, to fill in ~H → S's column.

H	S	~H	~H → S
T	T	F	T
T	F	F	T
F	T	T	T
F	F	T	F

And it is all done!

H	**S**	**~H**	**~H → S**
T	T	F	T
T	F	F	T
F	T	T	T
F	F	T	F

Truth Table Construction

Now suppose H is true but S is false. The truth table can be read to reveal the value of $\sim H \rightarrow S$.

H	**S**	**~H**	**~H → S**
T	T	F	T
T	F	F	T
F	T	T	T
F	F	T	F

So when I hate logic but I am not smart, this entire statement is still true.

Truth and Statements

A **contingent statement** is a statement that can either be true or false.

A **tautology** is a statement that is always true because of its logical form.

A **contradiction** is a statement that is always false because of its logical form.

Contingent Statements

The truth table for "If I do not hate logic, then I am smart" reveals that it is a contingent statement.

H	S	~H	~H → S
T	T	F	T
T	F	F	T
F	T	T	T
F	F	T	F

This shows that it may logically be either true or false.

Tautologies

The statement "It will rain tomorrow or it will not rain tomorrow" is a tautology.

R	$\sim R$	$R \vee \sim R$
T	F	T
F	T	T

No matter what, this statement is always true.

Contradictions

The statement "It will rain tomorrow and it will not rain tomorrow" is a contradiction.

R	~R	R & ~R
T	F	T
F	T	T

No matter what, this statement is always false.

Next Class…

We will do a workshop on translating English to the language of logic and constructing truth tables.