Introduction to Logical Reasoning

Basic Set Theory

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Sets and their Contents

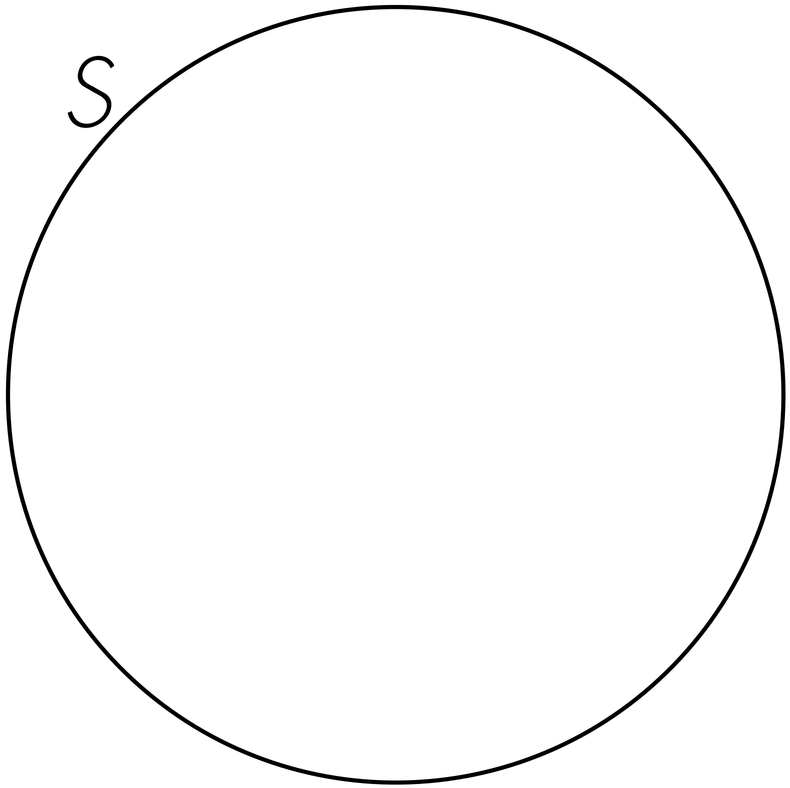
Set: A collection of "things". A set is denoted by italicized capital letters, e.g., $A, B, C, \ldots$.

Element: A "thing" that is in a set. An element is denoted by italicized lowercase letters, e.g., $x, y, z, \ldots$.

Empty set: The set that contains no elements. The empty set is denoted by $\varnothing$.

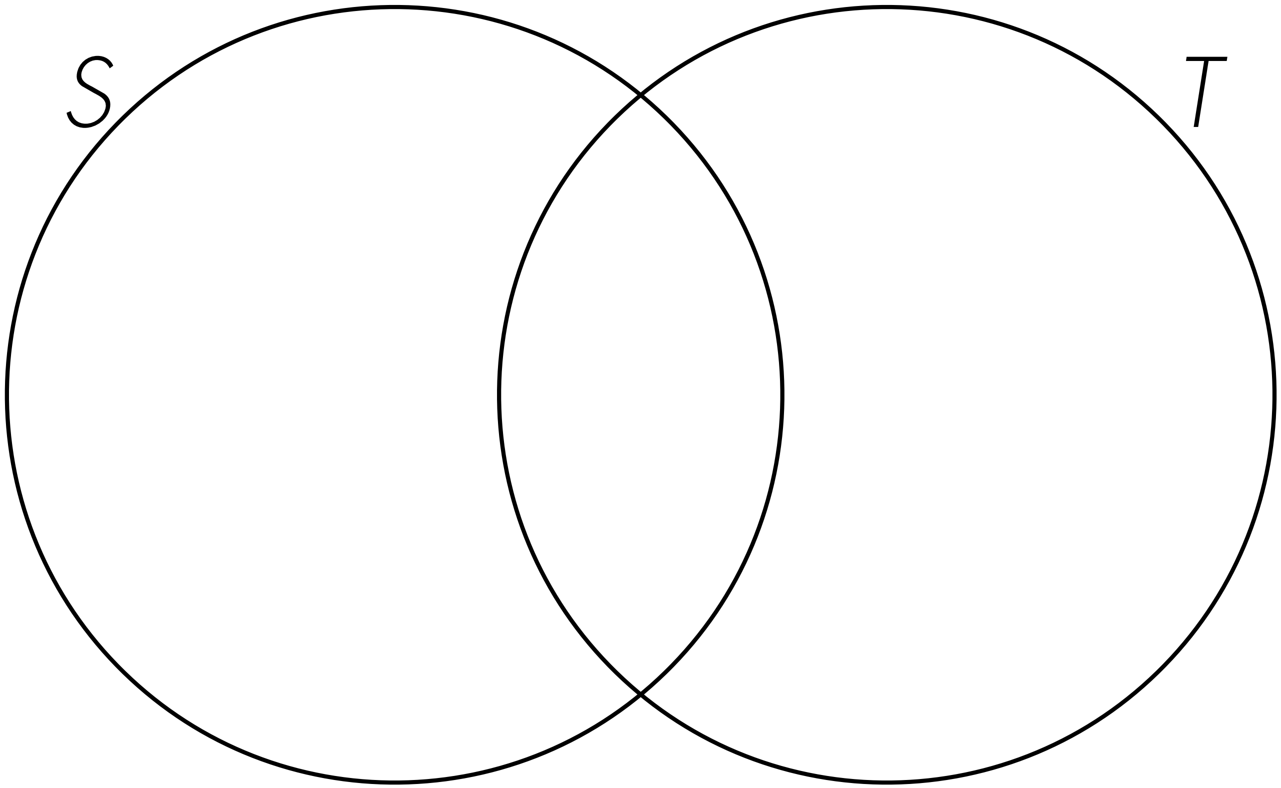
Venn Diagrams

Sets may may be diagrammed using circles. For instance, a set *S* may be diagrammed as follows:



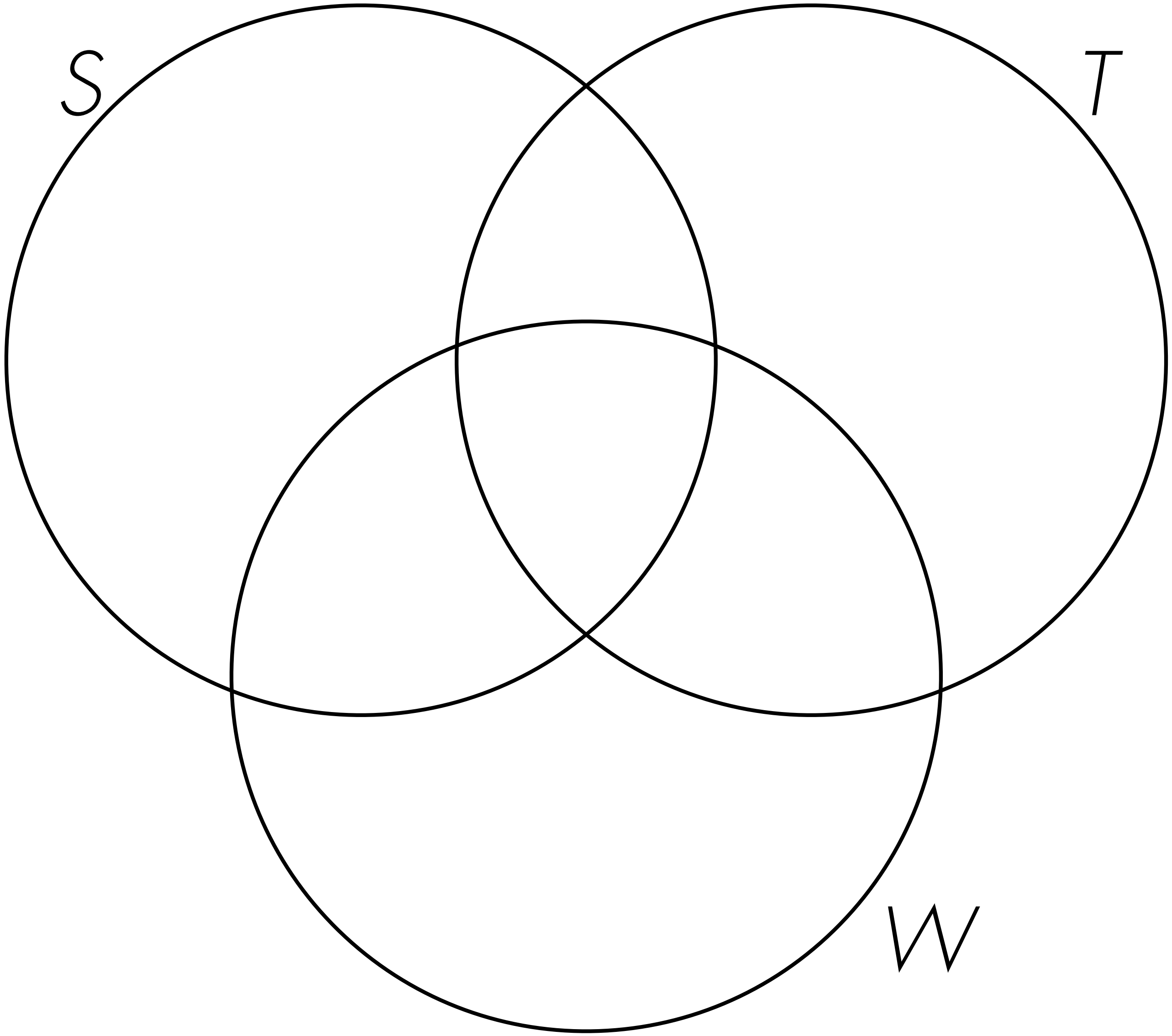
Venn Diagrams

Two sets, *S* and *T*, may be diagrammed together:



Venn Diagrams

Three sets, *S*, *T* and *W*, may be diagrammed together:



Sets and their Contents

$x \in S$ means that "x is an element of set S", while
$x \notin S$ means that "x is *not* an element of set S".

Venn Diagrams

Diagram of $x \in S$:

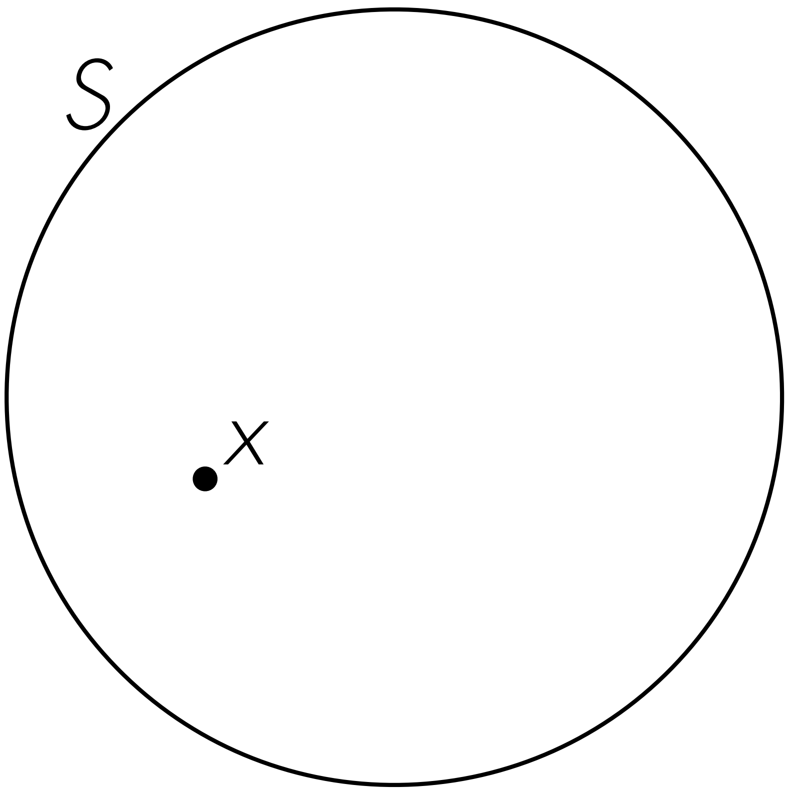
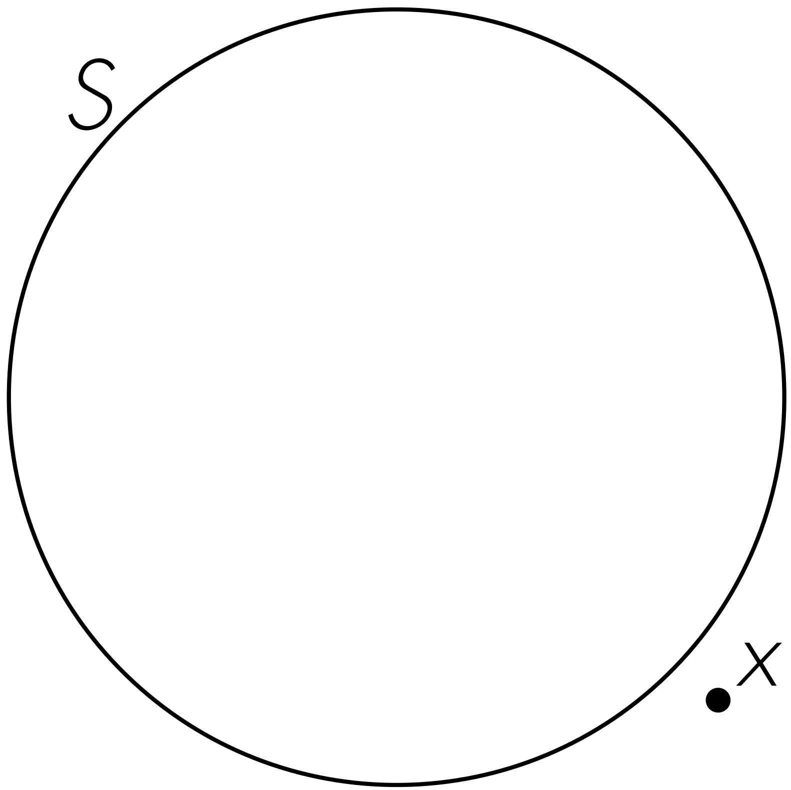


Diagram of $x \notin S$:



Sets and their Contents

The contents of a set can be shown in two ways:

1. The **roster** method: $S = \{1, 2, 3\}$, and

2. The **rule** method: $S = \{x \mid x \text{ is a whole number and } 1 \leq x \leq 3\}$.

Both represent the same set of numbers: the set of the numbers 1, 2, and 3.

Sets and Their Contents

The Roster Method: Explicitly writing out all the contents of a set.

For instance:

$S = \{2, 4, 6, 8\}$, or

$T = \{a, e, i, o, u\}$.

But what if the set has a hundred elements? Or what if it has an infinite number of elements?

Sets and Their Contents

The Rule Method: Devise a rule that specifies the contents of the set.

For instance:

$S = \{x \mid x \text{ is even and } 2 \leq x \leq 8\}$, or

$T = \{x \mid x \text{ is a letter of the English alphabet and } x \text{ is a vowel (excluding the letter 'y')}\}$.

Comparing Sets

Subset: For any sets S and T, $S \subseteq T$ (S is a subset of T) if, and only if, for every $x \in S$, $x \in T$.

Three properties concerning subsets:

1. For any set S, $S \subseteq S$,
2. For any set S, $\varnothing \subseteq S$,
3. For any set S, S has 2^n subsets.

Comparing Sets

Proper Subset: For any sets S and T, $S \subset T$ (S is a proper subset of T) if, and only if, $S \subseteq T$ and there exists at least one x, such that $x \in T$ but $x \notin S$.

Three properties concerning proper subsets:

1. For any set S, $S \not\subset S$,
2. For any set S, if $\varnothing \subset S$, then S is not empty.
3. For any set S, S has $2^n - 1$ proper subsets.

Comparing Sets

Set Equivalence: For any sets S and T, $S = T$ (S is equivalent to T) if, and only if, $S \subseteq T$ and $T \subseteq S$.

Venn Diagrams

Diagram of $S \subseteq T$:

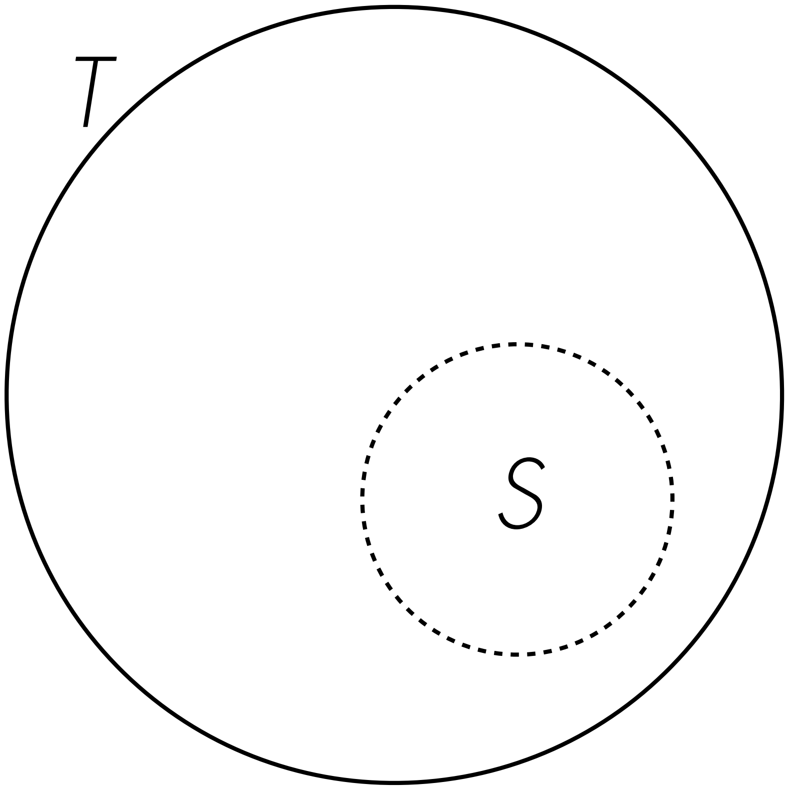
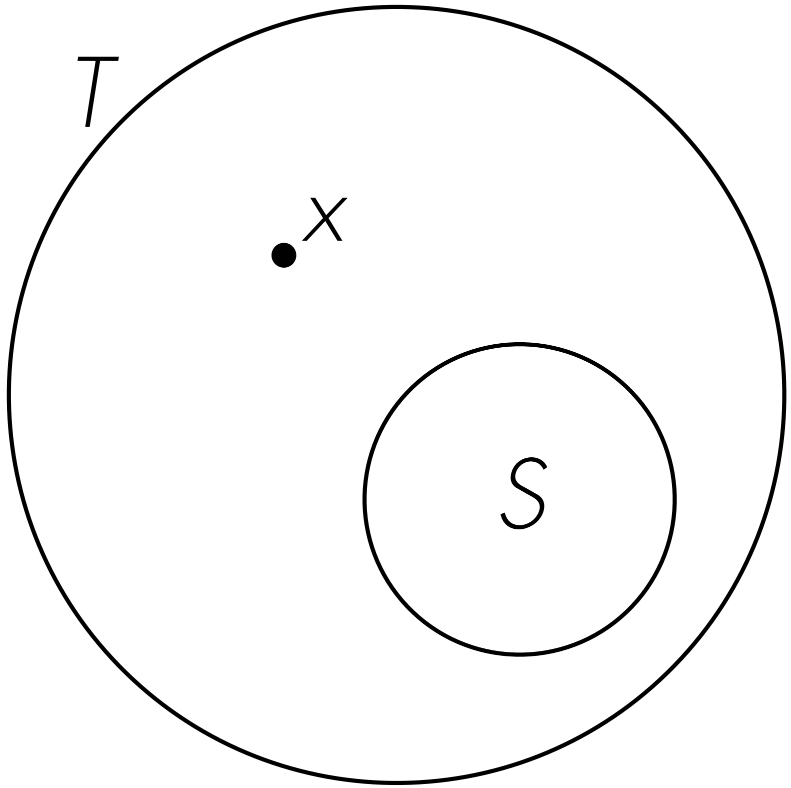


Diagram of $S \subset T$:



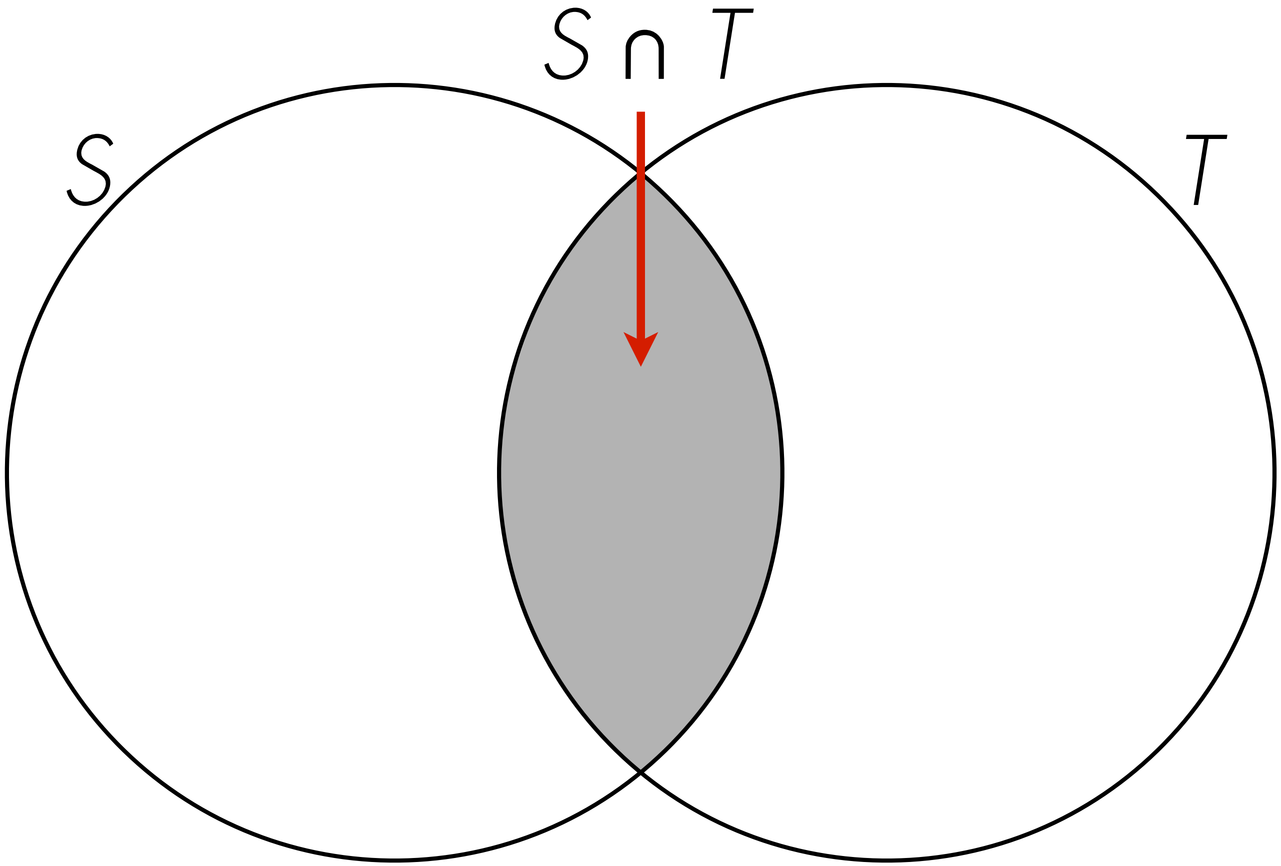
Set Operations

Set Intersection: For any sets S and T, there exists the set $S \cap T$ (the intersection of S and T), such that for any x, $x \in S \cap T$ if, and only if, $x \in S$ *and* $x \in T$.

Set Disjointness: For any sets S and T, S and T are disjoint if, and only if, $S \cap T = \varnothing$.

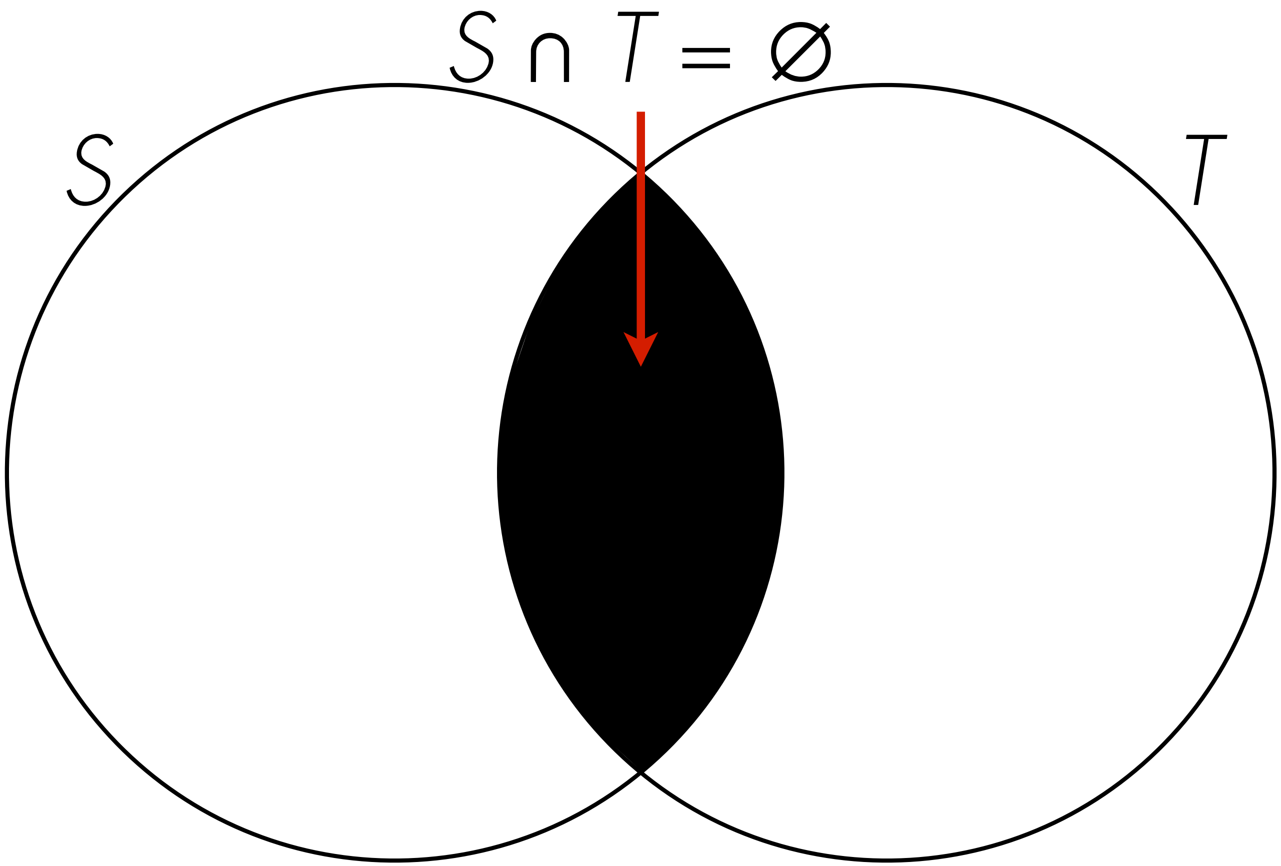
Venn Diagrams

The intersection of sets *S* and *T*:



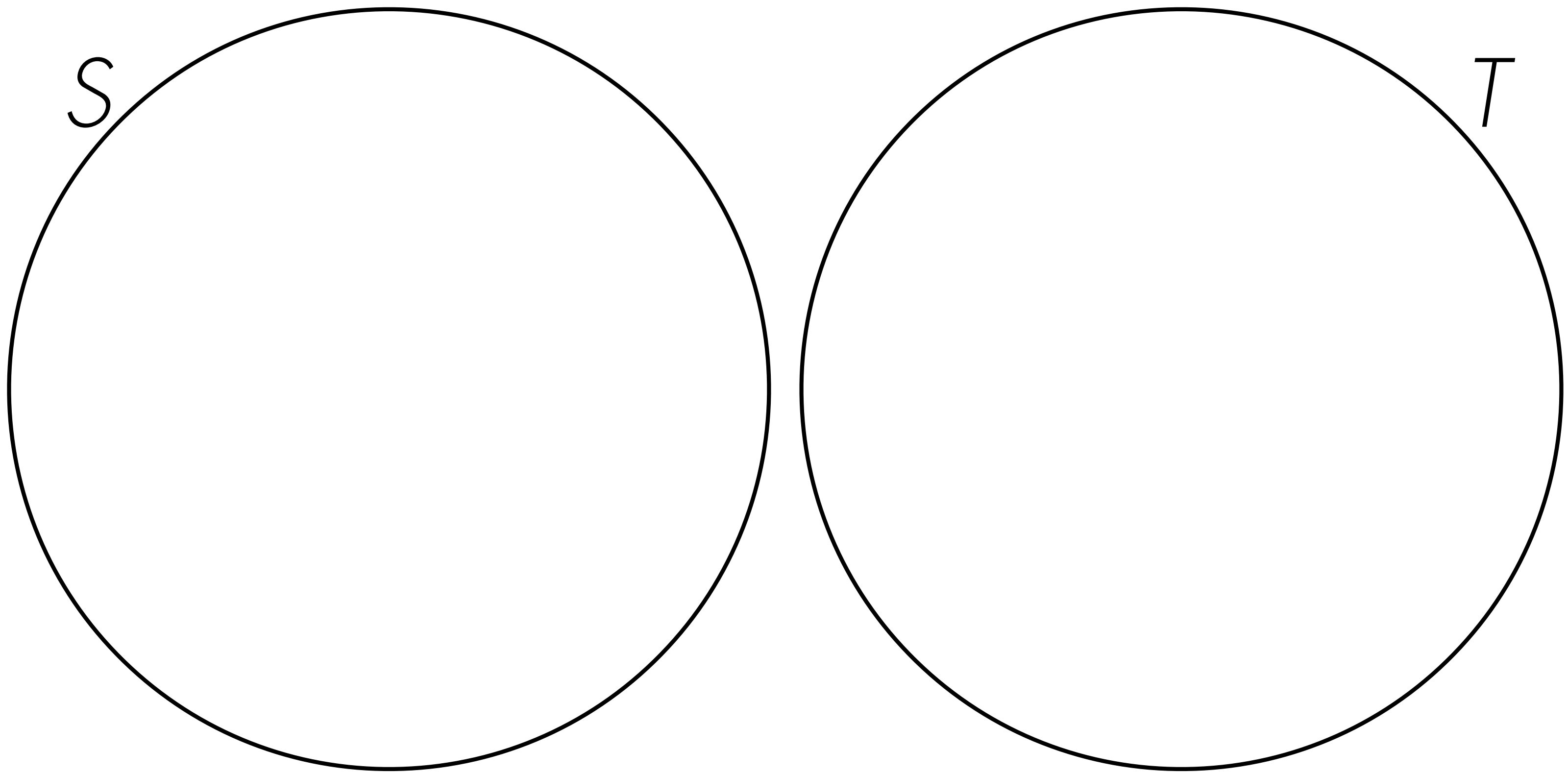
Venn Diagrams

Example of when sets *S* and *T* are disjoint:



Venn Diagrams

Example of when sets *S* and *T* are disjoint:

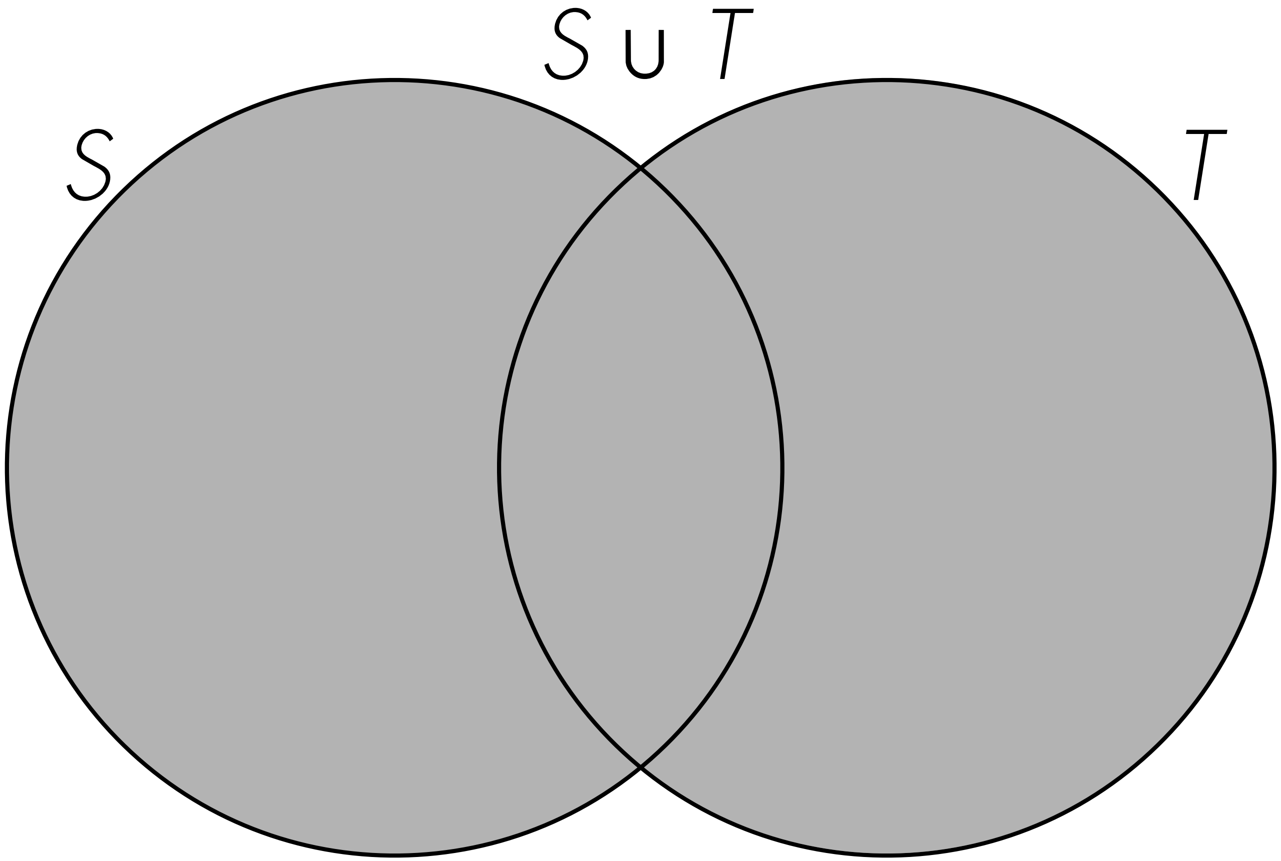


Set Operations

Set Union: For any sets S and T, there exists the set $S \cup T$ (the union of S and T), such that for any x, $x \in S \cup T$ if, and only if, $x \in S$ *or* $x \in T$.

Venn Diagrams

The union of sets *S* and *T*:



Set Operations

Set intersection and set union are **commutative**:

$S \cap T = T \cap S$, and

$S \cup T = T \cup S$.

They are also both **associative**:

$(S \cap T) \cap W = S \cap (T \cap W)$, and

$(S \cup T) \cup W = S \cup (T \cup W)$.

Set Operations

But be aware that sometimes:

$$(S \cap T) \cup W \neq S \cap (T \cup W).$$

This means the order of parentheses is extremely important!

For instance, let

$S = \{1, 2\}$,

$T = \{2, 3\}$, and

$W = \{4, 5\}$.

So $(S \cap T) \cup W = \{2, 4, 5\}$, but $S \cap (T \cup W) = \{2\}$.

Set Operations

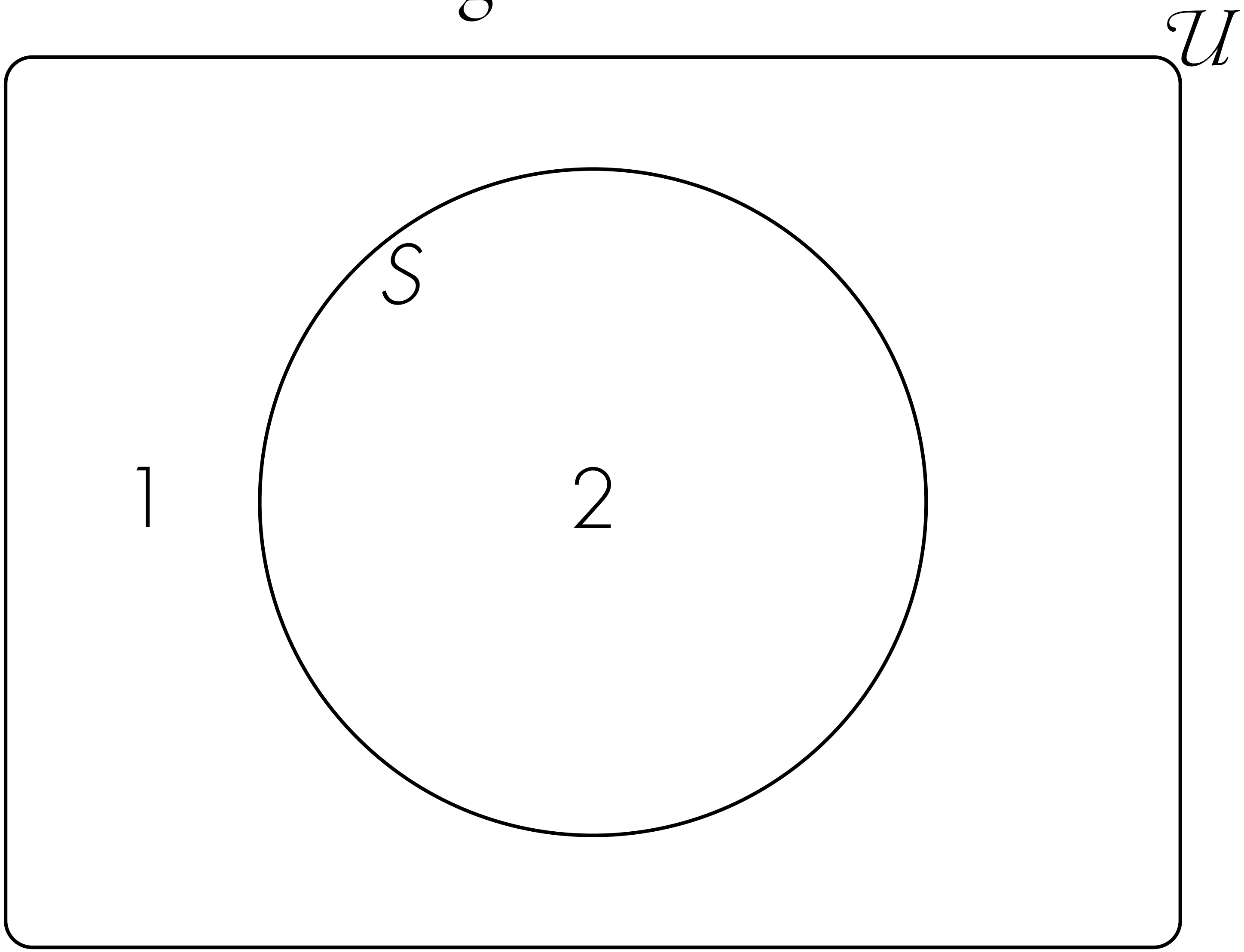
Set Complement: For any set S, there exists a set $\overline{S}$ (the complement of S), such that $S \cap \overline{S} = \varnothing$ and $S \cup \overline{S} = \mathcal{U}$.

Universal Set: For any set S, there exists a set $\mathcal{U}$ (the universal set) for S, such that $S \subseteq \mathcal{U}$ and $S \cap \overline{S} = \mathcal{U}$.

Venn Diagrams

This can be seen with a diagram:

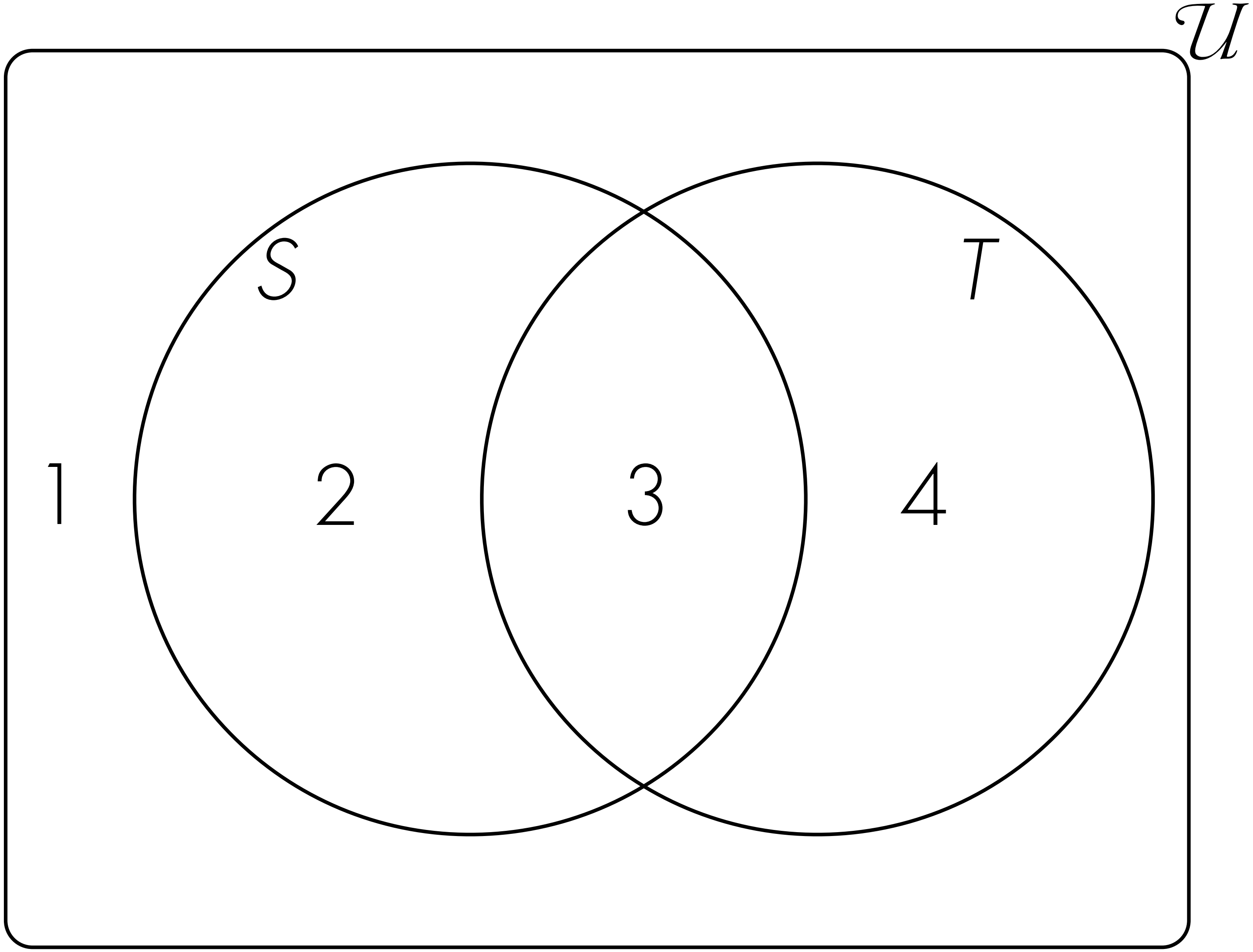
1: $\overline{S}$
2: S



Venn Diagrams

The possible sets with sets S and T:

1: $\overline{S} \cap \overline{T}$
2: $S \cap \overline{T}$
3: $S \cap T$
4: $\overline{S} \cap T$



Next Class…

We will have a workshop on working with sets.