A FAMILY OF VALUES FOR n-PERSON COOPERATIVE GAMES

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Abstract. A family of solution values is derived for n-person, cooperative, transferable utility games by relaxing the Axiom of Efficiency for the Shapley value. The Shapley value is shown to always be a member of this collection. In a special case, a geometric relationship between this family and the Shapley value is presented. Also, income redistribution and the role of dummy players in cooperative games is examined.

1 Introduction

Many economic models assume that people only pursue their own material interests and do not really care about social goals. There are, however, examples of economic behavior induced by altruism. These include voluntary reductions of water use during droughts, conservation of energy during fuel shortages, donations to public television stations, and many forms of charitable work. People care not only about their own well-being, but also about the well-being of others (see Rabin [10]).

The Shapley value [11] is a well-known solution concept for n-person cooperative games. This paper offers an extension to the Shapley value by relaxing these axioms and introducing the concept of a rationing function. In addition, a particular rationing function will be proposed along with a geometric and economic interpretation.

2 Preliminaries

This paper considers n-person cooperative games in characteristic function form. Formally, let $N = \{1, \ldots, n\}$ denote the set of players. Each set $S \subseteq N$ represents a coalition of players in $N$.

Definition 1 A game $(N, v)$ is in characteristic function form if $v$ is a real-valued function defined on subsets of the player set $N$ with $v(\emptyset) = 0$.

Often, the value $v(S)$ represents the maximum value to coalition $S$ in a two-player game played by $S$ and $N \setminus S$.

Definition 2 A game $(N, v)$ is superadditive if $v(S \cup T) \geq v(S) + v(T)$ for all $S$ and $T$ such that $S \cap T = \emptyset$.

Definition 3 (Friedman [3, 4]) The game $(N, v)$ has transferable utility if, for each $S \subseteq N$, the scalar value $v(S)$ can be freely apportioned among the members of $S$.

Definition 4 A carrier for a game $(N, v)$ is a coalition $T \subseteq N$ such that $v(T) \geq v(S \cap T)$ for any $S \subseteq N$.

This definition is slightly different from the one given by Shapley [11].

Definition 5 (Owen [9]) Let $(N, v)$ be an n-person game, and let $\pi \in \Pi(N)$. Then, the game $(N, \pi v)$ is defined as the game $(N, u)$, such that

$u(\{\pi(i_1), \pi(i_2), \ldots, \pi(i_{|S|})\}) = v(S)$

for any coalition $S = \{i_1, i_2, \ldots, i_{|S|}\}$.

Definition 6 (Friedman [3]) Let $(N, v)$ be an n-person game. The marginal value, $c_S(v)$ for coalition $S \subseteq N$ is given by

$c_S(v)(i) = \pi(i)$

for all $i \in N$, and

$c_S(v)(v) = v(S) - \sum_{L \subset S} c_L(v)$

for all $S \subseteq N$ with $|S| \geq 2$.

The marginal value of $S$ can also be computed by using the formula

$c_S(v) = \sum_{L \subseteq S} (-1)^{|S|-1} v(L)$.

3 Results

The economic strength of individual players of a cooperative game has been studied for many years (Shapley [11], Maschler [6], Coleman [1], Monderer et al. [7], and Ifarra and Usategui [5]). The best-known solution concept is the Shapley value [11], which has many desirable properties. For example, every superadditive characteristic function game has a unique value which satisfies Shapley’s axioms.

It will be shown that the Shapley value is one member of a family of solutions. This collection arises from a relaxation of the Shapley axioms.

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3.1 Relaxed axioms
Let \( \phi(v) = (\phi_1(v), \phi_2(v), \ldots, \phi_n(v)) \) be an \( n \)-dimensional vector satisfying the following three axioms:

**Axiom 1 (Symmetry)** For each \( \pi \in \Pi(N) \), \( \phi_{\pi_0}(\pi v) = \phi_i(v) \).

**Axiom 2 (Rationing)** For each carrier \( C \) of \((N,v)\)
\[
\sum_{i \in C} \phi_i(v) = g(C)v(C) \quad \text{with} \quad \frac{|C|}{n} \leq g(C) \leq 1.
\]

**Axiom 3 (Law of Aggregation)** For any two games \((N,v)\) and \((N,w)\)
\[
\phi(v + w) = \phi(v) + \phi(w).
\]

The function \( g(C) \) is called the rationing function. It can be any real-valued function defined on attributes of the carrier \( C \) with range \( [\frac{|C|}{n}, 1] \). Note that if the game \((N,v)\) is superadditive, then \( g(C) = 1 \) yields Shapley’s [11] original axioms.

3.2 A family of values
A particular choice of the rationing function \( g(C) \) produces an important set of solutions. Let \( N = \{1, \ldots, n\} \) and let \( c \equiv |C| \) for \( C \subseteq N \). Given the value of the parameter \( r \in [\frac{1}{2}, 1] \) consider the real-valued function
\[
g(C) \equiv g(c,r) = \frac{(n-c)r + (c-1)}{n-1}.
\]

The function \( g(C) \) specifies the distribution of revenue among the players of a game. Note that this function can be rewritten as
\[
g(c,r) = 1 - (1 - r) \frac{n - c}{n - 1}.
\]

Using this form of the rationing function results in the following:

**Theorem 1** Let \((N,v)\) be an \( n \)-person cooperative transferable utility game. Suppose \( v(N) = \max_{S \subseteq N} \{v(S)\} \).

For each \( r \in [\frac{1}{2}, 1] \), there exists a unique value, \( \phi_{s,r}(v) \), for each player \( i \) satisfying the three axioms. Moreover, this unique value is given by
\[
\phi_{s,r}(v) = \sum_{S \subseteq N \atop S \ni i} \left( 1 - \frac{|S|}{|N|} \right) p \quad \text{if} \quad \frac{1}{n-1} \leq \frac{|S|}{|N|} \leq 1
\]
where \( p = 1 - \frac{1}{n-1} \).

**Proof.** It can be easily verified that \( \phi_{s,r}(v) \) satisfies Axioms 1 and 3. In order to show that \( \phi_{s,r}(v) \) satisfies Axiom 2, first note that it is linear in \( r \) for \( r \in [\frac{1}{n-1}, 1] \). Also
\[
\phi_{s,r}(v) = \frac{v(N)}{n} \quad \text{for all} \quad i \in N.
\]

So,
\[
\phi_{s,r}(v) = \frac{1 - r}{n-1} \phi_{s,1}(v),
\]

Note that \( \sum_{i \in C} \phi_{s,1}(v) = v(C) \). Therefore,
\[
\phi_{s,r}(v) = \frac{(1 - r)v(N) + (n-1)\phi_{s,1}(v)}{n-1}.
\]

This is always true if \((N,v)\) is superadditive.

3.3 A geometric interpretation
Let \( a_{S,J} \) denote the coefficients of \( v(S) \) in Equation (1) for \( \phi_{s,r}(v) \) and let \( a_S = (a_{S,1}, \ldots, a_{S,n}) \). Table 1 lists the coefficients of \( c_S(v) \) for a 3-person cooperative transferable utility game. The values for the vectors \( a_S \) can be geometrically interpreted using barycentric coordinates in \( \mathbb{R}^2 \) as shown in Figure 1.

![Figure 1: Geometric representation of \( a_S \) for \( n = 3 \)](image)

Since \( v(N) = \max_{S \subseteq N} \{v(S)\} \), then, for any carrier \( C \), \( v(N) = v(N \cap C) = v(C) \). So \( v(N) = v(C) \). Hence, for those players within a carrier \( C \),
\[
\sum_{i \in C} \phi_{s,r}(v)
= \sum_{i \in C} \left[ (1 - r)v(N) + (nr - 1)\phi_{s,1}(v) \right]
= \left( \frac{1 - r}{n-1} \right) v(N) + \frac{nr - 1}{n-1} \sum_{i \in C} \phi_{s,1}(v)
= \frac{(n-c)r + (c-1)}{n-1} v(C)
= g(c,r)v(C).
\]

To show uniqueness, reverse the steps in Equations (2–6). In doing so, note that when \( r = 1 \) the values \( \phi_{s,1}(v) \) are the Shapley values [11] and provide the unique solution to
\[
\sum_{i \in C} v(C) = \sum_{i \in C} \phi_{s,1}(v).
\]

Therefore,
\[
\phi_{s,r}(v) = \frac{(1 - r)v(N) + (nr - 1)\phi_{s,1}(v)}{n-1}
\]

is the unique solution to
\[
\sum_{i \in C} \phi_{s,r}(v) = \frac{(n-c)r + (c-1)}{n-1} v(C).
\]

This is always true if \((N,v)\) is superadditive.
The core

The concept of a core will be used for the example in the next section.

Definition 7 (Owen [9]) An imputation for the n-person game \((N, v)\) is an \(n\)-dimensional vector \(x = (x_1, x_2, \ldots, x_n)\) satisfying

\[
\sum_{i \in N} x_i = v(N) \quad (7)
\]

\[
x_i \geq v(\{i\}) \quad \text{for all } i \in N. \quad (8)
\]

Definition 8 (Owen [9]) Let \(x\) and \(y\) be two imputations, and let \(S \subseteq N\) be a coalition. The imputation \(x\) dominates \(y\) through \(S\) if

\[
x_i > y_i \quad \text{for all } i \in S \quad (9)
\]

\[
\sum_{i \in S} x_i \leq v(S). \quad (10)
\]

Definition 9 (Owen [9]) The core, \(\mathcal{C}(v) \subseteq \mathbb{R}^n\), for the game \((N, v)\) is a set of imputations \(x\) such that any \(y \notin \mathcal{C}(v)\) does not dominate any \(x \in \mathcal{C}(v)\) through any \(S \subseteq N\).

The core contains those solutions that distribute the proceeds from the game so that all possible coalitions of players are satisfied. Given an imputation, \(x\), in the core, no coalition can form whose members can get more than \(x\). This property makes the core an intuitively desirable solution concept (see Edgeworth [2]).

5 An example

This solution approach can be applied to a problem suggested by Nowak and Radzik [8]. Consider a three-person game where

\[
v(\{1\}) = v(\{2\}) = 0, \quad v(\{3\}) = 1,
\]

\[
v(\{1, 2\}) = 3.5, \quad v(\{1, 3\}) = v(\{2, 3\}) = 0,
\]

\[
v(\{1, 2, 3\}) = 5.
\]

The Shapley value for this game is

\[
\phi(v) = \left(\frac{25}{11}, \frac{25}{11}, \frac{50}{11}\right).
\]

Note that the Shapley value will not necessarily satisfy the condition of individual rationality

\[
\phi(v) \geq v(\{i\})
\]

when the characteristic function \(v\) is not superadditive. That is the case here since \(\phi_S(v) < v(\{3\})\).

### Table 1: Coefficients of \(c_S(v)\) for a 3-person cooperative transferable utility game

<table>
<thead>
<tr>
<th>(c(1)(v))</th>
<th>(c(2)(v))</th>
<th>(c(3)(v))</th>
<th>(c(1,2)(v))</th>
<th>(c(1,3)(v))</th>
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<th>(c(1,2,3)(v))</th>
</tr>
</thead>
<tbody>
<tr>
<td>(\phi_1(v))</td>
<td>(\phi_2(v))</td>
<td>(\phi_3(v))</td>
<td>(\phi_{1,2}(v))</td>
<td>(\phi_{1,3}(v))</td>
<td>(\phi_{2,3}(v))</td>
<td>(\phi_{1,2,3}(v))</td>
</tr>
<tr>
<td>(a_{1,2} = (r, r, r))</td>
<td>(a_{2,3} = (r, r, r))</td>
<td>(a_{3,1} = (r, r, r))</td>
<td>(a_{1,3} = (r, r, r))</td>
<td>(a_{2,1} = (r, r, r))</td>
<td>(a_{1,2,3} = (\frac{1}{2}, \frac{1}{2}, \frac{1}{2}))</td>
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</tr>
</tbody>
</table>

### Figure 2: The family of values and the core

The solidarity value (Nowak and Radzik [8]) \(\psi(v)\) of this game is

\[
\psi(v) = \left(\frac{16}{24}, \frac{16}{24}, \frac{13}{24}\right)
\]

and is in the core of \((N, v)\). For every \(r \in \left[\frac{1}{3}, 1\right]\), the general form of the family of values is

\[
\phi_r(v) = \left(\frac{35 + 15r}{24}, \frac{35 + 15r}{24}, \frac{50 - 30r}{24}\right).
\]

The diagram in Figure 2 shows the relationship between the family of values and the core. Note that, in the diagram,

\[
A = \left(\frac{24}{11}, \frac{24}{11}, \frac{24}{11}\right) \quad \text{(the Shapley value)}
\]

\[
B = \left(\frac{5}{11}, \frac{5}{11}, \frac{5}{11}\right).
\]

Neither of these extreme values of the family of values is in the core for this game. However, those solutions for \(\frac{1}{3} \leq r \leq \frac{1}{2}\) are elements of the core.

6 The Role of Dummy Players

The previous sections have offered an extension of Shapley’s solution concept based on a relaxation of the Shapley axioms [11]. A key difference is in the treatment of those players who do not materially contribute to any coalition.

Definition 10 (von Neumann and Morgenstern [12]) Player \(i\) is a dummy player (or null player) in a game \((N, v)\) if and only if

\[
v(S \cup \{i\}) = v(S) + v(\{i\})
\]

for all \(S \subseteq N\) with \(i \notin S\).
For superadditive games, Shapley [11] provides the following:

**Theorem 2** For all \( i \in N \), \( \phi_i(v) \geq v(\{i\}) \). Also, \( \phi_i(v) = v(\{i\}) \) if and only if \( i \) is a dummy player.

In other words, dummy players in a game do not receive any benefit beyond their individual worth. (see von Neumann and Morgenstern [12]).

Nowak and Radzik [8] offer the following example related to social welfare and income redistribution: Players 1, 2, and 3 are brothers living together. Players 1 and 2 can make a profit of one unit, that is, \( v(\{1, 2\}) = 1 \). Player 3 is a disabled person and can contribute nothing to any coalition. Therefore, \( v(\{1, 2, 3\}) = 1 \). Also, \( v(\{1, 3\}) = v(\{2, 3\}) = 0 \) and \( v(\{i\}) = 0 \) for every Player \( i \).

The Shapley value of this game is

\[
\phi(v) = \left( \frac{1}{2}, \frac{1}{2}, 0 \right)
\]

and for the family of values, we get

\[
\phi_r(v) = \left( \frac{1 + r}{4}, \frac{1 + r}{4}, \frac{1 - r}{2} \right)
\]

for \( r \in \left[ \frac{1}{3}, 1 \right] \). Every \( r \) yields a solution satisfying individual rationality, but, in this case, \( \phi_r(v) \) belongs to the core only when it equals the Shapley value (\( r = 1 \)).

For this particular game, the solidarity value is a member of the family when \( r = \frac{5}{9} \). Nowak and Radzik propose this single value as a “better” solution for the game \((N, v)\) than its Shapley value. They suggest that it could be used to include subjective social or psychological aspects in a cooperative game.

### 7 Conclusion

This paper has presented an extension of the Shapley value using a rationing function to redistribute the value of carriers of the game. This allows decision-makers to incorporate additional social and economic factors when developing solutions.

The resulting family of values can provide solutions that belong the core, even though the Shapley value is not a member. Furthermore, on occasion, it can provide individually rational solutions for non-superadditive games.

### References


