Introduction to Logical Reasoning

Understanding Proofs by Natural Deduction

David Emmanuel Gray

Northwestern University in Qatar
Carnegie Mellon University in Qatar

The Nine Rules of Inference

1. *Modus Ponens* (M.P.)

   1. $p \to q$.
   2. p.

   ∴ q.

2. *Modus Tollens* (M.T.)

   1. $p \to q$.
   2. $\sim q$.

   ∴ $\sim p$.

3. Hypothetical Syllogism (H.S.)

   1. $p \to q$.
   2. $q \to r$.

   ∴ $p \to r$.

4. Disjunctive Syllogism (D.S.)

   1. $p \lor q$.
   2. $\sim p$.

   ∴ q.

5. Constructive Dilemma (C.D.)

   1. $(p \to q) \,\&\, (\mathrm{r} \to \mathrm{s})$.
   2. $p \lor r$.

   ∴ $q \lor s$.

6. Absorption (Abs.)

   1. $p \to q$.

   ∴ $p \to (p \,\&\, q)$.

7. Simplification (Simp.)

   1. $p \,\&\, q$.

   ∴ p.

8. Conjunction (Conj.)

   1. p.
   2. q.

   ∴ $p \,\&\, q$.

9. Addition (Add.)

   1. p.

   ∴ $p \lor q$.

Longer Formal Proofs

So far we have just seen arguments whose arguments only require one step in order to construct a proof of validity via natural deduction. That is, all that was needed was to recognize the pattern being used. However, most arguments will require proofs with more steps. Before diving into these, it is good to see how longer proofs look.

A Long Argument

Recall that last class I showed you the following proof of validity from natural deduction:

1. $A \rightarrow B$.
2. $B \rightarrow C$.
3. $C \rightarrow D$.
4. $\sim D$.
5. $A \vee E$.

$\therefore$ E.

6. $A \rightarrow C$. 1, 2; H.S.
7. $A \rightarrow D$. 6, 3; H.S.
8. $\sim A$. 7, 4; M.T.
9. E. 5, 8; D.S.

Understanding Formal Proofs

In starting to practice natural deduction, it is useful to begin by looking at correct formal proofs of validity, but with the explanation of each step left blank. We then fill in these blanks in the proof by trying to recognize which rule of inference can be used to get us to that step.

Proof 1

Fill in the blanks for the following proof of validity:

1. $A \& B$.
2. $(A \vee C) \rightarrow D$.

$\therefore$ $A \& D$.

3. A. ____________________
4. $A \vee C$. ____________________
5. D. ____________________
6. $A \& D$. ____________________

Proof 1

Fill in the blanks for the following proof of validity:

1. $A \& B.$
2. $(A \vee C) \rightarrow D.$

∴ $A \& D.$

3. $A.$ ____1; Simp.____
4. $A \vee C.$ ____3; Add.____
5. $D.$ ____2, 4; M.P.____
6. $A \& D.$ ____3; 5; Add.____

Proof 2

Fill in the blanks for the following proof of validity:

1. $(E \vee F) \,\&\, (G \vee H)$.
2. $(E \rightarrow G) \,\&\, (F \rightarrow H)$.
3. $\sim G$.

$\therefore$ H.

4. $E \vee F$. ____________________
5. $G \vee H$. ____________________
6. H. ____________________

Proof 2

Fill in the blanks for the following proof of validity:

1. $(E \vee F) \,\&\, (G \vee H)$.
2. $(E \rightarrow G) \,\&\, (F \rightarrow H)$.
3. $\sim G$.

$\therefore$ H.

4. $E \vee F$. ______1; Simp.______
5. $G \vee H$. ______2, 4; C.D.______
6. H. ______5, 3; D.S.______

Proof 3

Fill in the blanks for the following proof of validity:

1. $Q \rightarrow R$.
2. $\sim S \rightarrow (T \rightarrow U)$.
3. $S \vee (Q \vee T)$.
4. $\sim S$.

$\therefore$ $R \vee U$.

5. $T \rightarrow U$. ____________________
6. $(Q \rightarrow R) \,\&\, (T \rightarrow U)$. ____________________
7. $Q \vee T$. ____________________
8. $R \vee U$. ____________________

Proof 3

Fill in the blanks for the following proof of validity:

1. $Q \rightarrow R$.
2. $\sim S \rightarrow (T \rightarrow U)$.
3. $S \vee (Q \vee T)$.
4. $\sim S$.

$\therefore R \vee U$.

5. $T \rightarrow U$. ______2, 4; M.P.______
6. $(Q \rightarrow R) \& (T \rightarrow U)$. ______1, 5; Conj.______
7. $Q \vee T$. ______3, 4; D.S.______
8. $R \vee U$. ______6, 7; C.D.______

Learning Natural Deduction

There are only three ways to learn natural deduction:

1. Practice,
2. Practice, and
3. Practice.

If you do not practice this, then you will not be able to do it. I trust you now understand *modus ponens* and *modus tollens*, so you can follow the implications here.

Next Class…

We will do a workshop on doing simple formal proofs of validity.