Introduction to Logical Reasoning

Lecture #14

Introduction to Natural Deduction

Professor David Emmanuel Gray

Carnegie Mellon University in Qatar
Northwestern University in Qatar

A Long Argument

Consider the following argument:

1. $A \rightarrow B$.
2. $B \rightarrow C$.
3. $C \rightarrow D$.
4. $\sim D$.
5. $A \vee E$.

$\therefore$ E.

Constructing a truth table for this would be tedious! Since there are five simple positive statements involved, there would be $2^5 = 32$ rows!!

The Truth Table Monster!

A	B	C	D	Conclusion E	Premise 4 ~D	Premise 1 A → B	Premise 2 B → C	Premise 3 C → D	Premise 5 A ∨ E
T	T	T	T	T	F	T	T	T	T
T	T	T	T	F	F	T	T	T	T
T	T	T	F	T	T	T	T	F	T
T	T	T	F	F	T	T	T	F	T
T	T	F	T	T	F	T	F	T	T
T	T	F	T	F	F	T	F	T	T
T	T	F	F	T	T	T	F	T	T
T	T	F	F	F	T	T	F	T	T
T	F	T	T	T	F	F	T	T	T
T	F	T	T	F	F	F	T	T	T
T	F	T	F	T	T	F	T	F	T
T	F	T	F	F	T	F	T	F	T
T	F	F	T	T	F	F	T	T	T
T	F	F	T	F	F	F	T	T	T
T	F	F	F	T	T	F	T	T	T
T	F	F	F	F	T	F	T	T	T
F	T	T	T	T	F	T	T	T	T
F	T	T	T	F	F	T	T	T	F
F	T	T	F	T	T	T	T	F	T
F	T	T	F	F	T	T	T	F	F
F	T	F	T	T	F	T	F	T	T
F	T	F	T	F	F	T	F	T	F
F	T	F	F	T	T	T	F	T	T
F	T	F	F	F	T	T	F	T	F
F	F	T	T	T	F	T	T	T	T
F	F	T	T	F	F	T	T	T	F
F	F	T	F	T	T	T	T	F	T
F	F	T	F	F	T	T	T	F	F
F	F	F	T	T	F	T	T	T	T
F	F	F	T	F	F	T	T	T	F
F	F	F	F	T	T	T	T	T	T
F	F	F	F	F	T	T	T	T	F

A Shorter Form of Assessment

But there is a more "natural" way to show that this is deductively valid:

1. $A \rightarrow B$.
2. $B \rightarrow C$.
3. $C \rightarrow D$.
4. $\sim D$.
5. $A \vee E$.

$\therefore$ E.

6. $A \rightarrow C$. 1, 2; Hypothetical Syllogism.
7. $A \rightarrow D$. 6, 3; Hypothetical Syllogism.
8. $\sim A$. 7, 4; *Modus Tollens.*
9. E. 5, 8; Disjunctive Syllogism.

Natural Deduction

Natural deduction is a method of deriving the conclusion of a deductive argument by using rules of inference (or established argument patterns). With the right set of rules, it is possible to construct a formal proof of validity for any deductively valid argument. Once mastered, this is far more efficient, elegant, and illuminating than simply checking validity with a truth table.

For this course, we will focus on nine common rules of inference.

Modus Ponens (M.P.)

Recall that the pattern for M.P. says that affirming both (1) a hypothetical and (2) its antecedent allows you to also (∴) affirm its consequent:

1. $p \rightarrow q$.
2. p.

∴ q.

Examples of Using M.P.

The following is a valid argument. Use natural deduction to construct that argument's formal proof of validity. This proof will only require *one* step.

1. $A \rightarrow B$.
2. A.

$\therefore$ B.

Examples of Using M.P.

The following is a valid argument. Use natural deduction to construct that argument's formal proof of validity. This proof will only require *one* step.

1. $A \rightarrow B$.
2. A.

∴ B.

3. B. 1, 2; M.P.

We add a new numbered statement, stating the inference rule used to get it along with the number of the premises used with that rule. In this case, we get the argument's conclusion right away just by using M.P.

Examples of Using M.P.

The following is a valid argument. Use natural deduction to construct that argument's formal proof of validity. This proof will only require *one* step.

1. $A \rightarrow B$.
2. A.

$\therefore$ B.

3. B. 1, 2; M.P.

Just put A in for *p* and B in for *q*, and this has the same pattern as M.P. You saw this before, when we covered argument patterns.

Examples of Using M.P.

The following is a valid argument. Use natural deduction to construct that argument's formal proof of validity. This proof will only require *one* step.

1. $A \rightarrow B$.
2. A.

∴ B.

3. B. 1, 2; M.P.

The **first** number says which line in the proof is the first line for M.P. (the one affirming the hypothetical), while the **second** number tells us which line is the second line for M.P. (the one affirming the antecedent).

Examples of Using M.P.

The following is a valid argument. Use natural deduction to construct that argument's formal proof of validity. This proof will only require *one* step.

1. $A \rightarrow B$.
2. A.

$\therefore$ B.

3. B. 1, 2; M.P.

So this completes the proof, explaining how the conclusion follows from the premises.

Examples of Using M.P.

The following is a valid argument. Use natural deduction to construct that argument's formal proof of validity. This proof will only require *one* step.

1. C.
2. $C \rightarrow F$.

$\therefore$ F.

Examples of Using M.P.

The following is a valid argument. Use natural deduction to construct that argument's formal proof of validity. This proof will only require *one* step.

1. C.
2. $C \rightarrow F$.

∴ F.

3. F. 2, 1; M.P.

We just put C in for *p*, and put F in for *q*, and this then has the same pattern as M.P.

Examples of Using M.P.

The following is a valid argument. Use natural deduction to construct that argument's formal proof of validity. This proof will only require *one* step.

1. C.
2. $C \rightarrow F$.

∴ F.

3. F. 2, 1; M.P.

Even if the order of the premises is reversed, the rule still applies. Just put the number labels in *correct order* for the step in the proof. For M.P., the line number affirming the hypothetical always goes first.

Examples of Using M.P.

The following is a valid argument. Use natural deduction to construct that argument's formal proof of validity. This proof will only require *one* step.

1. C.
2. $C \rightarrow F$.

$\therefore$ F.

3. F.   2, 1; M.P.

So this completed the proof!

Examples of Using M.P.

The following is a valid argument. Use natural deduction to construct that argument's formal proof of validity. This proof will only require *one* step.

1. $\sim(D \& Z) \rightarrow (A \rightarrow D)$.
2. $\sim(D \& Z)$.

$\therefore$ $A \rightarrow D$.

Examples of Using M.P.

The following is a valid argument. Use natural deduction to construct that argument's formal proof of validity. This proof will only require *one* step.

1. $\sim(D \& Z) \rightarrow (A \rightarrow D)$.
2. $\sim(D \& Z)$.

$\therefore$ $A \rightarrow D$.

3. $A \rightarrow D$. 1, 2; M.P.

We just put $\sim(D \& Z)$ in for *p*, and put $A \rightarrow D$ in for *q*, and this then has the same pattern as M.P.

Examples of Using M.P.

The following is a valid argument. Use natural deduction to construct that argument's formal proof of validity. This proof will only require *one* step.

1. $\sim(D \& Z) \rightarrow (A \rightarrow D)$.
2. $\sim(D \& Z)$.

$\therefore$ $A \rightarrow D$.

3. $A \rightarrow D$. 1, 2; M.P.

So even if the statements are more complex, the rule still applies as long as the general pattern conforms to the rule of inference.

Familiar Rules of Inference

1. Modus Ponens (M.P.)

1. $p \rightarrow q$.
2. p.

$\therefore$ q.

2. Modus Tollens (M.T.)

1. $p \rightarrow q$.
2. $\sim q$.

$\therefore$ $\sim p$.

3. Hypothetical Syllogism (H.S.)

1. $p \rightarrow q$.
2. $q \rightarrow r$.

$\therefore$ $p \rightarrow r$.

4. Disjunctive Syllogism (D.S.)

1. $p \vee q$.
2. $\sim p$.

$\therefore$ q.

New Rules of Inference

1. Modus Ponens (M.P.)

1. $p \rightarrow q$.
2. p.

$\therefore$ q.

2. Modus Tollens (M.T.)

1. $p \rightarrow q$.
2. $\sim q$.

$\therefore$ $\sim p$.

3. Hypothetical Syllogism (H.S.)

1. $p \rightarrow q$.
2. $q \rightarrow r$.

$\therefore$ $p \rightarrow r$.

4. Disjunctive Syllogism (D.S.)

1. $p \vee q$.
2. $\sim p$.

$\therefore$ q.

5. Constructive Dilemma (C.D.)

1. $(p \rightarrow q) \,\&\, (r \rightarrow s)$.
2. $p \vee r$.

$\therefore$ $q \vee s$.

6. Absorption (Abs.)

1. $p \rightarrow q$.

$\therefore$ $p \rightarrow (p \,\&\, q)$.

7. Simplification (Simp.)

1. $p \,\&\, q$.

$\therefore$ p.

8. Conjunction (Conj.)

1. p.
2. q.

$\therefore$ $p \,\&\, q$.

9. Addition (Add.)

1. p.

$\therefore$ $p \vee q$.

Constructive Dilemma (C.D.)

1. $(p \rightarrow q) \,\&\, (r \rightarrow s)$.
2. $p \vee r$.

∴ $q \vee s$.

The idea is that (1) a dilemma is affirmed along with (2) the disjunction of the starting points for that dilemma. From these, we may therefore (∴) affirm the disjunction of the end points for that dilemma.

In Greek, "dilemma" means "two paths". So line 1 of the constructive dilemma asserts the conjunction of two "paths" (the hypotheticals): the first "path" goes from *p* to *q* and the second "path" goes from *r* to *s*. So if you know you start in *p* or *r*, then you know you will end up in *q* or *s*.

Absorption (Abs.)

$$\begin{array}{ll} 1. & p \rightarrow q. \\ \hline \therefore & p \rightarrow (p \,\&\, q). \end{array}$$

This is a rather technical, though very useful, rule.

The idea is that we know trivially that if p is true then p is true (i.e., it should be obvious that $p \rightarrow p$). Building on that, we (1) affirm any hypothetical, so we may therefore (∴) affirm that hypothetical, but with the antecedent p now also part of the consequent (as the first conjunct with the old consequent q now the second conjunct).

Simplification (Simp.)

1.	$p \;\&\; q$.
$\therefore$	p.

The idea is that (1) a conjunction is affirmed. From this, we may therefore (∴) affirm the first conjunct alone.

Conjunction (Conj.)

1. p.
2. q.

$\therefore$ $p \,\&\, q$.

The idea is that (1, 2) two statements are affirmed. From these, we may therefore (∴) affirm the conjunction combining both statements.

Addition (Add.)

1. p.

$\therefore$ $p \vee q$.

The idea is that (1) a statement is affirmed. From this, we may therefore ($\therefore$) affirm a disjunction with that statement as the first disjunct. The second disjunct may be *any* other statement.

People generally do not like this rule because it is like magic: that second disjunct q just appears right out of nowhere! Even so, this is perfectly logical. Assuming that p is true (which line 1 has us do) means that $p \vee q$ must also be true. Why? Because disjunction asserts the truth of *at least one* of its disjuncts, and, in this case, it must be p.

The Nine Rules of Inference

1. Modus Ponens (M.P.)

1. $p \rightarrow q.$
2. $p.$

$\therefore\ q.$

2. Modus Tollens (M.T.)

1. $p \rightarrow q.$
2. $\sim q.$

$\therefore\ \sim p.$

3. Hypothetical Syllogism (H.S.)

1. $p \rightarrow q.$
2. $q \rightarrow r.$

$\therefore\ p \rightarrow r.$

4. Disjunctive Syllogism (D.S.)

1. $p \vee q.$
2. $\sim p.$

$\therefore\ q.$

5. Constructive Dilemma (C.D.)

1. $(p \rightarrow q) \,\&\, (r \rightarrow s).$
2. $p \vee r.$

$\therefore\ q \vee s.$

6. Absorption (Abs.)

1. $p \rightarrow q.$

$\therefore\ p \rightarrow (p \,\&\, q).$

7. Simplification (Simp.)

1. $p \,\&\, q.$

$\therefore\ p.$

8. Conjunction (Conj.)

1. $p.$
2. $q.$

$\therefore\ p \,\&\, q.$

9. Addition (Add.)

1. $p.$

$\therefore\ p \vee q.$

Pattern Matching

Given all these rules, the first thing to practice is recognizing patterns in arguments. That is, when given an argument, can you see how the rules of inference might be applied.

Argument 1

The following is a valid argument. Use natural deduction to construct that argument's formal proof of validity. This proof will only require *one* step.

1. $(A \,\&\, C) \rightarrow C.$

$\therefore\ (A \,\&\, B) \rightarrow [(A \,\&\, B) \,\&\, C].$

Argument 2

The following is a valid argument. Use natural deduction to construct that argument's formal proof of validity. This proof will only require *one* step.

1. $(D \vee E) \,\&\, (F \vee G)$.

$\therefore$ $(D \vee E)$.

Learning Natural Deduction

There are only three ways to learn natural deduction:

1. Practice,
2. Practice, and
3. Practice.

If you do not practice this, then you will not be able to do it. I trust you now understand *modus ponens* and *modus tollens*, so you can follow the implications here.

Appendix: Different Symbols

The logical symbols used by Vaughn are sometimes different from those used by Copi and Cohen. I will stick to using the symbols from Vaughn, but here is a table for translating the various symbols:

Logical Operator	Vaughn	Copi & Cohen
Conjunction	& (ampersand)	• (dot)
Negation	~ (tilde)	~ (tilde)
Disjunction	∨ (wedge)	∨ (wedge)
Implication	→ (arrow)	⊃ (horseshoe)
Equivalence	None/Not Used	≡ (triple-bar)
Therefore	∴ (triple-dot)	∴ (triple-dot)

Next Class...

We will start to look at longer proofs of natural deduction and continue practicing pattern matching.

Also, please do not forget to turn in your response to the Lecture #14 Questionnaire on your way out.