Basic Set Theory

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Defintions

Definition: Set A set is any collection of "things". Your immediate family is a set. A shopping list is a set of items that you wish to buy when you go to the store. The cars in the dealership parking lot is a set. A set is typically denoted by a capital letter, such as A, B, C, S, and T.

The only thing that matters to a set is what is in it. There is no notion of order or 'how many' of a particular item. A thing that is in a set is called an **element** or member of the set. An elements of a set is typically denoted by a lower case letter, such as x, y, and z.

Definition: Empty Set There is one particularly special set, the empty set, that contains no elements (it's empty). The empty set is denoted by the symbol $\emptyset$.

Notation Membership in a set is usually denoted with a funny looking letter 'e'. If S is a set, and x is some "thing", then we say x is an element of S by writing $x \in S$. When x is not in S, we write $x \notin S$.

Notation The contents of a set is usually written out between curly braces. This notation can either explicitly list all the members of the set (i.e., $\{1, 2, 3\}$ being the set containing the numbers 1, 2, and 3), or can give a rule (i.e., $\{x|x$ is a whole number and $1 \leq x \leq 3\}$, which is also the set containing the numbers 1, 2, and 3. The bar $|$ means "such that" so that the notation reads "The set of all x such that x is a whole number between 1 and 3"). The empty set $\emptyset$ is written in this notion as $\{\}$.

So, for instance, the following are all the same sets, due to the fact that there is no concept of multiplicity or order among sets.

$$\{2, 3, 1\}, \{1, 1, 1, 2, 1, 3\}, \{1, 3, 2\}$$

$$\{2, 2, 1, 1, 3, 3\}, \{1, 2, 3\}, \{3, 2, 1\}$$

All the above sets contain just the elements 1, 2, and 3, and that is all that matters.

Definition: Subset For any sets S and T, S is a subset of T if, and only if, for every $x \in S$, $x \in T$. When S is a subset of T, this is denoted by $S \subseteq T$. If S is not a subset of T then this is denoted by $S \nsubseteq T$

For instance, the set of all apples is a subset of the set of all fruit. But the set of all fruit is not a subset of the set of all apples.

Fact For any set S, $S \subseteq S$.

For instance, the set of all apples is a subset of itself. This is because everything in the set of apples is also in the set of apples. Yes, this should be obviously true, but it is nonetheless still important to note.

Fact For any set S, $\emptyset \subseteq S$.

For instance, notice that everything in the empty set is also in the set of all apples. This may sound strange, but if you think about it for a moment it is undeniable. That is, there is (literally?) nothing you can name in the empty set that the set of all apples is missing. So the empty set must be a subset of the set of all apples.

Fact For any set S, if S has n elements in it, then S has 2^n subsets.

For instance, the set $\{2, 4, 6\}$ has three elements in it. Therefore it has $2^3 = 8$ subsets. These subsets may be listed out:

1. $\emptyset$ (don't forget this one!),
2. $\{1\}$,
3. $\{2\}$,
4. $\{3\}$,
5. $\{1, 2\}$,
6. $\{1, 3\}$,
7. $\{2, 3\}$, and
8. $\{1, 2, 3\}$ (don't forget this one either!).

Definition: Proper Subset For any sets S and T, S is a proper subset of T if, and only if, $S \subseteq T$ and there exists one element x, such that $x \in T$ but $x \notin S$. When S is a proper subset of T, this is denoted by $S \subset T$. If S is not a proper subset of T then this is denoted by $S \not\subset T$.

For instance, the set of all apples is a proper subset of the set of all fruit, this is because there are fruit that are not apples.

Fact For any set S, $S \not\subset S$.

So, for instance, the set of all apples is not a proper subset of itself. This is because the set of all apples does not contain anything more than the set of apples. Once again, this should be obviously true, but it is still important to notice.

Fact For any set S, if $\emptyset \subset S$ then S is not the empty set.

This one is a little more tricky, but upon reflection should not be too surprising. We know from a previous fact that the empty set is a subset of any other set. But if the empty set is a *proper* subset of some set S, then S must have at least one thing in it (which the empty set does not have). So S cannot be empty, and thus it is not the empty set.

Fact For any set S, if S has n elements in it, then S has $2^n - 1$ proper subsets.

For instance, the set $\{2, 4, 6\}$ has three elements in it. Therefore it has $2^3 - 1 = 7$ subsets. These subsets may be listed out:

1. $\emptyset$,
2. $\{1\}$,
3. $\{2\}$,
4. $\{3\}$,
5. $\{1, 2\}$,
6. $\{1, 3\}$, and
7. $\{2, 3\}$.

Which set from the previous list of subsets is missing from this list of *proper* subsets? (Answer: $\{2, 4, 6\}$.) Why is this set not included on this list? (Answer: a set cannot be a proper subset of itself.)

Definition: Set Equivalence For any two sets S and T, S is equivalent to T if, and only if, $S \subseteq T$ and $T \subseteq S$. The usual symbol for equality, $=$, is used to denote set equivalence. So S is equivalent to T is denoted by $S = T$.

This might seem to be a funny way to define equivalence, but it does work. For instance, let set S be $\{1, 2, 3\}$ and let set T be $\{3, 2, 2, 1\}$. Notice that everything in S is also in T, so $S \subseteq T$. Furthermore, everything in T is also in S. So $T \subseteq S$. This means that according to the definition of set equivalence, $S = T$. This is in accord with what was said earlier about sets: duplication of elements and the order in which elements are presented within a set does not matter. So while this might seem to be a rather roundabout way to define set equivalence, it correct captures the idea about what makes two different looking sets identical.

Set Operations

Definition: Intersection For any sets A and B, there exists $A \cap B$ (the intersection of A and B), such that for any x, $x \in A \cap B$ if, and only if, $x \in A$ and $x \in B$.

Put informally, intersection is just telling you what elements the two sets have in common. For instance, consider the set of all fruit and the set of all food that you enjoy eating. The intersection of these two sets is simply the set of all fruits that you enjoy eating. For your colon's sake, I hope that this is not the empty set.

Definition: Set Disjointness For any sets S and T, S and T are disjoint if, and only if, $S \cap T = \emptyset$.

In other words, two sets are disjoint when they have absolutely no elements in common. For instance, the set of all apples and the set of all oranges are disjoint because these two sets have nothing in common. (I am obviously assuming it is impossible for any apple to also be an orange. Maybe some crazy genetically modified fruit exists violating this assumption, though I doubt it.)

Definition: Union For For any sets A and B, there exists $A \cup B$ (the union of A and B), such that for any x, $x \in A \cup B$ if, and only if, $x \in A$ or $x \in B$.

Informally, the union of two sets is just collects the contents of both sets together into a single set. So, for instance, the union of the set of all apples and the union of the set of all oranges is simply the set of all apples and oranges.

Fact Set intersection and set union are both **commutative** operations. That is, for all sets S and T, the following hold:

- $S \cap T = T \cap S$, and
- $S \cup T = T \cup S$.

This means that the order in which the sets are presented, with either a $\cap$ or a $\cup$ between them, does not matter for set intersection or set union.

Fact Set intersection and set union are both **associative** operations. That is, for all sets S, T, and W, the following hold:

- $(S \cap T) \cap W = T \cap (S \cap W)$, and
- $(S \cup T) \cup W = S \cup (T \cup W)$.

This means that the location of the parentheses is flexible when dealing with either *all* $\cap$'s or *all* $\cup$'s.

Fact In certain cases, the following may hold:

- $(S \cap T) \cup W \neq S \cap (T \cup W)$

This means that if there are *both* $\cap$'s and $\cup$'s involved, you need to be really careful where you put the parentheses!

So, for instance, consider that $(\{1, 2\} \cap \{2, 3\}) \cup \{4, 5\} = \{2, 4, 5\}$ (take a moment to verify that this is correct). Also notice that $\{1, 2\} \cap (\{2, 3\} \cup \{4, 5\}) = \{2\}$ (again, take a moment to verify that this is correct). Obviously $\{2, 4, 5\} \neq \{2\}$. Therefore, the following is the case here:

- $(\{1, 2\} \cap \{2, 3\}) \cup \{4, 5\} \quad \neq \quad \{1, 2\} \cap (\{2, 3\} \cup \{4, 5\})$

Once again, I repeat: **when dealing with both $\cap$'s and $\cup$'s, you need to be really careful where you put the parentheses!**

Two Mutually Recursive Notions

Definition: Set Complement For any set S, there exists a set $\bar{S}$ (the complement of S), such that $S \cap \bar{S} = \emptyset$ and $S \cup \bar{S} = \mathcal{U}$.

Definition: Universal Set For any set S, there exists a set $\mathcal{U}$ (the universal set) for S, such that $S \subseteq \mathcal{U}$ and $S \cup \bar{S} = \mathcal{U}$.

Informally, the idea of set complement should be pretty easy. The complement of a set S is basically a set containing everything *not* in S. The concept of a universal set $\mathcal{U}$ is necessary to give a boundary to what could possibility be in $\bar{S}$.

For instance, suppose I have the set $\{2, 4, 6\}$. What is the complement of this set? Well, that all depends on what the particular universal set $\mathcal{U}$ is involved here. If $\mathcal{U}$ is the set of literally everything, then the compliment of $\{2, 4, 6\}$ would look like $\{1, 1.5, 3, 5, \text{apples}, \text{ oranges}, \text{ you}, \text{ your mother}, \text{ etc. etc.}\ldots\}$. In other words, this complement is not very meaningful, since there is nothing sensible unifying these things. However, if in this case $\mathcal{U} = \{1, 2, 3, 4, 5, 6, 7, 8, 9, 10\}$, then the complement of $\{2, 4, 6\}$ is bounded in a sensible way. That is, this complement is then {1, 3, 5, 7, 8, 9, 10}.

Consequently, when dealing with set complement, you need to be clear on what the relevant universal set $\mathcal{U}$ that you are using.