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MIXED CONVOLVED ACTION PRINCIPLES FOR DYNAMICS OF LINEAR POROELASTIC CONTINUA

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ABSTRACT

Although Lagrangian and Hamiltonian analytical mechanics represent perhaps the most remarkable expressions of the dynamics of a mechanical system, these approaches also come with limitations. In particular, there is inherent difficulty to represent dissipative processes and the restrictions placed on end point variations are not consistent with the definition of initial value problems. The present work on poroelastic media extends the recent formulation of a mixed convolved action to address a continuum dynamical problem with dissipation through the development of a new variational approach. The action in this proposed approach is formed by replacing the inner product in Hamilton's principle with a time convolution. As a result, dissipative processes can be represented in a natural way and the required constraints on the variations are consistent with the actual initial and boundary conditions of the problem. The variational formulations developed here employ temporal impulses of velocity, effective stress, pore pressure and pore fluid mass flux as primary variables in this mixed approach, which also uses convolution operators and fractional calculus to achieve the desired characteristics. The resulting mixed convolved action is formulated in both the time and frequency domains to develop two new stationary principles for dynamic poroelasticity. In addition, the first variation of the action provides a temporally well-balanced weak form that leads to a new family of finite element methods in time, as well as space.

INTRODUCTION

While Hamilton's principle of stationary action has long been regarded as perhaps the most elegant formulation describing the dynamics of a physical system, it also has notable shortcomings, mainly the inability to model dissipative phenomena and the inconsistency of variations with respect to the specified initial conditions. In order to accommodate irreversible phenomena, a Rayleigh dissipation function can be introduced, along with a prescribed set of rules for taking the variations. While these methods have enjoyed great success for a range of problems, it is well known that such formulations do not lead to true variational principles in a strict mathematical sense.

In order to resolve these main shortcomings of Hamilton's principle, the concept of mixed convolved action (MCA) has been developed in recent work for linear lumped parameter single degree of freedom dynamical systems [1,2] and linear elastodynamic continua [3]. Here we extend the stationary principle of mixed convolved action to consider the dynamic response of linear poroelastic media, based upon the Biot theory [4-6]. Notably, this novel formulation is demonstrated to recover all of the governing partial differential equations, boundary conditions and initial conditions of Biot poroelasticity as the Euler-Lagrange equations associated with the mixed convolved action functional. This MCA functional is written in terms of mixed impulsive variables, fractional derivatives and the convolution of convolutions. Thus, a single scalar functional encapsulates all of the conservative and non-conservative aspects of dynamic poroelastic response and a stationary principle is derived without the need for ad hoc assumptions concerning the variations.

Beyond the theoretical significance of the principle of stationary mixed convolved action for linear poroelastic dynamic response, these concepts lead directly to the development of novel computational methods involving finite element representations over both space and time. The present paper includes the theoretical formulation in terms of primary mixed variables, which include the impulses of velocity, pressure, stress and flux. From analysis of the mixed convolved action, the former two variables require C^0 continuity over space, while the latter two may be defined with C^{-1} spatial continuity. Thus, for a two-dimensional numerical implementation, displacement and pressure impulse can be defined using standard three-node linear triangular finite elements, with stress impulse and relative pore fluid displacement defined by independent constants over each finite element. Meanwhile, all four primary impulsive variables require C^0 temporal continuity. Consequently, linear temporal shape functions can be used to represent all of the primary variables over each time step.

The focus of the present work is on the theoretical formulations needed to develop new stationary variational principles for dynamic poroelasticity in both the time and frequency domains. For the former case, a brief description of the corresponding space and time finite element methods is also provided.

GOVERNING EQUATIONS

In this section, we direct attention to the continuum problem for poroelasticity involving energy dissipation and develop for the first time a pure variational statement for an irreversible process. As an initial example, we consider viscous flow of a pore fluid as the dissipative process and develop a mixed convolved action formalism for infinitesimal poroelasticity. In a way, this formalism can be regarded as the evolution of previous work on the poroelastic problem in Reference [7], which used instead an inner product action variation based upon Lagrangian energy and Rayleigh dissipation functionals.

For a continuum governed by infinitesimal poroelasticity theory, let v_i and σ_{ij}^e represent the velocity and effective stress of the solid skeleton, respectively. Meanwhile, for the pore fluid, let p and q_i denote the pore pressure and the average velocity relative to the solid skeleton, respectively. Then, the impulses of these four quantities are defined as u_i, J_{ij}, π and w_i, respectively, where

$$u_i(t) = \int_0^t v_i(t)dt \tag{1a}$$

$$J_{ij}(t) = \int_0^t \sigma_{ij}^e(t)dt = \int_0^t C_{ijkl}\varepsilon_{kl}(t)dt \tag{1b}$$

$$\pi(t) = \int_0^t p(t)dt \tag{1c}$$

$$w_i(t) = \int_0^t q_i(t)dt \tag{1d}$$

Here, u_i is the solid skeleton displacement and w_i represents the average pore fluid displacement relative to the solid skeleton. A number of dynamic poroelastic formulations are written in terms of u_i and w_i as primary variables, including Biot [6], Predeleanu [8] and Manolis and Beskos [9]. However, following the approach taken in Reference [7], we instead consider mixed formulations written in terms of all four variables. Naturally, in the corresponding rate form, we have for these variables

$$\dot{u}_i = v_i \tag{2a}$$

$$\dot{J}_{ij} = \sigma_{ij}^e = C_{ijkl}\varepsilon_{kl} \tag{2b}$$

$$\dot{\pi} = p \tag{2c}$$

$$\dot{w}_i = q_i \tag{2d}$$

where σ_{ij}^e denotes the effective stress, ε_{ij} represents the total strain tensor and C_{ijkl} is the linear elastic constitutive tensor for the solid skeleton written in terms of drained properties. Meanwhile, the total stress σ_{ij} can be written in terms of the effective stress and pore pressure as

$$\sigma_{ij} = \sigma_{ij}^e - \beta_{ij} p \tag{3}$$

with β_{ij} representing a constitutive tensor for anisotropic poroelastic media relating to compressibility of the two-phase mixture, which reduces to $\beta_{ij} = \beta\delta_{ij}$ for the isotropic case.

In terms of these mixed variables, the governing differential equations for Biot dynamic poroelastic response over the domain Ω take the following form:

$$\rho_o \ddot{u}_k + \rho_f \ddot{w}_k - B_{ijk}\left(\dot{J}_{ij} - \beta_{ij}\dot{\pi}\right) = \overline{f}_k \tag{4a}$$

$$A_{ijkl}\ddot{J}_{kl} - B_{ijk}\dot{u}_k = 0 \tag{4b}$$

$$\frac{1}{Q}\ddot{\pi} + B_i\dot{w}_i + \beta_{ij}B_{ijk}\dot{u}_k = \overline{\Gamma} \tag{4c}$$

$$\frac{\rho_f}{n}\ddot{w}_j + \rho_f\ddot{u}_j + \lambda_{ij}\dot{w}_i + B_j\dot{\pi} = 0 \tag{4d}$$

where ρ_s and ρ_f represent the mass density of the solid and fluid, respectively, while ρ_o is the mass density of the solid-fluid mixture, such that

$$\rho_o = (1-n)\rho_s + n\rho_f \tag{5}$$

Furthermore, n is the porosity and Q is the Biot parameter to account for compressibility of the two phase mixture. In addition, $\overline{f}_k$ represents a specified body force density, while $\overline{\Gamma}$ is a specified volumetric body source rate. The constitutive tensors A_{ijkl} and λ_{ij} are the inverses of the elastic moduli of the solid skeleton C_{ijkl} and the permeability κ_{ij}, respectively. The permeability, in turn, can be written as $\kappa_{ij} = k_{ij}/\eta$, where k_{ij} and η represent the specific permeability and pore fluid viscosity, respectively. Finally, B_i and B_{ijk} represent differential operators that are defined as

$$B_i = \frac{\partial}{\partial x_i} \tag{6a}$$

$$B_{ijk} = \frac{1}{2}\left(\delta_{ik}\delta_{jq} + \delta_{iq}\delta_{jk}\right)\frac{\partial}{\partial x_q} \tag{6b}$$

Notice that equation (4a) represents linear momentum balance, (4b) is the linear elastic effective stress-strain constitutive relation in rate form and (4c) is the pore fluid balance equation with

$$\dot{\zeta} = -\dot{w}_{i,i} + \overline{\Gamma} \tag{7}$$

as the fluid content rate. The remaining governing equation (4d) represents an extended Darcy's law for pore fluid flow.

In addition to the governing differential equations, boundary conditions must be specified. For the simplest form, these can be written:

$$u_k = \overline{u}_k \quad \text{on} \quad \Gamma_v \tag{8a}$$

$$\dot{J}_{kj} n_j - \beta_{kj} \dot{\pi} n_j = \sigma_{ij} n_j = \overline{t}_k \quad \text{on} \quad \Gamma_t \tag{8b}$$

$$\pi = \overline{\pi} \quad \text{on} \quad \Gamma_p \tag{8c}$$

$$\dot{w}_i n_i = \overline{q} \quad \text{on} \quad \Gamma_q \tag{8d}$$

where $\overline{u}_k$ and $\overline{\pi}$ represent essential boundary conditions of displacement and pore pressure impulse applied on the surfaces Γ_v and Γ_p, respectively. Meanwhile, for the natural boundary conditions, $\overline{t}_k$ are the tractions specified on the portion of the surface Γ_t, while $\overline{q}$ represents the specified normal relative fluid volume discharge on Γ_q.

Then, to complete the definition of the Biot poroelastic problem, initial conditions are required. In mixed variables, these take the following form at time zero:

$$\rho_o \dot{u}_k(0) + \rho_f \dot{w}_k(0) - B_{ijk} \left(J_{ij}(0) - \beta_{ij} \pi(0) \right) = \overline{j}_k(0) \tag{9a}$$

$$A_{ijkl} \dot{J}_{kl}(0) - B_{ijk} u_k(0) = 0 \tag{9b}$$

$$\frac{1}{Q} \dot{\pi}(0) + B_i w_i(0) + \beta_{ij} B_{ijk} u_k(0) = \overline{\Upsilon}(0) \tag{9c}$$

$$\frac{\rho_f}{n} \dot{w}_j(0) + \rho_f \dot{u}_j(0) + \lambda_{ij} w_i(0) + B_j \pi(0) = 0 \tag{9d}$$

where $\overline{j}_k$ and $\overline{\Upsilon}$ are the impulses of $\overline{f}_k$ and $\overline{\Gamma}$, respectively.

TIME DOMAIN VARIATIONAL FORMULATION

In Reference [7], an underlying action is defined implicitly for Biot dynamic poroelasticity by identifying Lagrangian and dissipation functions, which are then integrated over time to provide an inner product-based variational formulation. Within this Mixed Lagrangian Formalism (MLF) [10-12], in the presence of dissipative effects, the action is never written in explicit form. Instead, special restricted variations are introduced following the Rayleigh dissipation approach to write the stationarity of the action, which then may be used to produce effective numerical algorithms for dynamical problems.

However, as first noted by Gurtin [13-15] and Tonti [16-19], the use of an inner product operator over time is more consistent with boundary value problems, rather than initial value problems. Tonti, in particular, emphasized that the more appropriate operators are convolution based. Afterwards, Oden and Reddy [20] derived a number of convolution-based formulations for continuum problems, including elastodynamics and thermoelasticity, with the latter including the effects of dissipation. In all of this work, the objective was to recover the governing partial differential equations as the Euler-Lagrange equations of the variational formulation.

In the present work, we strive to recover the complete definition of the initial/boundary value problem, including the governing partial differential equations, boundary conditions and initial conditions, as the Euler-Lagrange equations of a single scalar action functional, which we denote as the Mixed Convolved Action (MCA). We should perhaps mention that, while this action is defined in explicit form, it is not possible to define Lagrange energy or Rayleigh dissipation state functions, because the temporal inner product has been replaced by convolution. Interestingly, if we transform the problem to the frequency or Laplace domain, then the classical Lagrangian formulation becomes a convolution, whereas the mixed convolved action transforms to a simple product.

As noted in the Introduction, previous formulations have been developed for lumped parameter systems [1,2] and for elastodynamic continua [3]. Here, we extend the MCA approach defined in Reference [3] for dynamic poroelastic response. The mixed convolved action for this case can be written:

$$\begin{aligned}
I_{C_p} &= \int_\Omega \left[\tfrac{1}{2} \dot{u}_k * (1-n) \rho_s \dot{u}_k \right] d\Omega \\
&+ \int_\Omega \left[\tfrac{1}{2} \left(\dot{u}_k + \frac{\dot{w}_k}{n} \right) * n \rho_f \left(\dot{u}_k + \frac{\dot{w}_k}{n} \right) \right] d\Omega \\
&- \int_\Omega \left[\tfrac{1}{2} \dot{J}_{ij} * A_{ijkl} \dot{J}_{kl} \right] d\Omega \\
&- \int_\Omega \left[\tfrac{1}{2} \dot{\pi} * \frac{1}{Q} \dot{\pi} - \tfrac{1}{2} \breve{w}_i * \lambda_{ij} \breve{w}_j \right] d\Omega \\
&+ \int_\Omega \left[\tfrac{1}{2} \left(\breve{J}_{ij} * B_{ijk} \breve{u}_k - \breve{u}_k * B_{ijk} \breve{J}_{ij} \right) \right] d\Omega \\
&- \int_\Omega \left[\tfrac{1}{2} \left(\beta_{ij} \breve{\pi} * B_{ijk} \breve{u}_k - \breve{u}_k * B_{ijk} \beta_{ij} \breve{\pi} \right) \right] d\Omega \\
&+ \int_\Omega \left[\tfrac{1}{2} \left(\breve{w}_i * B_i \breve{\pi} - \breve{\pi} * B_i \breve{w}_i \right) \right] d\Omega \\
&- \int_\Omega \left[\breve{u}_k * \breve{\overline{j}}_k \right] d\Omega + \int_\Omega \left[\breve{\pi} * \breve{\overline{\Upsilon}} \right] d\Omega \\
&- \int_{\Gamma_t} \tfrac{1}{2} \left[\breve{u}_k * \breve{\overline{\tau}}_k \right] d\Gamma + \int_{\Gamma_v} \tfrac{1}{2} \left[\breve{\tau}_k * \breve{\overline{u}}_k \right] d\Gamma \\
&- \int_{\Gamma_q} \tfrac{1}{2} \left[\breve{\pi} * \breve{\overline{w}} \right] d\Gamma + \int_{\Gamma_p} \tfrac{1}{2} \left[\breve{w} * \breve{\overline{\pi}} \right] d\Gamma
\end{aligned} \tag{10}$$

with $w = w_i n_i$ and $\tau_i = J_{ji} n_j$. Here and in what follows, the superposed breve symbol represents a left Riemann-Liouville semi-derivative [21,22], defined as

$$\breve{f} = \left(\mathscr{D}_{0^+}^{1/2} f \right)(t) \equiv \frac{1}{\sqrt{\pi}} \frac{d}{dt} \int_0^t \frac{f(\tau)}{(t-\tau)^{1/2}} d\tau \tag{11}$$

for any suitably continuous function $f(t)$, where the non-italic π is the ratio of the circumference to the diameter of a circle.

Notice from (11) that the semi-derivative operator also involves a convolution of the function $f(t)$ with a kernel $1/\sqrt{\pi t}$, so that many of the terms in the mixed convolved action in (10) are actually convolutions of convolutions. In particular, the first of those terms, with action density $\tfrac{1}{2} \breve{w}_i * \lambda_{ij} \breve{w}_j$, models the viscous dissipation. Here, we have the convolution of the semi-derivative of the relative displacement of the pore fluid $\breve{w}_i$ with itself through the inverse permeability tensor λ_{ij}. This captures the history dependence of these irreversible processes, which is something that cannot be done within the classical Lagrangian inner product framework. Interestingly, all of the other terms involving semi-derivatives can be written with balanced orders of the time derivatives across the two distinct variables. As we shall see, this not only leads to a weak form with ideal properties, but also permits recovery of the complete initial/boundary value problem of dynamic poroelasticity.

The next step is to enforce stationarity of the mixed convolved action (10) by setting the first variation equal to zero. Despite the presence of both first- and semi-derivatives with respect to time, this operation is easily performed and the result can be written as follows:

$$
\begin{aligned}
\delta I_{C_p} &= \int_\Omega \left[\delta \dot{u}_k * (1-n)\rho_s \dot{u}_k \right] d\Omega \\
&+ \int_\Omega \left[\left(\delta \dot{u}_k + \frac{\delta \dot{w}_k}{n} \right) * n\rho_f \left(\dot{u}_k + \frac{\dot{w}_k}{n} \right) \right] d\Omega \\
&- \int_\Omega \left[\delta \dot{J}_{ij} * A_{ijkl} \dot{J}_{kl} \right] d\Omega \\
&- \int_\Omega \left[\delta \dot{\pi} * \frac{1}{Q} \dot{\pi} - \delta \breve{w}_i * \lambda_{ij} \breve{w}_j \right] d\Omega \\
&+ \int_\Omega \left[\tfrac{1}{2} \left(\delta \breve{J}_{ij} * B_{ijk} \breve{u}_k - \breve{u}_k * B_{ijk} \delta \breve{J}_{ij} \right) \right] d\Omega \\
&+ \int_\Omega \left[\tfrac{1}{2} \left(\breve{J}_{ij} * B_{ijk} \delta \breve{u}_k - \delta \breve{u}_k * B_{ijk} \breve{J}_{ij} \right) \right] d\Omega \\
&- \int_\Omega \left[\tfrac{1}{2} \left(\beta_{ij} \delta \breve{\pi} * B_{ijk} \breve{u}_k - \breve{u}_k * B_{ijk} \beta_{ij} \delta \breve{\pi} \right) \right] d\Omega \\
&- \int_\Omega \left[\tfrac{1}{2} \left(\beta_{ij} \breve{\pi} * B_{ijk} \delta \breve{u}_k - \delta \breve{u}_k * B_{ijk} \beta_{ij} \breve{\pi} \right) \right] d\Omega \\
&+ \int_\Omega \left[\tfrac{1}{2} \left(\delta \breve{w}_i * B_i \breve{\pi} - \breve{\pi} * B_i \delta \breve{w}_i \right) \right] d\Omega \\
&+ \int_\Omega \left[\tfrac{1}{2} \left(\breve{w}_i * B_i \delta \breve{\pi} - \delta \breve{\pi} * B_i \breve{w}_i \right) \right] d\Omega \\
&- \int_\Omega \left[\delta \breve{u}_k * \breve{\bar{j}}_k \right] d\Omega + \int_\Omega \left[\delta \breve{\pi} * \breve{\bar{\Upsilon}} \right] d\Omega \\
&- \int_{\Gamma_t} \tfrac{1}{2} \left[\delta \breve{u}_k * \breve{\bar{\tau}}_k \right] d\Gamma + \int_{\Gamma_v} \tfrac{1}{2} \left[\delta \breve{\tau}_k * \breve{\bar{u}}_k \right] d\Gamma \\
&- \int_{\Gamma_q} \tfrac{1}{2} \left[\delta \breve{\pi} * \breve{\bar{w}} \right] d\Gamma + \int_{\Gamma_p} \tfrac{1}{2} \left[\delta \breve{w} * \breve{\bar{\pi}} \right] d\Gamma = 0
\end{aligned}
\tag{12}
$$

By using classical and fractional integration-by-parts, we will show that (12) does indeed reproduce all of the elements of the initial/boundary value problem, but first let us establish the weak form to be used as the foundation for a corresponding time-space finite element method for dynamic poroelastic response. Examining the temporal derivatives in (12), we notice that terms appear in which first derivatives of all four field variables (e.g., u_k, J_{ij}, π, w_i) are convoluted with first derivatives of their variations. Consequently, integration-by-parts cannot reduce the maximum level of the temporal derivatives and we will require C^0 continuity of all variables in time. Note, however, that these variables are impulses of velocity, stress, pore pressure and relative fluid velocity, so that the continuity requirements only apply to these impulses. Thus, displacement and relative fluid displacement must be continuous in time, but the usual stress and pore pressure fields may be C^{-1} continuous (or discontinuous) in time.

On the other hand, the spatial derivatives in (12) are confined to terms involving variable pairs, including u_k - J_{ij}, u_k - π and π - w_i pairs. This means that there is an opportunity to reduce the continuity requirements on one variable in each pair. In order to best accomplish this objective, we must choose to perform spatial integration-by-parts to shift all derivatives from J_{ij}, δJ_{ij}, w_i and δw_i to the pair variable in each case. Then, u_k and π will require C^0 continuity in space, while J_{ij} and w_i will need to maintain only C^{-1} continuity. After performing all of these recommended integration-by-parts operations, the weak form becomes:

$$
\begin{aligned}
\delta I_{C_p} &= \int_\Omega \left[\delta \dot{u}_k * (1-n)\rho_s \dot{u}_k \right] d\Omega \\
&+ \int_\Omega \left[\left(\delta \dot{u}_k + \frac{\delta \dot{w}_k}{n} \right) * n\rho_f \left(\dot{u}_k + \frac{\dot{w}_k}{n} \right) \right] d\Omega \\
&- \int_\Omega \left[\delta \dot{J}_{ij} * A_{ijkl} \dot{J}_{kl} \right] d\Omega \\
&- \int_\Omega \left[\delta \dot{\pi} * \frac{1}{Q} \dot{\pi} - \delta \breve{w}_i * \lambda_{ij} \breve{w}_j \right] d\Omega \\
&+ \int_\Omega \left[\left(\delta \breve{J}_{ij} - \beta_{ij} \delta \breve{\pi} \right) * B_{ijk} \breve{u}_k \right] d\Omega \\
&+ \int_\Omega \left[\left(\breve{J}_{ij} - \beta_{ij} \breve{\pi} \right) * B_{ijk} \delta \breve{u}_k \right] d\Omega \\
&+ \int_\Omega \left[\delta \breve{w}_i * B_i \breve{\pi} + \breve{w}_i * B_i \delta \breve{\pi} \right] d\Omega \\
&- \int_\Omega \left[\delta \breve{u}_k * \breve{\bar{j}}_k \right] d\Omega + \int_\Omega \left[\delta \breve{\pi} * \breve{\bar{\Upsilon}} \right] d\Omega \\
&- \int_{\Gamma_t} \tfrac{1}{2} \left[\delta \breve{u}_k * \left(\breve{\bar{\tau}}_k + \breve{\tau}_k \right) \right] d\Gamma \\
&+ \int_{\Gamma_v} \tfrac{1}{2} \left[\delta \breve{\tau}_k * \left(\breve{\bar{u}}_k - \breve{u}_k \right) \right] d\Gamma \\
&- \int_{\Gamma_q} \tfrac{1}{2} \left[\delta \breve{\pi} * \left(\breve{\bar{w}} + \breve{w} \right) \right] d\Gamma \\
&+ \int_{\Gamma_p} \tfrac{1}{2} \left[\delta \breve{w} * \left(\breve{\bar{\pi}} - \breve{\pi} \right) \right] d\Gamma = 0
\end{aligned}
\tag{13}
$$

In a subsequent section, we will briefly discuss the discretization of (13) toward development of a time and space finite element method. Interestingly, for simple spatial and temporal variations, all of the integrals appearing in (13), including those involving fractional derivatives, can be evaluated in closed form.

However, before moving on to that discussion, let us recover the strong form of the problem by shifting all spatial and temporal derivatives from the variations (δu_k, δJ_{ij}, $\delta \pi$ and δw_i) to the real fields (u_k, J_{ij}, π and w_i) by using classical and fractional integration-by-parts for convolutions. All of the required formulas are defined in References [21, 22, 1], making this a systematic procedure. After some algebraic manipulation, the result can be written as follows:

$$
\begin{aligned}
\delta I_{C_p} = & \int_\Omega \delta u_k * \begin{bmatrix} \rho_o \ddot{u}_k + \rho_f \ddot{w}_k \\ - B_{ijk}\left(\dot{J}_{ij} - \beta_{ij}\dot{\pi}\right) - \overline{f}_k \end{bmatrix} d\Omega \\
& - \int_\Omega \delta J_{ij} * \left[A_{ijkl} \ddot{J}_{kl} - B_{ijk} \dot{u}_k \right] d\Omega \\
& - \int_\Omega \delta \pi * \left[\tfrac{1}{Q} \ddot{\pi} + B_i \dot{w}_i + \beta_{ij} B_{ijk} \dot{u}_k - \overline{\Gamma} \right] d\Omega \\
& + \int_\Omega \delta w_i * \left[\tfrac{\rho_f}{n} \ddot{w}_j + \rho_f \ddot{u}_j + \lambda_{ij} \dot{w}_i + B_j \dot{\pi} \right] d\Omega \\
& + \int_\Omega \delta u_k(t) \begin{bmatrix} \rho_o \dot{u}_k(0) + \rho_f \dot{w}_k(0) \\ - B_{ijk}\left(J_{ij}(0) - \beta_{ij}\pi(0)\right) - \overline{j}_k(0) \end{bmatrix} d\Omega \\
& - \int_\Omega \delta u_k(0) \left[\rho_o \dot{u}_k(t) + \rho_f \dot{w}_k(t) \right] d\Omega \\
& - \int_\Omega \delta J_{ij}(t) \left[A_{ijkl} \dot{J}_{kl}(0) - B_{ijk} u_k(0) \right] d\Omega \\
& + \int_\Omega \delta J_{ij}(0) \left[A_{ijkl} \dot{J}_{kl}(t) \right] d\Omega \\
& - \int_\Omega \delta \pi(t) \begin{bmatrix} \tfrac{1}{Q} \dot{\pi}(0) + B_i w_i(0) \\ + \beta_{ij} B_{ijk} u_k(0) - \overline{\Upsilon}(0) \end{bmatrix} d\Omega \\
& + \int_\Omega \delta \pi(0) \left[\tfrac{1}{Q} \dot{\pi}(t) \right] d\Omega \\
& + \int_\Omega \delta w_i(t) \begin{bmatrix} \tfrac{\rho_f}{n} \dot{w}_j(0) + \rho_f \dot{u}_j(0) \\ + \lambda_{ij} w_i(0) + B_j \pi(0) \end{bmatrix} d\Omega \\
& - \int_\Omega \delta w_i(0) \left[\tfrac{\rho_f}{n} \dot{w}_j(t) + \rho_f \dot{u}_j(t) \right] d\Omega \\
& + \int_{\Gamma_t} \tfrac{1}{2} \left[\delta u_k * t_k - \delta u_k * \overline{t}_k \right] d\Gamma \\
& + \int_{\Gamma_t} \tfrac{1}{2} \left[\delta u_k(t) \tau_k(0) - \delta u_k(t) \overline{\tau}_k(0) \right] d\Gamma \\
& + \int_{\Gamma_v} \tfrac{1}{2} \left[\delta u_k * t_k \right] d\Gamma + \int_{\Gamma_v} \tfrac{1}{2} \left[\delta u_k(t) \tau_k(0) \right] d\Gamma \\
& - \int_{\Gamma_v} \tfrac{1}{2} \left[\delta \tau_k * v_k - \delta \tau_k * \overline{v}_k \right] d\Gamma \\
& - \int_{\Gamma_v} \tfrac{1}{2} \left[\delta \tau_k(t) u_k(0) - \delta \tau_k(t) \overline{u}_k(0) \right] d\Gamma \\
& - \int_{\Gamma_\tau} \tfrac{1}{2} \left[\delta \tau_k * v_k \right] d\Gamma - \int_{\Gamma_\tau} \tfrac{1}{2} \left[\delta \tau_k(t) u_k(0) \right] d\Gamma \\
& + \int_{\Gamma_q} \tfrac{1}{2} \left[\delta \pi * q - \delta \pi * \overline{q} \right] d\Gamma \\
& + \int_{\Gamma_q} \tfrac{1}{2} \left[\delta \pi(t) w(0) - \delta \pi(t) \overline{w}(0) \right] d\Gamma \\
& + \int_{\Gamma_p} \tfrac{1}{2} \left[\delta \pi * q \right] d\Gamma - \int_{\Gamma_p} \tfrac{1}{2} \left[\delta \pi(t) w(0) \right] d\Gamma \\
& - \int_{\Gamma_p} \tfrac{1}{2} \left[\delta w * p - \delta w * \overline{p} \right] d\Gamma \\
& - \int_{\Gamma_p} \tfrac{1}{2} \left[\delta w(t) \pi(0) - \delta w(t) \overline{\pi}(0) \right] d\Gamma \\
& - \int_{\Gamma_q} \tfrac{1}{2} \left[\delta w * p \right] d\Gamma + \int_{\Gamma_q} \tfrac{1}{2} \left[\delta w(t) \pi(0) \right] d\Gamma = 0
\end{aligned} \tag{14}
$$

again with $w = w_i n_i$, $q = q_i n_i$, $\tau_i = J_{ji} n_j$ and $t_i = \sigma_{ji} n_j$.

From (14) for arbitrary variations, we have as the Euler-Lagrange equations:

Governing partial differential equations

$$\rho_o \ddot{u}_k + \rho_f \ddot{w}_k - B_{ijk}\left(\dot{J}_{ij} - \beta_{ij}\dot{\pi}\right) = \overline{f}_k \tag{15a}$$

$$A_{ijkl} \ddot{J}_{kl} - B_{ijk} \dot{u}_k = 0 \tag{15b}$$

$$\frac{1}{Q} \ddot{\pi} + B_i \dot{w}_i + \beta_{ij} B_{ijk} \dot{u}_k = \overline{\Gamma} \tag{15c}$$

$$\frac{\rho_f}{n} \ddot{w}_j + \rho_f \ddot{u}_j + \lambda_{ij} \dot{w}_i + B_j \dot{\pi} = 0 \tag{15d}$$

for $x \in \Omega,\ \tau \in (0,t)$

Initial conditions over the spatial domain

$$\rho_o \dot{u}_k(0) + \rho_f \dot{w}_k(0) - B_{ijk}\left(J_{ij}(0) - \beta_{ij}\pi(0)\right) = \overline{j}_k(0) \tag{16a}$$

$$A_{ijkl} \dot{J}_{kl}(0) - B_{ijk} u_k(0) = 0 \tag{16b}$$

$$\frac{1}{Q} \dot{\pi}(0) + B_i w_i(0) + \beta_{ij} B_{ijk} u_k(0) = \overline{\Upsilon}(0) \tag{16c}$$

$$\frac{\rho_f}{n} \dot{w}_j(0) + \rho_f \dot{u}_j(0) + \lambda_{ij} w_i(0) + B_j \pi(0) = 0 \tag{16d}$$

for $x \in \Omega$

Boundary conditions over entire time span

$$t_k = \overline{t}_k \qquad x \in \Gamma_t \tag{17a}$$

$$v_k = \overline{v}_k \qquad x \in \Gamma_v \tag{17b}$$

$$q = \overline{q} \qquad x \in \Gamma_q \tag{17c}$$

$$p = \overline{p} \qquad x \in \Gamma_p \tag{17d}$$

for $\tau \in (0,t)$

Boundary conditions at time zero

$$\tau_k(0) = \overline{\tau}_k(0) \qquad x \in \Gamma_t \tag{18a}$$

$$u_k(0) = \overline{u}_k(0) \qquad x \in \Gamma_v \tag{18b}$$

$$w(0) = \overline{w}(0) \qquad x \in \Gamma_q \tag{18c}$$

$$\pi(0) = \overline{\pi}(0) \qquad x \in \Gamma_p \tag{18d}$$

In addition, the variations are constrained by the following:

Zero variations for specified boundary conditions

$$\delta\tau_k = 0 \qquad x \in \Gamma_t,\ \tau \in (0,t) \tag{19a}$$

$$\delta u_k = 0 \qquad x \in \Gamma_v,\ \tau \in (0,t) \tag{19b}$$

$$\delta w = 0 \qquad x \in \Gamma_q,\ \tau \in (0,t) \tag{19c}$$

$$\delta\pi = 0 \qquad x \in \Gamma_p,\ \tau \in (0,t) \tag{19d}$$

Zero end time variations for specified boundary conditions

$$\delta\tau_k(t) = 0 \qquad x \in \Gamma_t \tag{20a}$$

$$\delta u_k(t) = 0 \qquad x \in \Gamma_v \tag{20b}$$

$$\delta w(t) = 0 \qquad x \in \Gamma_q \tag{20c}$$

$$\delta\pi(t) = 0 \qquad x \in \Gamma_p \tag{20d}$$

Zero variations at initial time

$$\delta u_k(0) = 0 \tag{21a}$$

$$\delta J_{ij}(0) = 0 \tag{21b}$$

$$\delta\pi(0) = 0 \tag{21c}$$

$$\delta w_i(0) = 0 \tag{21d}$$

$$\text{for } x \in \Omega$$

This demonstrates that the Euler-Lagrange equations associated with the mixed convolved action, specified in (10), provide all of the relations that define the initial/boundary value problem of Biot dynamic poroelasticity.

As a result, we have now established a *Principle of Stationary Mixed Convolved Action* for a Linear Poroelastic Continuum undergoing infinitesimal deformation. This may be stated as follows: Of all the possible trajectories $\{u_k(\tau), J_{ij}(\tau), \pi(\tau), w_i(\tau)\}$ of the system during the time interval $(0,t)$, the one that renders the action I_{C_p} in (10) stationary, corresponds to the solution of the initial/boundary value problem. Thus, the stationary trajectory satisfies the balance laws of linear momentum (15a) and mass flow (15c), along with the linear elastic effective stress-strain constitutive relationship (15b) and the extended Darcy law (15d) in the domain Ω over the entire time interval. In addition, the traction (17a), velocity (17b), mass flux (17c) and pressure (17d) boundary conditions are satisfied throughout the time interval, while also complying with the initial conditions defined by (16a-d) in Ω and (18a-d) on the appropriate portions of the bounding surface. Furthermore, the possible trajectories under consideration during the variational process are constrained precisely by their need to satisfy the specified boundary and initial conditions of the problem in the form of (19a-d), (20a-d) and (21a-d).

Therefore, we are able to define a single real scalar functional, based upon convolution and fractional derivatives, which encapsulates all of the governing differential equations, along with the boundary and initial conditions, for linear dynamic poroelasticity. Furthermore, this represents the first true variational formulation for a dissipative poroelastic continuum.

FREQUENCY DOMAIN VARIATIONAL FORMULATION

As mentioned previously, the mixed convolved action also has some interesting characteristics in the frequency domain. Introducing a time harmonic response of the field variables directly into (10) or by performing a Fourier transform, one can write the following action at frequency ω :

$$\begin{aligned}
\tilde{I}_{C_p} &= \int_\Omega \left[-\tfrac{1}{2}\omega^2 \tilde{u}_k (1-n)\rho_s \tilde{u}_k \right] d\Omega \\
&+ \int_\Omega \left[-\tfrac{1}{2}\omega^2 \left(\tilde{u}_k + \frac{\tilde{w}_k}{n} \right) n\rho_f \left(\tilde{u}_k + \frac{\tilde{w}_k}{n} \right) \right] d\Omega \\
&- \int_\Omega \left[-\tfrac{1}{2}\omega^2 \tilde{J}_{ij} A_{ijkl} \tilde{J}_{kl} \right] d\Omega \\
&- \int_\Omega \left[-\tfrac{1}{2}\omega^2 \tilde{\pi} \frac{1}{Q} \tilde{\pi} - \tfrac{1}{2} i\omega \tilde{w}_i \lambda_{ij} \tilde{w}_j \right] d\Omega \\
&+ \int_\Omega \left[\tfrac{1}{2} i\omega \left(\tilde{J}_{ij} B_{ijk} \tilde{u}_k - \tilde{u}_k B_{ijk} \tilde{J}_{ij} \right) \right] d\Omega \\
&- \int_\Omega \left[\tfrac{1}{2} i\omega \left(\beta_{ij} \tilde{\pi} B_{ijk} \tilde{u}_k - \tilde{u}_k B_{ijk} \beta_{ij} \tilde{\pi} \right) \right] d\Omega \\
&+ \int_\Omega \left[\tfrac{1}{2} i\omega \left(\tilde{w}_i B_i \tilde{\pi} - \tilde{\pi} B_i \tilde{w}_i \right) \right] d\Omega \\
&- \int_\Omega i\omega \left[\tilde{u}_k \tilde{\bar{f}}_k \right] d\Omega + \int_\Omega i\omega \left[\tilde{\pi}\, \tilde{\bar{\Upsilon}} \right] d\Omega \\
&- \int_{\Gamma_t} \tfrac{1}{2} i\omega \left[\tilde{u}_k \tilde{\bar{\tau}}_k \right] d\Gamma + \int_{\Gamma_v} \tfrac{1}{2} i\omega \left[\tilde{\tau}_k \tilde{\bar{u}}_k \right] d\Gamma \\
&- \int_{\Gamma_q} \tfrac{1}{2} i\omega \left[\tilde{\pi}\, \tilde{\bar{w}} \right] d\Gamma + \int_{\Gamma_p} \tfrac{1}{2} i\omega \left[\tilde{w}\, \tilde{\bar{\pi}} \right] d\Gamma
\end{aligned} \tag{22}$$

where the superposed tilde denotes the Fourier transform of the variable.

Of course, the convolutions present in the time domain action I_{C_p} are transformed to a simple product at each frequency in $\tilde{I}_{C_p}$. This leads directly to a frequency domain variational formulation for dynamic poroelasticity, which will be developed next.

We begin by taking the first variation of (22). Then, after several integration-by-parts operations, the following weak form can be defined:

$$\begin{aligned}
\delta\tilde{I}_{C_p} &= -\int_\Omega \left[\delta\tilde{u}_k \omega^2 (1-n)\rho_s \tilde{u}_k \right] d\Omega \\
&- \int_\Omega \left[\left(\delta\tilde{u}_k + \frac{\delta\tilde{w}_k}{n} \right) \omega^2 n\rho_f \left(\tilde{u}_k + \frac{\tilde{w}_k}{n} \right) \right] d\Omega \\
&+ \int_\Omega \left[\delta\tilde{J}_{ij} \omega^2 A_{ijkl} \tilde{J}_{kl} \right] d\Omega \\
&+ \int_\Omega \left[\delta\tilde{\pi}\ \omega^2 \frac{1}{Q} \tilde{\pi} + \delta\tilde{w}_i\ i\omega\ \lambda_{ij} \tilde{w}_j \right] d\Omega \\
&+ \int_\Omega \left[\left(\delta\tilde{J}_{ij} - \beta_{ij}\delta\tilde{\pi} \right) i\omega\ B_{ijk} \tilde{u}_k \right] d\Omega \\
&+ \int_\Omega \left[\left(\tilde{J}_{ij} - \beta_{ij}\tilde{\pi} \right) i\omega\ B_{ijk} \delta\tilde{u}_k \right] d\Omega
\end{aligned}$$

$$
\begin{aligned}
&+\int_{\Omega}\left[\delta\tilde{w}_i\; i\omega\; B_i\tilde{\pi}+\tilde{w}_i\; i\omega\; B_i\delta\tilde{\pi}\right]\, d\Omega \\
&-\int_{\Omega}\left[\delta\tilde{u}_k\; i\omega\;\tilde{\bar{f}}_k\right]\, d\Omega+\int_{\Omega}\left[\delta\tilde{\pi}\; i\omega\;\tilde{\bar{\Upsilon}}\right]\, d\Omega \\
&-\int_{\Gamma_t}\tfrac{1}{2}\left[\delta\tilde{u}_k\; i\omega\left(\tilde{\bar{\tau}}_k+\tilde{\tau}_k\right)\right]\, d\Gamma \\
&+\int_{\Gamma_v}\tfrac{1}{2}\left[\delta\tilde{\tau}_k\; i\omega\left(\tilde{\bar{u}}_k-\tilde{u}_k\right)\right]\, d\Gamma \\
&-\int_{\Gamma_q}\tfrac{1}{2}\left[\delta\tilde{\pi}\; i\omega\left(\tilde{\bar{w}}+\tilde{w}\right)\right]\, d\Gamma \\
&-\int_{\Gamma_p}\tfrac{1}{2}\left[\delta\tilde{w}\; i\omega\left(\tilde{\bar{\pi}}-\tilde{\pi}\right)\right]\, d\Gamma=0
\end{aligned}
\tag{23}
$$

Notice that this weak form requires only C^{-1} spatial continuity of the stress impulse and relative pore fluid displacement amplitudes represented by $\tilde{J}_{ij}$ and $\tilde{w}_i$, respectively, while both skeleton displacement and pore pressure impulse amplitudes must maintain C^0 continuity. Consequently, (23) can provide the foundation for a frequency domain finite element method for dynamic poroelasticity.

On the other hand, if we perform integration-by-parts on (23) to isolate all of the field variable variations from spatial derivatives, then the Euler-Lagrange equations of (22) will emerge. This form of the stationarity of $\tilde{I}_{C_P}$ can be written:

$$
\begin{aligned}
\delta I_{C_P}=&-\int_{\Omega}\delta\tilde{u}_k\left[\begin{array}{l}\omega^2\rho_o\tilde{u}_k+\omega^2\rho_f\tilde{w}_k\\ +i\omega\; B_{ijk}\left(\tilde{J}_{ij}-\beta_{ij}\tilde{\pi}\right)+\tilde{\bar{f}}_k\end{array}\right] d\Omega \\
&+\int_{\Omega}\delta\tilde{J}_{ij}\left[\omega^2 A_{ijkl}\tilde{J}_{kl}+i\omega\; B_{ijk}\tilde{u}_k\right] d\Omega \\
&+\int_{\Omega}\delta\tilde{\pi}\left[\omega^2\tfrac{1}{Q}\tilde{\pi}-i\omega\; B_i\tilde{w}_i-i\omega\;\beta_{ij}B_{ijk}\tilde{u}_k+\tilde{\bar{\Gamma}}\right] d\Omega \\
&-\int_{\Omega}\delta\tilde{w}_i\left[\omega^2\left(\tfrac{\rho_f}{n}\tilde{w}_j+\rho_f\tilde{u}_j\right)-i\omega\left(\lambda_{ij}\tilde{w}_i+B_j\tilde{\pi}\right)\right] d\Omega \\
&+\int_{\Gamma_t}\tfrac{1}{2}\left[\delta\tilde{u}_k\left(\tilde{t}_k-\tilde{\bar{t}}_k\right)\right] d\Gamma+\int_{\Gamma_v}\tfrac{1}{2}\left[\delta\tilde{u}_k\;\tilde{t}_k\right] d\Gamma \\
&-\int_{\Gamma_v}\tfrac{1}{2}\left[\delta\tilde{\tau}_k\left(\tilde{v}_k-\tilde{\bar{v}}_k\right)\right] d\Gamma-\int_{\Gamma_t}\tfrac{1}{2}\left[\delta\tilde{\tau}_k\;\tilde{v}_k\right] d\Gamma \\
&+\int_{\Gamma_q}\tfrac{1}{2}\left[\delta\tilde{\pi}\left(\tilde{q}-\tilde{\bar{q}}\right)\right] d\Gamma+\int_{\Gamma_p}\tfrac{1}{2}\left[\delta\tilde{\pi}\;\tilde{q}\right] d\Gamma \\
&+\int_{\Gamma_p}\tfrac{1}{2}\left[\delta\tilde{w}\left(\tilde{p}-\tilde{\bar{p}}\right)\right] d\Gamma+\int_{\Gamma_q}\tfrac{1}{2}\left[\delta\tilde{w}\;\tilde{p}\right] d\Gamma=0
\end{aligned}
\tag{24}
$$

Then, for arbitrary variations, the following strong form is recovered:

Governing partial differential equations in frequency domain

$$\omega^2\rho_o\tilde{u}_k+\omega^2\rho_f\tilde{w}_k+i\omega\; B_{ijk}\left(\tilde{J}_{ij}-\beta_{ij}\tilde{\pi}\right)+\tilde{\bar{f}}_k=0 \tag{25a}$$

$$\omega^2 A_{ijkl}\tilde{J}_{kl}+i\omega\; B_{ijk}\tilde{u}_k=0 \tag{25b}$$

$$\omega^2\tfrac{1}{Q}\tilde{\pi}-i\omega\; B_i\tilde{w}_i-i\omega\;\beta_{ij}B_{ijk}\tilde{u}_k+\tilde{\bar{\Gamma}}=0 \tag{25c}$$

$$\omega^2\left(\tfrac{\rho_f}{n}\tilde{w}_j+\rho_f\tilde{u}_j\right)-i\omega\left(\lambda_{ij}\tilde{w}_i+B_j\tilde{\pi}\right)=0 \tag{25d}$$

for $x\in\Omega$

Boundary conditions in frequency domain

$$\tilde{t}_k=\tilde{\bar{t}}_k \qquad x\in\Gamma_t \tag{26a}$$

$$\tilde{v}_k=\tilde{\bar{v}}_k \qquad x\in\Gamma_v \tag{26b}$$

$$\tilde{q}=\tilde{\bar{q}} \qquad x\in\Gamma_q \tag{26c}$$

$$\tilde{p}=\tilde{\bar{p}} \qquad x\in\Gamma_p \tag{26d}$$

along with the following restrictions on the variations

Zero variations for specified boundary conditions

$$\delta\tilde{\tau}_k=0 \qquad x\in\Gamma_t, \tag{27a}$$

$$\delta\tilde{u}_k=0 \qquad x\in\Gamma_v, \tag{27b}$$

$$\delta\tilde{w}=0 \qquad x\in\Gamma_q, \tag{27c}$$

$$\delta\tilde{\pi}=0 \qquad x\in\Gamma_p, \tag{27d}$$

This establishes the *Principle of Stationary Mixed Convolved Action* for a Linear Poroelastic Continuum undergoing time harmonic infinitesimal deformation. This new principle may be stated as follows: Of all the possible solutions $\{\tilde{u}_k(\omega),\tilde{J}_{ij}(\omega),\tilde{\pi}(\omega),\tilde{w}_i(\omega)\}$ of the system at frequency ω, the one that renders the action $\tilde{I}_{C_P}$ in (22) stationary, corresponds to the solution of the time harmonic boundary value problem. Thus, the stationary trajectory satisfies the balance laws of linear momentum (25a) and mass flow (25c), along with the linear elastic effective stress-strain constitutive relationship (25b) and the extended Darcy law (25d) in the domain Ω at frequency ω. In addition, the traction (26a), velocity (26b), mass flux (26c) and pressure (26d) specified conditions are satisfied on the appropriate portions of the bounding surface. During the variational process, the possible trajectories under consideration are constrained only by their need to satisfy the specified boundary conditions of the problem, as defined in (27a-d).

FINITE ELEMENT FORMULATIONS

While the focus of the present work is to define new variational formulations based upon the concept of mixed convolved action, a few words can be said concerning the development of corresponding finite element methods for dissipative dynamic continua. Beyond the theoretical significance of defining the Principle of Mixed Convolved Action for dynamic poroelastic media, the weak form of (13) enables formulation of a space-time finite element approach, because unlike Lagrangian inner product formulations there is no restriction on the variations at the end of the time interval. This is one key result of the mixed convolved action methodology.

In order to create an initial finite element formulation, simple spatial and temporal representations can be adopted, for example, by selecting three-node elements for planar poroelastic problems, having a linear variation of displacement u_i and pore pressure impulse π over the element with C^0 continuity across elements. On the other hand, stress impulse J_{ij} and relative fluid displacement w_i can be assumed constant within each element and thus discontinuous (or C^{-1} continuous) across elements.

For the temporal variations of all four field variables, only C^0 continuity is needed and thus linear temporal shape functions are fully appropriate. Interestingly, despite the presence of fractional derivatives, all temporal functions appearing in the weak form (13) can be evaluated in closed form and ultimately have very simple form, as defined previously in References [1-3].

Additionally, the weak form in (23) can be used as a starting point for a frequency domain finite element method for dynamic poroelasticity. After performing spatial discretization, this weak form may be written:

$$\left\{\delta\tilde{\mathbf{u}}^T \quad \delta\tilde{\mathbf{w}}^T \quad \delta\tilde{\mathbf{J}}^T \quad \delta\tilde{\boldsymbol{\pi}}^T\right\} \begin{bmatrix} \omega^2\mathbf{M}_{uu} & \omega^2\mathbf{M}_{uw} & -i\omega\mathbf{B}_{Ju}^T & i\omega\mathbf{B}_{\pi u}^T \\ \omega^2\mathbf{M}_{uw}^T & \omega^2\mathbf{M}_{ww}-i\omega\boldsymbol{\Lambda}_{ww} & \mathbf{0} & -i\omega\mathbf{B}_{\pi w}^T \\ -i\omega\mathbf{B}_{Ju} & \mathbf{0} & -\omega^2\mathbf{A}_{JJ} & \mathbf{0} \\ i\omega\mathbf{B}_{\pi u} & -i\omega\mathbf{B}_{\pi w} & \mathbf{0} & -\omega^2\mathbf{A}_{\pi\pi} \end{bmatrix} \begin{Bmatrix} \tilde{\mathbf{u}} \\ \tilde{\mathbf{w}} \\ \tilde{\mathbf{J}} \\ \tilde{\boldsymbol{\pi}} \end{Bmatrix} = \left\{\delta\tilde{\mathbf{u}}^T \quad \delta\tilde{\mathbf{w}}^T \quad \delta\tilde{\mathbf{J}}^T \quad \delta\tilde{\boldsymbol{\pi}}^T\right\} \begin{Bmatrix} \tilde{\bar{f}} \\ \tilde{\bar{p}} \\ -\tilde{\bar{v}} \\ -\tilde{\bar{\gamma}} \end{Bmatrix} \tag{28}$$

where the individual matrices can be developed from the corresponding terms in (23) using standard finite element technologies. For arbitrary variations, this provides a set of complex linear algebraic equations of the form:

$$\begin{bmatrix} \omega^2\mathbf{M}_{uu} & \omega^2\mathbf{M}_{uw} & -i\omega\mathbf{B}_{Ju}^T & i\omega\mathbf{B}_{\pi u}^T \\ \omega^2\mathbf{M}_{uw}^T & \omega^2\mathbf{M}_{ww}-i\omega\boldsymbol{\Lambda}_{ww} & \mathbf{0} & -i\omega\mathbf{B}_{\pi w}^T \\ -i\omega\mathbf{B}_{Ju} & \mathbf{0} & -\omega^2\mathbf{A}_{JJ} & \mathbf{0} \\ i\omega\mathbf{B}_{\pi u} & -i\omega\mathbf{B}_{\pi w} & \mathbf{0} & -\omega^2\mathbf{A}_{\pi\pi} \end{bmatrix} \begin{Bmatrix} \tilde{\mathbf{u}} \\ \tilde{\mathbf{w}} \\ \tilde{\mathbf{J}} \\ \tilde{\boldsymbol{\pi}} \end{Bmatrix} = \begin{Bmatrix} \tilde{\bar{f}} \\ \tilde{\bar{p}} \\ -\tilde{\bar{v}} \\ -\tilde{\bar{\gamma}} \end{Bmatrix} \tag{29}$$

which after applying a set of well-defined prescribed boundary conditions can be solved the response at frequency ω. Note that the variables $\tilde{\mathbf{J}}$ and $\tilde{\mathbf{w}}$ only require C^{-1} continuity. If interpolation is element-by-element, then these variables may be eliminated at the element level and do not need to be assembled. This then provides in essence a $\tilde{\mathbf{u}}$ - $\tilde{\boldsymbol{\pi}}$ hybrid finite element methodology for frequency domain poroelasticity.

CONCLUSIONS

Starting with the idea first proposed by Gurtin and Tonti of substituting convolution for inner product operators as the basis for variational formulations for dynamical systems, we present a mixed convolved action approach for Biot poroelasticity. The action functional involves a mixed set of impulsive variables, including skeleton displacement, relative pore fluid displacement, stress impulse and pore pressure impulse, which are selected to provide a well-defined and balanced structure to the formulation. As a result, the mixed convolved action functional, although algebraically complicated, encapsulates all of the governing partial differential equations, boundary conditions and initial conditions of the poroelastic problem. Furthermore, new time and space finite element methods can be developed systematically from this framework that will inherit key characteristics of the underlying problem, such as energy conservation for cases without viscous dissipation. A frequency domain variational formulation also is developed that has a convenient structure in the sense that the time domain convolution transforms to a simple product. For future work, it will be interesting to explore the development of new computational algorithms and to seek a fundamental understanding of the physical meaning of the convolved action.

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