Mixed convolved action principles in linear continuum dynamics

Gary F. Dargush · Bradley T. Darrall · Jinkyu Kim · Georgios Apostolakis

Abstract The paper begins with an overview of several of the classical integral formulations of elastodynamics, which highlights the natural appearance of temporal convolutions in the reciprocal theorem for such problems. This leads first to the formulation of a principle of virtual convolved action, as an extension of the principle of virtual work to dynamical problems. Then, to overcome the key shortcomings of Hamilton's principle, the concept of mixed convolved action is developed for linear dynamical problems within the context of continuum solid mechanics. This new approach is broadly applicable to both reversible and irreversible phenomena without the need for special treatments, such as the artificial definition of Rayleigh dissipation functionals. The focus here is on linear elastic and viscoelastic media, which in the latter case is represented by classical Kelvin-Voigt and Maxwell models. Remarkably, for each problem type, the stationarity of the mixed convolved action provides not only the governing partial differential equations, but also the specified boundary and initial conditions, as its Euler-Lagrange equations. Thus, the entire initial/boundary value problem definition is encapsulated in a scalar mixed convolved action functional written in terms of displacements and stress impulses. The resulting formulations possess an elegant structure that provides a versatile framework for development of novel computational methods, involving finite element representations in both space and time. We present perhaps the simplest approach by employing linear three-node triangular elements for two-dimensional analysis, along with linear shape functions over the temporal domain. Numerical examples are included to verify the formulation and to explore concepts of stress wave attenuation.

Keywords Variational methods · Hamilton's principle · Euler-Lagrange equations · Elasticity · Viscoelasticity · Finite element methods · Implicit methods

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1 Introduction

Hamilton's principle has long provided a fundamental basis for theoretical dynamics [1-4]. However, we know that there are two major shortcomings; first, irreversible processes cannot be accommodated in a purely variational manner and, second, variations are not consistent with the specified initial conditions.

Irreversible phenomena can be brought into the framework of Hamilton's principle, following an approach originally proposed by Rayleigh for mechanical damping of a viscous nature, by introducing a quadratic dissipation function [5-7]. While this is not a true variational method in a strict mathematical sense, it can provide an attractive foundation for computations. For example, using this approach, Mixed Lagrangian Formulations have been developed recently for dissipative systems for plasticity [8, 9], for contact and fracture [10, 11], and for thermoelasticity [12-14]. Alternatively, generalized bracket formalisms have been established to address a broad range of dissipative processes [15-20].

However, in order to overcome both limitations of Hamilton's principle, convolution-based temporal operators are needed. Gurtin [21-23] was the first to introduce such formulations for continuum problems of viscoelasticity and elastodynamics, while Tonti [24, 25] provides an insightful assessment of variational methods for dynamical problems in general and advocates for the use of the convolutional bilinear form. Oden and Reddy [26] extend the formulations of both Gurtin and Tonti to a large class of boundary and initial value problems in mechanics, especially for Hellinger-Reissner type mixed principles. The present work builds on all of these previous efforts employing temporal convolutions and on the recent work for single-degree-of-freedom dynamical systems [27, 28]. Here we propose a new set of action principles, based upon impulsive mixed variables, fractional derivatives and the convolution of convolutions to produce an elegant theoretical structure for linear initial/boundary value problems within the mechanics of solid continua. We consider both reversible (elastic) and irreversible (viscoelastic) phenomena within a common framework.

The remainder of the paper is organized as follows. Sect. 2 provides an overview of the governing equations and several integral theorems for elastodynamics to serve as background and motivation for the present work. A new principle of virtual convolved action is also introduced. In Sects. 3

and 4, we focus on two specific dynamical problems of interest involving infinitesimal deformations in reversible and irreversible mechanics, respectively. Sect. 3 addresses conservative problems of elastodynamics, while the irreversible phenomena dealt with in Sect. 4 include classical Kelvin-Voigt and Maxwell viscoelastic models. New mixed convolved action principles are developed for each of these problems from a time domain perspective. Then, in Sect. 5, we formulate finite element methods for all of these problems in two-dimensions under conditions of plane stress and plane strain. The finite element discretization takes place over space and time using linear three-node triangular elements and linear temporal shape functions, respectively. A number of examples are considered in Sect. 6. The first two are aimed toward verification of the mixed convolved action formulation and numerical implementation through comparison with analytical solutions and results from well-established time-integration algorithms. The latter two illustrate ideas for stress wave attenuation via impedance tailoring in simple impulse-loaded protective system geometries. Afterward, we finish by providing some conclusions in Sect. 7. Appendix A includes relevant background on fractional calculus, especially related to convolution and integration-by-parts operations that are essential for the present development.

2 Overview of elastodynamic theory

Let us begin with an overview of the governing equations and integral theorems for the dynamic response of a linear elastic continuum undergoing infinitesimal deformation within the domain Ω with bounding surface Γ [29-32]. First, by considering the balance of linear momentum, we may write

$$\rho_o \ddot{u}_i - \sigma_{ji,j} - \overline{f}_i = 0 \tag{1}$$

where u_i represents the displacement field, σ_{ij} is the stress tensor, $\overline{f}_i$ is the applied body force per unit volume and ρ_o is the mass density. Standard indicial notation will be used throughout. Thus, summation is implied over repeated indices, spatial derivatives are denoted by indices after the comma and superposed dots represent partial derivatives with respect to time.

In classical elasticity, by ignoring couple-stresses, the balance of angular momentum leads to symmetry of the stress tensor σ_{ij}, which is then related to the symmetric strain tensor ε_{ij} through the constitutive relations

$$\varepsilon_{ij} - A_{ijkl}\sigma_{kl} = 0 \tag{2}$$

where

$$\varepsilon_{ij} = \tfrac{1}{2}\left(u_{i,j} + u_{j,i}\right) \tag{3}$$

Equations (1)-(3) above, along with initial conditions over Ω and boundary conditions on Γ, define the elastodynamic initial/boundary value problem in differential form.

A number of integral forms have been developed over the years for this elastodynamic problem, including the principle of virtual work, Hamilton's principle and several varieties of the reciprocal theorem.

To derive the principle of virtual work, one may begin by multiplying Eq. (1) by a virtual displacement field δu_i and then integrating over the domain. Thus,

$$\int_{\Omega}\left(\rho_o\,\ddot{u}_i - \sigma_{ji,j} - \overline{f}_i\right)\delta u_i\; d\Omega = 0 \tag{4}$$

After invoking the divergence theorem and assuming a kinematically compatible virtual field, Eq. (4) can be expressed as the balance of internal and external virtual work, as

$$\delta W_{\text{internal}} = \delta W_{\text{external}} \tag{5}$$

$$\int_{\Omega}\left(\rho_o\,\ddot{u}_i\delta u_i + \sigma_{ji}\delta\varepsilon_{ij}\right) d\Omega = \int_{\Gamma_t}\overline{t}_i\delta u_i\; d\Gamma + \int_{\Omega}\overline{f}_i\delta u_i\; d\Omega \tag{6}$$

where Γ_t represents the portion of the boundary on which tractions $\overline{t}_i$ are specified, while $\delta u_i = 0$ on the remainder of the boundary Γ_v having displacements prescribed. The overbars denote specified values, while tractions, in general, are defined at any location on the boundary, as

$$t_i = \sigma_{ji}\, n_j \tag{7}$$

with n_i representing the unit outward normal vector.

Several points should be emphasized concerning this statement of virtual work for the dynamical problem. First of all, Eq. (6) has been developed without consideration of constitutive relations and therefore is not restricted to elastic media. Secondly, integration is performed only over the spatial domain. Equation (6) is valid at any instant of time, but there is no comparable temporal integration. Also, through the use of integration-by-parts, Eq. (6) can be viewed as a weak form in space, while retaining a strong form in time with the appearance of the acceleration $\ddot{u}_i$. Consequently, computational methods based on this principle of virtual work for dynamics often use finite element representations in space, but are limited to non-variational finite difference or Newmark type approaches in time.

On the other hand, Hamilton's principle for elastodynamics integrates a Lagrangian density function L over both space and time, where $L = T - V$ with T and V representing the kinetic and potential energy densities, respectively. More specifically, for elastodynamics, the resulting functional can be written

$$\begin{aligned} I_{H_L} &= \int_\Omega \int_0^t \left[\tfrac{1}{2} \dot{u}_k(\tau) \rho_o \dot{u}_k(\tau) - \tfrac{1}{2} \varepsilon_{ij}(\tau) C_{ijkl} \varepsilon_{kl}(\tau) \right] d\tau \, d\Omega \\ &+ \int_{\Gamma_t} \int_0^t \left[u_k(\tau) \overline{t}_k(\tau) \right] d\tau \, d\Gamma + \int_\Omega \int_0^t \left[u_k(\tau) \overline{f}_k(\tau) \right] d\tau \, d\Omega \end{aligned} \tag{8}$$

where C_{ijkl} is the elastic constitutive tensor inverse to A_{ijkl} and the last two integrals represent the contributions from the applied surface tractions and body forces, respectively. Notice that in Eq. (8), the dependence of the field variables on time is made explicit, while the spatial dependence is assumed, as in Eq. (4) and (6). This allows one to focus on the temporal aspects, which in this case involve inner products of field variables, either with each other or with applied loadings, over time.

Hamilton's principle states that of all the possible permissible solutions, the one that makes I_{H_L} in Eq. (8) stationary corresponds to the solution to the governing equations of elastodynamics. Let us work through this derivation to emphasize the associated requirements. Taking the first variation of Eq. (8) provides the following relation for stationarity:

$$\delta I_{H_L} = \int_{\Omega}\int_0^t \left[\delta\dot{u}_k(\tau)\rho_o\dot{u}_k(\tau) - \delta\varepsilon_{ij}(\tau)C_{ijkl}\varepsilon_{kl}(\tau)\right] d\tau\, d\Omega$$
$$+\int_{\Gamma_t}\int_0^t \left[\delta u_k(\tau)\overline{t}_k(\tau)\right] d\tau\, d\Gamma + \int_{\Omega}\int_0^t \left[\delta u_k(\tau)\,\overline{f}_k(\tau)\right] d\tau\, d\Omega = 0 \tag{9}$$

where, of course, there is no variation of the specified traction boundary values and body forces. Next, integration-by-parts and the divergence theorem are used to shift derivatives from the variations to the real fields for the first two terms. Thus,

$$\int_{\Omega}\int_0^t \left[\delta\dot{u}_k(\tau)\rho_o\dot{u}_k(\tau)\right] d\tau\, d\Omega = \int_{\Omega}\left[\delta u_k(\tau)\rho_o\dot{u}_k(\tau)\,|_0^t\right] d\Omega$$
$$-\int_{\Omega}\int_0^t \left[\delta u_k(\tau)\rho_o\ddot{u}_k(\tau)\right] d\tau\, d\Omega \tag{10.1}$$

$$\int_{\Omega}\int_0^t \left[\delta e_{ij}(\tau)C_{ijkl}\varepsilon_{kl}(\tau)\right] d\tau\, d\Omega = \int_{\Gamma}\int_0^t \left[\delta u_k(\tau)t_k(\tau)\right] d\tau\, d\Gamma$$
$$-\int_{\Omega}\int_0^t \left[\delta u_i(\tau)\left(C_{ijkl}\varepsilon_{kl}(\tau)\right)_{,j}\right] d\tau\, d\Omega \tag{10.2}$$

After substituting Eqs. (10.1) and (10.2) into (9) and collecting terms, we find:

$$\delta I_{H_L} = -\int_{\Omega}\int_0^t \left[\delta u_k(\tau)\left(\rho_o\ddot{u}_k(\tau) - \left(C_{ijkl}\varepsilon_{kl}(\tau)\right)_{,j} - \overline{f}_k(\tau)\right)\right] d\tau\, d\Omega$$
$$-\int_{\Gamma_t}\int_0^t \left[\delta u_k(\tau)\left(t_k(\tau) - \overline{t}_k(\tau)\right)\right] d\tau\, d\Gamma + \int_{\Omega}\left[\delta u_k(\tau)\rho_o\dot{u}_k(\tau)\,|_0^t\right] d\Omega = 0 \tag{11}$$

For arbitrary variations δu_i over the domain Ω, the first integral in Eq. (11) imposes the governing differential equations of linear momentum balance Eq. (1), specialized for an elastic body, at each point in Ω and each instant of time. The second integral enforces the traction boundary conditions, respectively, at each instant of time for any point on the surface Γ_t. Thus, these are Euler-Lagrange equations associated with the functional I_{H_L}. However, it is the remaining spatial integral in Eq. (11) that is problematic, because this requires, in general, that the variations of displacement must vanish throughout Ω at the beginning and end of the time interval. How can one assume zero variations at the end time t, when the corresponding displacement field is typically a primary unknown of the initial/boundary value problem?

Perhaps, we should mention also that despite nearly two centuries since the introduction of Hamilton's principle, the physical meaning and significance of L has never been made clear. Why should one choose the difference between the kinetic and potential energies as the basis for this principle? Furthermore, in applications, the incompatibility between the end point requirements in Hamilton's principle and the specification of initial conditions is often de-emphasized or completely ignored.

Complementary and mixed versions of Hamilton's principle can be developed, as well, by applying Legendre transforms to convert to stress-related variables. Of particular note is the mixed Lagrangian formalism (MLF) of Sivaselvan and Reinhorn [8], for which the functional I_{H_S} can be written:

$$\begin{aligned} I_{H_S} &= \int_\Omega \int_0^t \left[\tfrac{1}{2} \dot{u}_k(\tau) \rho_o \dot{u}_k(\tau) + \tfrac{1}{2} \dot{J}_{ij}(\tau) A_{ijkl} \dot{J}_{kl}(\tau) + J_{ij}(\tau) \dot{\varepsilon}_{ij}(\tau) \right] d\tau \, d\Omega \\ &+ \int_{\Gamma_t} \int_0^t \left[u_k(\tau) \overline{t}_k(\tau) \right] d\tau \, d\Gamma + \int_\Omega \int_0^t \left[u_k(\tau) \overline{f}_k(\tau) \right] d\tau \, d\Omega \end{aligned} \tag{12}$$

where J_{ij} represents the impulse of stress. Thus,

$$J_{ij}(t) = \int_0^t \sigma_{ij}(\tau) d\tau = \int_0^t C_{ijkl} \varepsilon_{kl}(\tau) d\tau \tag{13}$$

By enforcing stationarity of I_{H_S}, performing integration-by-parts and invoking the divergence theorem, one can demonstrate the following Euler-Lagrange equations are associated with this functional:

$$\rho_o \ddot{u}_i - B_{ijk} \dot{J}_{ij} - \overline{f}_i = 0 \tag{14.1}$$

$$-A_{ijkl} \ddot{J}_{kl} + B_{ijk} \dot{u}_k = 0 \tag{14.2}$$

over the domain Ω, where the constitutive relation Eq. (14.2) is now written in rate form and the differential equilibrium and strain operator

$$B_{ijk} = \tfrac{1}{2} \left(\delta_{ik} \delta_{jq} + \delta_{iq} \delta_{jk} \right) \tfrac{\partial}{\partial x_q} \tag{15}$$

Meanwhile, the traction boundary conditions are enforced on Γ_t as additional Euler-Lagrange equations.

The elegant symmetry and symplectic character of Eqs. (14) motivate the use of stress impulse, rather than stress, within this mixed Lagrangian formalism. From Eqs. (14), we recognize displacement and stress impulse as analogous field variables or, alternatively, velocity and stress. Similar to other approaches based upon Hamilton's principle, the end point constraint issue precludes the use of finite element formulations in time and the MLF computational approaches typically make use of ideas from discrete variational calculus [33, 8, 9].

The final category of integral forms that must be discussed here is the reciprocal theorems. Following the fundamental work by Betti to derive the reciprocal theorem for elastostatic problems, a number of corresponding relations have been developed for the dynamical case. For example, consider two states of an elastic body, defined by $\left\{u_i, \sigma_{ij}, f_i\right\}$ and $\left\{\tilde{u}_i, \tilde{\sigma}_{ij}, \tilde{f}_i\right\}$ at times t and $\tilde{t}$, respectively. Then, one can write [31]

$$\int_{\Omega} \left(f_i - \rho_o \, \ddot{u}_i \right) \tilde{u}_i \, d\Omega + \int_{\Gamma} t_i \, \tilde{u}_i \, d\Gamma = \int_{\Omega} \left(\tilde{f}_i - \rho_o \, \ddot{\tilde{u}}_i \right) u_i \, d\Omega + \int_{\Gamma} \tilde{t}_i \, u_i \, d\Gamma \tag{16}$$

This can be obtained by multiplying Eqs. (1) and (2) for the first state by displacement and stress of the second state, respectively, and integrating over Ω. Equation (16) is then produced after invoking integration-by-parts and the divergence theorem, which enables the cancellation of the contributions of stress in the domain. Note that Eq. (16) resembles the elastostatic version, except for the addition of the inertial contributions throughout the domain. Can these extra terms be eliminated so that the reciprocal theorem for elastodynamics retains the spirit and simplicity of Betti's original elastostatic version?

Graffi [34] addressed this issue by introducing several versions of the reciprocal theorem that involve integration over time. Keeping in mind the nature of initial value problems, the most interesting of these formulations involves temporal convolutions. Let us deviate slightly from the original work of Graffi by starting from the governing differential equations in the form of Eqs. (14) with $\left\{u_i, J_{ij}, f_i\right\}$ and $\left\{\tilde{u}_i, \tilde{J}_{ij}, \tilde{f}_i\right\}$ representing two solutions that now extend over time from zero to t. Convolving Eqs. (14.1) and (14.2) with $\tilde{u}_i$ and $\tilde{J}_{ij}$, respectively, we must have

$$\int_{\Omega} \left(\left[\rho_o \, \ddot{u}_i - B_{ijk} \dot{J}_{ij} - f_i \right] * \tilde{u}_i \right)(t) \, d\Omega + \int_{\Omega} \left(\left[-A_{ijkl} \ddot{J}_{kl} + B_{ijk} \dot{u}_k \right] * \tilde{J}_{ij} \right)(t) \, d\Omega = 0 \tag{17}$$

where the Riemann convolution of two functions is defined as

$$\left(u * v\right)(t) = \int_0^t u(\tau)v(t-\tau)d\tau \tag{18}$$

After following the usual procedure of applying integration-by-parts, the divergence theorem and some algebraic manipulation, Eq. (17) can be rewritten in the form:

$$\begin{aligned}\int_\Omega f_i * \tilde{u}_i \, d\Omega + \int_\Gamma t_i * \tilde{u}_i \, d\Gamma + \int_\Omega \left[\rho_o \dot{u}_i(0)\tilde{u}_i(t) - \rho_o \dot{u}_i(t)\tilde{u}_i(0) \right] d\Omega \\ = \int_\Omega \tilde{f}_i * u_i \, d\Omega + \int_\Gamma \tilde{t}_i * u_i \, d\Gamma + \int_\Omega \left[\rho_o \dot{\tilde{u}}_i(0) u_i(t) - \rho_o \dot{\tilde{u}}_i(t) u_i(0) \right] d\Omega\end{aligned} \tag{19}$$

The additional domain integrals in Eq. (19), involving the products of initial and final conditions of displacement and velocity, represent the contributions from inertia expressed in terms of values at the beginning and ending of the time interval.

The reciprocal theorem, as expressed in Eq. (19), is often used as the basis for boundary integral representations and, ultimately, boundary element methods. This can be accomplished by selecting one of the states, say $\left\{\tilde{u}_i, \tilde{J}_{ij}, \tilde{f}_i\right\}$, to be the infinite space point force (or fundamental) solution of elastodynamics. For simpler lumped parameter dynamical systems, the analogous approach is associated with the Duhamel integral [35]. Thus, formulations based on temporal convolutions are well-established for dynamical problems. As a final note regarding Eq. (19), we should mention that the convolution based reciprocal theorem is in fact an action principle. Physically, we may say that the convolved action of one system of forces on the displacements of the other is equal to the convolved action of the forces of the second system on the displacements of the first.

Let us conclude this section by coming full circle to introduce a general principle of virtual convolved action for continuum dynamics. As we shall see, this can be derived without regard to a constitutive model and thus is valid for linear or nonlinear, rate independent or dependent materials undergoing infinitesimal deformation.

To derive this new principle, we may begin by convolving Eq. (1) with a virtual displacement field δu_i over time and then integrating over the spatial domain. Thus,

$$\int_{\Omega}\left(\rho_o\,\ddot{u}_i-\sigma_{ji,j}-\overline{f}_i\right)*\delta u_i\;d\Omega=0 \tag{20}$$

After invoking the divergence theorem and assuming a kinematically compatible virtual field with $\delta u_i = 0$ on Γ_v and at time zero, Eq. (20) can be expressed as the *Principle of Virtual Convolved Action*, which balances the internal and external contributions, as follows:

$$\delta A_{\text{internal}} = \delta A_{\text{external}} \tag{21}$$

$$\begin{aligned}\int_{\Omega}\left(\rho_o\,\dot{u}_i*\delta\dot{u}_i+\sigma_{ji}*\delta\varepsilon_{ij}\right)d\Omega=\int_{\Gamma_t}\overline{t}_i*\delta u_i\;d\Gamma+\int_{\Omega}\overline{f}_i*\delta u_i\;d\Omega\\+\int_{\Omega}\rho_o\,\dot{u}_i(0)\delta u_i(t)\;d\Omega\end{aligned} \tag{22}$$

In a way, this extends the principle of virtual work Eq. (6) to dynamics in much the same manner that the reciprocal theorem of Eq. (16) is broadened to Eq. (19) through the introduction of temporal convolutions.

3 Mixed convolved action for elastic continua

Beginning in this section, we again consider the elastodynamic response of a continuum undergoing infinitesimal deformation, but now focus on the development of new mixed convolved action stationary principles. The classical continuum approach based upon Hamilton's principle is developed in Kirchhoff [36] and later in Love [37]. As noted previously, convolution-based continuum formulations appear first in Gurtin [21-23] and Tonti [24]. A broad range of these principles are collected and then generalized by Oden and Reddy [26] to Hellinger-Reissner type mixed principles. Here we apply the ideas from References [8, 9, 13, 27, 28], along with relations from fractional calculus to formulate a mixed variational principle that recovers the governing partial differential equations, along with all initial and boundary conditions for the infinitesimal deformation linear elastodynamic problem.

Following the mixed Lagrangian formalism of Sivaselvan and Reinhorn [8], we use displacement u_i and impulse of elastic stresses J_{ij} as primary variables, but motivated by the concepts discussed in Sect. 2, we convert the inner products appearing in the functional I_{H_S} of Eq. (12) to temporal convolutions. We could develop this new convolved action by starting from the governing equations of elastodynamics, as was done in Sect. 2. However, for a change of pace, in this section, we work the other way around by stating the mixed convolved action I_{C_E} for elastodynamics and then demonstrating that this recovers all of the governing equations, along with initial and boundary conditions, as its Euler-Lagrange equations. This single real scalar action functional can be written in terms of convolutions and fractional derivatives of impulse variables, while recalling that action itself represents the impulse of energy. Let the *mixed convolved action* for elastodynamics be written as follows:

$$\begin{aligned} I_{C_E} &= \int_{\Omega} \left[\tfrac{1}{2} \dot{u}_k * \rho_o \dot{u}_k - \tfrac{1}{2} \dot{J}_{ij} * A_{ijkl} \dot{J}_{kl} \right] d\Omega \\ &\quad + \int_{\Omega} \left[\tfrac{1}{2} \left(\breve{J}_{ij} * B_{ijk} \breve{u}_k - \breve{u}_k * B_{ijk} \breve{J}_{ij} \right) \right] d\Omega \\ &\quad - \int_{\Omega} \left[\breve{u}_k * \breve{\bar{j}}_k \right] d\Omega \\ &\quad - \int_{\Gamma_t} \tfrac{1}{2} \left[\breve{u}_k * \breve{\bar{\tau}}_k \right] d\Gamma + \int_{\Gamma_v} \tfrac{1}{2} \left[\breve{\tau}_k * \breve{\bar{u}}_k \right] d\Gamma \end{aligned} \tag{23}$$

where a superposed breve is used to represent a left Riemann-Liouville semi-derivative with respect to time. Although several different forms of semi-derivatives are available, here the Riemann-Liouville definition yields the desired Euler-Lagrange equations, as will be shown below. Appendix A provides a brief overview of fractional calculus, emphasizing the aspects most relevant to the present work.

Additionally, as a reminder, we use an overbar to denote quantities not subject to variation. In particular, in Eq. (23), $\bar{j}_k$ represents the impulse of the applied body force density $\bar{f}_k$, while $\bar{\tau}_k$ is the impulse of $\bar{t}_k$, the applied surface tractions on a portion of the surface designated as Γ_t. In a similar way, $\bar{u}_k$ represents the enforced surface displacements on Γ_v, with τ_k as the impulse of the resulting reactive tractions t_k on that surface. Here, we assume the boundary conditions are defined, such that $\Gamma_v \cup \Gamma_t = \Gamma$ and $\Gamma_v \cap \Gamma_t = \varnothing$.

Comparing I_{H_S} of Eq. (12) with I_{C_E} of Eq. (23), we notice several changes in addition to the conversion of temporal inner products to convolutions. The coupling term in Eq. (12) between stress impulses and strain rates is now rewritten to provide more balance in the derivatives over space, by separating into two terms, and in time, by introducing semi-derivatives. The semi-derivatives appearing in the body force and boundary traction contributions have a similar purpose. Notice also that in Eq. (23) there is a contribution from enforced boundary displacements on Γ_v. All of these changes are intended to maintain the elegant structure that appears in Eqs. (14) for displacement and stress impulse variables. Just as in the reciprocal theorem of Eq. (19), each term represents a convolved action that conveys the essence of an evolving dynamical system. Within Eq. (23), these represent in order the inertial, complementary stress, displacement-stress impulse, body force, applied traction and enforced displacement convolved actions.

The first variation of the mixed convolved action in Eq. (23) becomes

$$\begin{aligned}\delta I_{C_E} &= \int_\Omega \left[\delta\dot{u}_k * \rho_o \dot{u}_k - \delta\dot{J}_{ij} * A_{ijkl}\dot{J}_{kl} \right] d\Omega \\ &+ \int_\Omega \left[\tfrac{1}{2}\left(\delta\breve{J}_{ij} * B_{ijk}\breve{u}_k - \delta\breve{u}_k * B_{ijk}\breve{J}_{ij} + B_{ijk}\delta\breve{u}_k * \breve{J}_{ij} - B_{ijk}\delta\breve{J}_{ij} * \breve{u}_k \right) \right] d\Omega \\ &- \int_\Omega \left[\delta\breve{u}_k * \breve{\breve{j}}_k \right] d\Omega - \int_{\Gamma_t} \tfrac{1}{2}\left[\delta\breve{u}_k * \breve{\breve{\tau}} \right] d\Gamma + \int_{\Gamma_v} \tfrac{1}{2}\left[\delta\breve{\tau}_k * \breve{\breve{u}}_k \right] d\Gamma \end{aligned} \tag{24}$$

We need to perform temporal integration-by-parts on all of the terms in Eq. (24). However, we also require a spatial integration-by-parts operation on the terms involving $B_{ijk}\delta\breve{u}_k$ and $B_{ijk}\delta\breve{J}_{ij}$. For this development within the classical size-independent theory, we make use of the symmetry of stresses $\dot{J}_{kj}$ and the Cauchy definition of surface traction, where $t_k = \dot{\tau}_k = \dot{J}_{jk}n_j$. We also must recognize that similar relations hold for the left half-order time derivatives of these variables. The reformulation for $B_{ijk}\delta\breve{u}_k$ proceeds as follows:

$$\begin{aligned}\int_\Omega \left[B_{ijk}\delta\breve{u}_k * \breve{J}_{ij} \right] d\Omega &= \int_\Omega \left[\tfrac{1}{2}\left(\delta\breve{u}_{i,j} + \delta\breve{u}_{j,i} \right) * \breve{J}_{ij} \right] d\Omega \\ &= \int_\Omega \left[\delta\breve{u}_{k,j} * \breve{J}_{kj} \right] d\Omega \\ &= \int_\Omega \left[\delta\breve{u}_k * \breve{J}_{kj} \right]_{,j} d\Omega - \int_\Omega \left[\delta\breve{u}_k * \breve{J}_{kj,j} \right] d\Omega \\ &= \int_\Gamma \left[\delta\breve{u}_k * \breve{J}_{jk}n_j \right] d\Gamma - \int_\Omega \left[\delta\breve{u}_k * \breve{J}_{jk,j} \right] d\Omega \\ &= \int_\Gamma \left[\delta\breve{u}_k * \breve{\tau}_k \right] d\Gamma - \int_\Omega \left[\delta\breve{u}_k * \breve{J}_{jk,j} \right] d\Omega \\ &= \int_\Gamma \left[\delta\breve{u}_k * \breve{\tau}_k \right] d\Gamma - \int_\Omega \left[\delta\breve{u}_k * B_{ijk}\breve{J}_{ij} \right] d\Omega \end{aligned} \tag{25.1}$$

Similarly, for $B_{ijk}\delta\breve{J}_{ij}$, we have

$$\int_{\Omega}\left[B_{ijk}\delta\breve{J}_{ij}*\breve{u}_k\right]d\Omega=\int_{\Gamma}\left[\delta\breve{\tau}_k*\breve{u}_k\right]d\Gamma-\int_{\Omega}\left[\delta\breve{J}_{ij}*B_{ijk}\breve{u}_k\right]d\Omega \tag{25.2}$$

After substituting Eqs. (25.1) and (25.2) into Eq. (24), we have

$$\begin{aligned}\delta I_{C_E}=&\int_{\Omega}\left[\delta\dot{u}_k*\rho_o\dot{u}_k-\delta\dot{J}_{ij}*A_{ijkl}\dot{J}_{kl}\right]d\Omega\\&+\int_{\Omega}\left[\delta\breve{J}_{ij}*B_{ijk}\breve{u}_k-\delta\breve{u}_k*B_{ijk}\breve{J}_{ij}\right]d\Omega\\&+\int_{\Gamma}\tfrac{1}{2}\left[\delta\breve{u}_k*\breve{\tau}_k-\delta\breve{\tau}_k*\breve{u}_k\right]d\Gamma\\&-\int_{\Omega}\left[\delta\breve{u}_k*\breve{\bar{j}}_k\right]d\Omega\\&-\int_{\Gamma_t}\tfrac{1}{2}\left[\delta\breve{u}_k*\breve{\bar{\tau}}_k\right]d\Gamma+\int_{\Gamma_v}\tfrac{1}{2}\left[\delta\breve{\tau}_k*\breve{\bar{u}}_k\right]d\Gamma\end{aligned} \tag{26}$$

Next, we must perform temporal integration-by-parts to shift all time derivatives from the variations to the real fields. For example, classical integration-by-parts must be used on the two terms on the first line of Eq. (26), while fractional integration-by-parts is needed to shift the semi-derivatives of the variations in all of the remaining integrals in Eq. (26). The required fractional calculus relations are provided in Appendix A, where the Riemann-Liouville definitions provide exactly the desired terms, as will be demonstrated in the following.

After performing these necessary temporal integration-by-parts operations, the stationarity of the mixed convolved action may be written

$$\begin{aligned}\delta I_{C_E}=&\int_{\Omega}\left[\delta u_k*\rho_o\ddot{u}_k+\delta u_k(t)\rho_o\dot{u}_k(0)-\delta u_k(0)\rho_o\dot{u}_k(t)\right]d\Omega\\&-\int_{\Omega}\left[\delta J_{ij}*A_{ijkl}\ddot{J}_{kl}+\delta J_{ij}(t)A_{ijkl}\dot{J}_{kl}(0)-\delta J_{ij}(0)A_{ijkl}\dot{J}_{kl}(t)\right]d\Omega\\&+\int_{\Omega}\left[\delta J_{ij}*B_{ijk}\dot{u}_k+\delta J_{ij}(t)B_{ijk}u_k(0)-\delta u_k*B_{ijk}\dot{J}_{ij}-\delta u_k(t)B_{ijk}J_{ij}(0)\right]d\Omega\\&+\int_{\Gamma}\tfrac{1}{2}\left[\delta u_k*\dot{\tau}_k+\delta u_k(t)\tau_k(0)-\delta\tau_k*\dot{u}_k-\delta\tau_k(t)u_k(0)\right]d\Gamma\\&-\int_{\Omega}\left[\delta u_k*\bar{f}_k+\delta u_k(t)\bar{j}_k(0)\right]d\Omega\\&-\int_{\Gamma_t}\tfrac{1}{2}\left[\delta u_k*\bar{t}_k+\delta u_k(t)\bar{\tau}_k(0)\right]d\Gamma\\&+\int_{\Gamma_v}\tfrac{1}{2}\left[\delta\tau_k*\bar{v}_k+\delta\tau_k(t)\bar{u}_k(0)\right]d\Gamma=0\end{aligned} \tag{27}$$

Collecting terms according to the variations, we then may write

$$
\begin{aligned}
\delta I_{C_E} = & \int_{\Omega} \delta u_k * \left[\rho_o \ddot{u}_k - B_{ijk} \dot{J}_{ij} - \overline{f}_k \right] d\Omega + \int_{\Omega} \delta J_{ij} * \left[-A_{ijkl} \ddot{J}_{kl} + B_{ijk} \dot{u}_k \right] d\Omega \\
& + \int_{\Omega} \delta u_k(t) \left[\rho_o \dot{u}_k(0) - B_{ijk} J_{ij}(0) - \overline{j}_k(0) \right] d\Omega - \int_{\Omega} \delta u_k(0) \left[\rho_o \dot{u}_k(t) \right] d\Omega \\
& + \int_{\Omega} \delta J_{ij}(t) \left[-A_{ijkl} \dot{J}_{kl}(0) + B_{ijk} u_k(0) \right] d\Omega - \int_{\Omega} \delta J_{ij}(0) \left[-A_{ijkl} \dot{J}_{kl}(t) \right] d\Omega \\
& + \int_{\Gamma_t} \tfrac{1}{2} \left[\delta u_k * t_k - \delta u_k * \overline{t}_k \right] d\Gamma + \int_{\Gamma_t} \tfrac{1}{2} \left[\delta u_k(t) \tau_k(0) - \delta u_k(t) \overline{\tau}_k(0) \right] d\Gamma \\
& + \int_{\Gamma_v} \tfrac{1}{2} \left[\delta u_k * t_k \right] d\Gamma + \int_{\Gamma_v} \tfrac{1}{2} \left[\delta u_k(t) \tau_k(0) \right] d\Gamma \\
& - \int_{\Gamma_v} \tfrac{1}{2} \left[\delta \tau_k * v_k - \delta \tau_k * \overline{v}_k \right] d\Gamma - \int_{\Gamma_v} \tfrac{1}{2} \left[\delta \tau_k(t) u_k(0) - \delta \tau_k(t) \overline{u}_k(0) \right] d\Gamma \\
& - \int_{\Gamma_t} \tfrac{1}{2} \left[\delta \tau_k * v_k \right] d\Gamma - \int_{\Gamma_t} \tfrac{1}{2} \left[\delta \tau_k(t) u_k(0) \right] d\Gamma = 0
\end{aligned}
\tag{28}
$$

With arbitrary variations over space and time, from Eq. (28), we have the following set of Euler-Lagrange equations, representing linear momentum balance and elastic constitutive behavior, respectively,

$$\rho_o \ddot{u}_k - B_{ijk} \dot{J}_{ij} = \overline{f}_k \qquad x \in \Omega,\ \tau \in (0,t) \tag{29.1}$$

$$-A_{ijkl} \ddot{J}_{kl} + B_{ijk} \dot{u}_k = 0 \qquad x \in \Omega,\ \tau \in (0,t) \tag{29.2}$$

natural and essential boundary conditions, respectively,

$$t_k = \overline{t}_k \qquad x \in \Gamma_t,\ \tau \in (0,t) \tag{30.1}$$

$$v_k = \overline{v}_k \qquad x \in \Gamma_v,\ \tau \in (0,t) \tag{30.2}$$

initial conditions over the domain

$$\rho_o \dot{u}_k(0) - B_{ijk} J_{ij}(0) = \overline{j}_k(0) \qquad x \in \Omega \tag{31.1}$$

$$-A_{ijkl} \dot{J}_{kl}(0) + B_{ijk} u_k(0) = 0 \qquad x \in \Omega \tag{31.2}$$

and initial conditions on the boundary

$$\tau_k(0) = \overline{\tau}_k(0) \qquad x \in \Gamma_t \tag{32.1}$$

$$u_k(0) = \bar{u}_k(0) \qquad x \in \Gamma_v \tag{32.2}$$

with the variations defined, such that

$$\delta\tau_k = 0 \qquad x \in \Gamma_t,\ \tau \in (0,t) \tag{33.1}$$

$$\delta u_k = 0 \qquad x \in \Gamma_v,\ \tau \in (0,t) \tag{33.2}$$

$$\delta u_k(0) = 0 \qquad x \in \Omega \tag{34.1}$$

$$\delta J_{ij}(0) = 0 \qquad x \in \Omega \tag{34.2}$$

$$\delta\tau_k(t) = 0 \qquad x \in \Gamma_t \tag{35.1}$$

$$\delta u_k(t) = 0 \qquad x \in \Gamma_v \tag{35.2}$$

Consequently, we have established a *Principle of Stationary Mixed Convolved Action for a Linear Elastodynamic Continuum* undergoing infinitesimal deformation, which may be stated as follows: Of all the possible trajectories $\{u_k(\tau), J_{ij}(\tau)\}$ of the system during the time interval $(0,t)$, the one that renders the action I_{C_E} in Eq. (23) stationary, corresponds to the solution of the initial/boundary value problem. Thus, the stationary trajectory satisfies the equations of motion Eq. (29.1) and constitutive relations Eq. (29.2) in the domain Ω, along with the traction Eq. (30.1) and velocity Eq. (30.2) boundary conditions, throughout the time interval, while also satisfying the initial conditions defined by Eqs. (31.1) and (31.2) in Ω and Eqs. (32.1) and (32.2) on the appropriate portions of the bounding surface. Furthermore, the possible trajectories under consideration during the variational process are constrained only by their need to satisfy the specified initial and boundary conditions of the problem in the form of Eqs. (33), (34) and (35).

We should note that it may not be possible to obtain all of these conditions for the mixed initial/boundary value problem of elastodynamics without involving the combination of mixed variables, convolution and fractional derivatives. However, due to special properties of the convolution of Riemann-Liouville fractional derivatives, alternative forms of the mixed convolved action can be written using complementary order fractional derivatives. This is discussed in more detail for the single degree-of-freedom case in Reference [27] and also Reference [28], where a computational method for mixed convolved action is developed.

4 Mixed convolved action for viscoelastic continua

In this section, we shift attention to a continuum problem involving energy dissipation and develop for the first time pure variational statements in irreversible solid mechanics, which capture all aspects of the initial/boundary value problem. As the first example, we consider dissipative processes in viscoelastic media and develop mixed convolved action formalisms for infinitesimal deformations. To formulate such a *mixed convolved action* for dynamic viscoelasticity, let us begin with the corresponding action for elastodynamics in Eq. (23) and then add terms for the viscoelastic contributions.

4.1 Kelvin-Voigt velocity-dependent viscoelasticity

As an extension of the development in Reference [28] for a single-degree-of-freedom Kelvin-Voigt element, we may write for the continuum:

$$I_{C_K} = I_{C_E} + \int_{\Omega} \left[\tfrac{1}{2} \breve{u}_j * C_{jk} \, \breve{u}_k \right] d\Omega \tag{36}$$

Here C_{jk} is the symmetric second order constitutive tensor associated with velocity-dependent viscoelastic response. Notice that this additional contribution involves the self-convolution of semi-derivatives of displacement, which conveys the path-dependent dissipative nature of viscoelastic behavior. This is in contrast to the self-convolutions of velocities and stresses that appear in the conservative mixed convolved action I_{C_E}, which carries over from the elastodynamic case, as shown in Eq. (36).

The first variation of the mixed convolved action in Eq. (36) becomes

$$\begin{aligned} \delta I_{C_K} = & \int_{\Omega} \left[\delta \dot{u}_k * \rho_o \dot{u}_k + \delta \breve{u}_j * C_{jk} \, \breve{u}_k - \delta \dot{J}_{ij} * A_{ijkl} \dot{J}_{kl} \right] d\Omega \\ & + \int_{\Omega} \left[\tfrac{1}{2} \left(\delta \breve{J}_{ij} * B_{ijk} \breve{u}_k - \delta \breve{u}_k * B_{ijk} \breve{J}_{ij} + B_{ijk} \delta \breve{u}_k * \breve{J}_{ij} - B_{ijk} \delta \breve{J}_{ij} * \breve{u}_k \right) \right] d\Omega \\ & - \int_{\Omega} \left[\delta \breve{u}_k * \breve{\bar{j}}_k \right] d\Omega - \int_{\Gamma_t} \tfrac{1}{2} \left[\delta \breve{u}_k * \breve{\bar{\tau}} \right] d\Gamma + \int_{\Gamma_v} \tfrac{1}{2} \left[\delta \breve{\tau}_k * \breve{\bar{u}}_k \right] d\Gamma \end{aligned} \tag{37}$$

Next, we perform classical and Riemann-Liouville fractional integration-by-parts on the appropriate terms in Eq. (37) in a manner similar to what was done for the elastodynamic case in Sect. 3. Then, after collecting terms according to the variations, the statement for stationary action I_{C_K} can be written

$$
\begin{aligned}
\delta I_{C_K} &= \int_\Omega \delta u_k * \left[\rho_o \ddot{u}_k + C_{jk}\dot{u}_j - B_{ijk}\dot{J}_{ij} - \overline{f}_k \right] d\Omega + \int_\Omega \delta J_{ij} * \left[-A_{ijkl}\ddot{J}_{kl} + B_{ijk}\dot{u}_k \right] d\Omega \\
&+ \int_\Omega \delta u_k(t)\left[\rho_o \dot{u}_k(0) + C_{jk}u_j(0) - B_{ijk}J_{ij}(0) - \overline{j}_k(0) \right] d\Omega - \int_\Omega \delta u_k(0)\left[\rho_o \dot{u}_k(t) \right] d\Omega \\
&+ \int_\Omega \delta J_{ij}(t)\left[-A_{ijkl}\dot{J}_{kl}(0) + B_{ijk}u_k(0) \right] d\Omega - \int_\Omega \delta J_{ij}(0)\left[-A_{ijkl}\dot{J}_{kl}(t) \right] d\Omega \\
&+ \int_{\Gamma_t} \tfrac{1}{2}\left[\delta u_k * t_k - \delta u_k * \overline{t}_k \right] d\Gamma + \int_{\Gamma_t} \tfrac{1}{2}\left[\delta u_k(t)\tau_k(0) - \delta u_k(t)\overline{\tau}_k(0) \right] d\Gamma \\
&+ \int_{\Gamma_v} \tfrac{1}{2}\left[\delta u_k * t_k \right] d\Gamma + \int_{\Gamma_v} \tfrac{1}{2}\left[\delta u_k(t)\tau_k(0) \right] d\Gamma \\
&- \int_{\Gamma_v} \tfrac{1}{2}\left[\delta \tau_k * v_k - \delta \tau_k * \overline{v}_k \right] d\Gamma - \int_{\Gamma_v} \tfrac{1}{2}\left[\delta \tau_k(t)u_k(0) - \delta \tau_k(t)\overline{u}_k(0) \right] d\Gamma \\
&- \int_{\Gamma_t} \tfrac{1}{2}\left[\delta \tau_k * v_k \right] d\Gamma - \int_{\Gamma_t} \tfrac{1}{2}\left[\delta \tau_k(t)u_k(0) \right] d\Gamma = 0
\end{aligned}
\tag{38}
$$

Notice that this is identical to the variation of δI_{C_E} in Eq. (28), except for the first and third volume integrals, which now contain an additional term involving C_{jk} associated with Kelvin-Voigt viscoelasticity.

Then, by assuming arbitrary variations over space and time, we uncover the following set of Euler-Lagrange equations, representing linear momentum balance with velocity-dependent Kelvin-Voigt contributions and elastic constitutive behavior, respectively,

$$\rho_o \ddot{u}_k + C_{jk}\dot{u}_j - B_{ijk}\dot{J}_{ij} = \overline{f}_k \qquad x \in \Omega,\ \tau \in (0,t) \tag{39.1}$$

$$-A_{ijkl}\ddot{J}_{kl} + B_{ijk}\dot{u}_k = 0 \qquad x \in \Omega,\ \tau \in (0,t) \tag{39.2}$$

natural and essential boundary conditions, respectively,

$$t_k = \overline{t}_k \qquad x \in \Gamma_t,\ \tau \in (0,t) \tag{40.1}$$

$$v_k = \overline{v}_k \qquad x \in \Gamma_v,\ \tau \in (0,t) \tag{40.2}$$

initial conditions over the domain, including the initial viscoelastic contribution,

$$\rho_o \dot{u}_k(0) + C_{jk} u_j(0) - B_{ijk} J_{ij}(0) = \bar{j}_k(0) \qquad x \in \Omega \tag{41.1}$$

$$-A_{ijkl} \dot{J}_{kl}(0) + B_{ijk} u_k(0) = 0 \qquad x \in \Omega \tag{41.2}$$

and initial conditions on the boundary

$$\tau_k(0) = \bar{\tau}_k(0) \qquad x \in \Gamma_t \tag{42.1}$$

$$u_k(0) = \bar{u}_k(0) \qquad x \in \Gamma_v \tag{42.2}$$

These are precisely the equations that govern the behavior of a Kelvin-Voigt viscoelastic medium. In addition, from Eq. (38), the variations must be defined, such that

$$\delta \tau_k = 0 \qquad x \in \Gamma_t,\ \tau \in (0,t) \tag{43.1}$$

$$\delta u_k = 0 \qquad x \in \Gamma_v,\ \tau \in (0,t) \tag{43.2}$$

$$\delta u_k(0) = 0 \qquad x \in \Omega \tag{44.1}$$

$$\delta J_{ij}(0) = 0 \qquad x \in \Omega \tag{44.2}$$

$$\delta \tau_k(t) = 0 \qquad x \in \Gamma_t \tag{45.1}$$

$$\delta u_k(t) = 0 \qquad x \in \Gamma_v \tag{45.2}$$

These correspond exactly to the known boundary and initial conditions of the Kelvin-Voigt velocity-dependent viscoelastic problem.

Therefore, we have established a *Principle of Stationary Mixed Convolved Action for a Linear Kelvin-Voigt Viscoelastic Continuum* undergoing infinitesimal deformation. This principle may be stated as follows: Of all the possible trajectories $\{u_k(\tau), J_{ij}(\tau)\}$ of the system during the time interval $(0,t)$, the one that renders the action I_{C_K} in Eq. (36) stationary, corresponds to the solution of the initial/boundary value problem. Thus, the stationary trajectory satisfies the equations of motion Eq. (39.1) and constitutive relations Eq. (39.2) in the domain Ω, along with the traction

Eq. (40.1) and velocity Eq. (40.2) boundary conditions, throughout the time interval, while also satisfying the initial conditions defined by Eqs. (41.1) and (41.2) in Ω and Eqs. (42.1) and (42.2) on the appropriate portions of the bounding surface. Furthermore, the possible trajectories under consideration during the variational process are constrained only by the need to satisfy the specified boundary and initial conditions of the problem in the form of Eqs. (43), (44) and (45).

4.2 Maxwell stress-dependent viscoelasticity

In a similar manner, we may consider Maxwell viscoelastic constitutive models within the mixed convolved action formalism. For this case, the action can be written as follows:

$$I_{C_M} = I_{C_E} - \int_\Omega \left[\tfrac{1}{2} \breve{J}_{ij} * D_{ijkl} \; \breve{J}_{kl} \right] d\Omega \tag{46}$$

where D_{ijkl} is the fourth order constitutive tensor associated with Maxwell viscoelastic response and all of the other terms are directly from the elastic case in Eq. (23).

Next, taking the first variation of the mixed convolved action from Eq. (46), we have

$$\begin{aligned} \delta I_{C_M} &= \int_\Omega \left[\delta \dot{u}_k * \rho_o \dot{u}_k - \delta \dot{J}_{ij} * A_{ijkl} \dot{J}_{kl} - \delta \breve{J}_{ij} * D_{ijkl} \; \breve{J}_{kl} \right] d\Omega \\ &+ \int_\Omega \left[\tfrac{1}{2} \left(\delta \breve{J}_{ij} * B_{ijk} \breve{u}_k - \delta \breve{u}_k * B_{ijk} \breve{J}_{ij} + B_{ijk} \delta \breve{u}_k * \breve{J}_{ij} - B_{ijk} \delta \breve{J}_{ij} * \breve{u}_k \right) \right] d\Omega \\ &- \int_\Omega \left[\delta \breve{u}_k * \breve{\bar{j}}_k \right] d\Omega - \int_{\Gamma_t} \tfrac{1}{2} \left[\delta \breve{u}_k * \breve{\bar{\tau}} \right] d\Gamma + \int_{\Gamma_v} \tfrac{1}{2} \left[\delta \breve{\tau}_k * \breve{\bar{u}}_k \right] d\Gamma \end{aligned} \tag{47}$$

After performing classical and Riemann-Liouville fractional integration-by-parts on the appropriate terms in Eq. (47), collecting terms according to the variations, and assuming arbitrary variations over space and time, we obtain the following set of Euler-Lagrange equations. These represent linear momentum balance and Maxwell viscoelastic constitutive behavior, respectively,

$$\rho_o \ddot{u}_k - B_{ijk} \dot{J}_{ij} = \bar{f}_k \qquad x \in \Omega, \; \tau \in (0,t) \tag{48.1}$$

$$-A_{ijkl} \ddot{J}_{kl} - D_{ijkl} \dot{J}_{kl} + B_{ijk} \dot{u}_k = 0 \qquad x \in \Omega, \; \tau \in (0,t) \tag{48.2}$$

natural and essential boundary conditions, respectively,

$$t_k = \bar{t}_k \qquad x \in \Gamma_t, \; \tau \in (0,t) \tag{49.1}$$

$$v_k = \overline{v}_k \qquad x \in \Gamma_v,\ \tau \in (0,t) \tag{49.2}$$

initial conditions over the domain, including the initial viscoelastic contribution,

$$\rho_o \dot{u}_k(0) - B_{ijk} J_{ij}(0) = \overline{j}_k(0) \qquad x \in \Omega \tag{50.1}$$

$$-A_{ijkl} \dot{J}_{kl}(0) - D_{ijkl} J_{kl}(0) + B_{ijk} u_k(0) = 0 \qquad x \in \Omega \tag{50.2}$$

and initial conditions on the boundary

$$\tau_k(0) = \overline{\tau}_k(0) \qquad x \in \Gamma_t \tag{51.1}$$

$$u_k(0) = \overline{u}_k(0) \qquad x \in \Gamma_v \tag{51.2}$$

These are precisely the equations that govern the behavior of a Maxwell viscoelastic medium.

As in the previous cases, the variations must be defined, such that

$$\delta \tau_k = 0 \qquad x \in \Gamma_t,\ \tau \in (0,t) \tag{52.1}$$

$$\delta u_k = 0 \qquad x \in \Gamma_v,\ \tau \in (0,t) \tag{52.2}$$

$$\delta u_k(0) = 0 \qquad x \in \Omega \tag{53.1}$$

$$\delta J_{ij}(0) = 0 \qquad x \in \Omega \tag{53.2}$$

$$\delta \tau_k(t) = 0 \qquad x \in \Gamma_t \tag{54.1}$$

$$\delta u_k(t) = 0 \qquad x \in \Gamma_v \tag{54.2}$$

which correspond exactly to the known boundary and initial conditions of the Maxwell viscoelastic problem.

Therefore, we have established a *Principle of Stationary Mixed Convolved Action for a Linear Maxwell Viscoelastic Continuum* undergoing infinitesimal deformation, which may be stated as follows: Of all possible trajectories $\{u_k(\tau), J_{ij}(\tau)\}$ of the system during the time interval $(0,t)$, the one that renders the action I_{C_M} in Eq. (46) stationary, corresponds to the solution of the initial/boundary value problem. In this case, the stationary trajectory satisfies the equations of motion Eq. (48.1) and viscoelastic constitutive relations Eq. (48.2) in the domain Ω, along with the traction Eq. (49.1) and velocity Eq. (49.2) boundary conditions, throughout the time interval. This trajectory also satisfies the initial conditions defined by Eqs. (50.1) and (50.2) in Ω and Eqs. (51.1) and (51.2) on the appropriate portions of the bounding surface Γ. Furthermore, within the variational process, the possible trajectories under consideration are constrained only by the need to satisfy the specified boundary and initial conditions of the problem in the form of Eqs. (52), (53) and (54).

4.3 Mixed convolved action for dissipative systems

In this section, two examples of the application of the mixed convolved action formalism to viscoelastic media have been provided. For both of these cases, and many others, the development of variational principles for dissipative phenomena is quite straightforward within this new approach, based upon mixed impulsive variables, Riemann-Liouville fractional calculus, and the convolution of convolutions. The remainder of the paper focuses on the development of finite element methods and the numerical solution of several initial/boundary value problems.

5 Weak forms and finite elements in space and time

5.1 Weak form and stationarity of mixed convolved action

In the previous two sections, we demonstrate that the mixed convolved action defined by Eqs. (23), (36) and (46) can recover the associated governing partial differential equations, along with the boundary conditions and initial conditions, which describe completely the initial-boundary value problem. Next, we define an appropriate weak form to serve as the foundation for a finite element method in space and time. For this, we consider the general dissipative problem involving both Kelvin-Voigt and Maxwell viscoelastic contributions, realizing that fully conservative elastodynamics is simply a special case having C_{jk} and D_{ijkl} as null constitutive tensors.

Let us begin by defining the mixed convolved action for this combined viscoelastic problem, which from Eqs. (36) and (46) can be written:

$$I_{C_V} = I_{C_E} + \int_{\Omega} \left[\tfrac{1}{2}\, \breve{u}_j * C_{jk}\, \breve{u}_k \right] d\Omega - \int_{\Omega} \left[\tfrac{1}{2}\, \breve{J}_{ij} * D_{ijkl}\, \breve{J}_{kl} \right] d\Omega \tag{55}$$

The first variation of Eq. (55) can be used as the weak form for a finite element method. Of course, this weak form is not unique, because various integration-by-parts operations may be invoked. In particular, one may tailor the weak form to provide desirable symmetries and continuity requirements. For the present work, we choose to seek stationarity of the following weak form:

$$\begin{aligned} \delta I_{C_V} &= \int_{\Omega} \left[\delta\dot{u}_k * \rho_o \dot{u}_k + \delta\breve{u}_j * C_{jk}\, \breve{u}_k - \delta\dot{J}_{ij} * A_{ijkl} \dot{J}_{kl} - \delta\breve{J}_{ij} * D_{ijkl}\, \breve{J}_{kl} \right] d\Omega \\ &+ \int_{\Omega} \left[\delta\breve{J}_{ij} * B_{ijk} \breve{u}_k + B_{ijk} \delta\breve{u}_k * \breve{J}_{ij} \right] d\Omega \\ &- \int_{\Omega} \left[\delta\breve{u}_k * \bar{\breve{j}}_k \right] d\Omega - \int_{\Gamma_t} \tfrac{1}{2} \left[\delta\breve{u}_k * \left(\bar{\breve{\tau}}_k + \breve{\tau}_k \right) \right] d\Gamma \\ &+ \int_{\Gamma_v} \tfrac{1}{2} \left[\delta\breve{\tau}_k * \left(\bar{\breve{u}}_k - \breve{u}_k \right) \right] d\Gamma = 0 \end{aligned} \tag{56}$$

with $\delta u_k = 0$ on Γ_v and $\delta\tau_k = 0$ on Γ_t. Notice the appearance of terms involving body forces and surface tractions in analogy with similar contributions in the usual weak form for quasistatic problems developed from, for example, the principle of minimum potential energy. The difference here is that the problem is dynamical, which requires fractional derivatives and convolutions over time. In addition, to create temporal balance between the kinematic and kinetic representations, the present weak form in Eq. (56) is written in terms of mixed impulsive variables.

Closer examination of Eq. (56) reveals that first order spatial derivatives are applied on displacements u_k through the operator B_{ijk}, while no spatial derivatives operate on the stress impulses J_{kl}. Consequently, for a convergent discretization of Eq. (56), displacement field variables must be C^0 continuous in space, while stress impulses require only C^{-1} continuity, or in other words, these latter field variables may be discontinuous in space. For the temporal representation, first order time derivatives are present for both displacement and stress impulse field variables and, thus, C^0 continuity must be provided over the time domain. This means the impulse of stress must be continuous in time, whereas stress may exhibit discontinuities in both time and space.

A further matter to emphasize concerning Eq. (56) is the complete absence of temporal end point constraints. On the other hand, any direct application of Hamilton's principle would require zero variations of the field variables at the beginning and end of the time interval, which is problematic for the formulation of a finite element method in time. The previous variational approaches of Kane et al. [38, 39], Marsden and West [40], Sivaselvan and Reinhorn [8], Sivaselvan et al. [9] and References [12-14] all attempt to circumvent this issue by invoking ideas from discrete variational calculus [33, 41] and by shifting the evaluation points to the interior of the temporal domain. On the other hand, here with the weak form emanating from the mixed convolved action, there is no such restriction and temporal finite element approaches are easily constructed.

With all of the above in mind, we begin with simple elements and a low order representation of the field variables in space and time for illustrative purposes. Many more elaborate and higher order formulations are possible. For plane stress and plane strain dynamic analysis, we choose three-node triangular elements with two degrees-of-freedom at each node and linear variations of displacement over the element. In addition, stress impulses are assumed as spatially-constant element-based variables. Both displacements u_k and stress impulses J_{kl} are described by linear shape functions in time. Identical spatial and temporal representations are used for the corresponding variations δu_k and δJ_{kl} and all integrals in Eq. (56) are evaluated analytically, including those involving the semi-derivatives, over a time interval from 0 to Δt.

5.2 Spatial discretization of weak form

Let us start by performing integration in space over an arbitrary three-node triangular element in the x_1 - x_2 (or x - y) plane for each of the volumetric terms in Eq. (56). For this we use a consistent representation of each term, meaning in this case that we take only the term with constant variation [42]. After this spatial discretization and integration, we may write

$$\int_{\Omega_e} \delta \dot{u}_k * \rho_o \dot{u}_k \, d\Omega = \delta \dot{\mathbf{u}}^T * \mathbf{M} \, \dot{\mathbf{u}} \tag{57.1}$$

$$\int_{\Omega_e} \delta \breve{u}_j * C_{jk} \, \breve{u}_k \, d\Omega = \delta \breve{\mathbf{u}}^T * \mathbf{C} \, \breve{\mathbf{u}} \tag{57.2}$$

$$\int_{\Omega_e} \delta \dot{J}_{ij} * A_{ijkl} \, \dot{J}_{kl} \, d\Omega = \delta \dot{\mathbf{J}}^T * \mathbf{A} \, \dot{\mathbf{J}} \tag{57.3}$$

$$\int_{\Omega_e} \delta \breve{J}_{ij} * D_{ijkl}\ \breve{J}_{kl}\ d\Omega = \delta \breve{\mathbf{J}}^T * \mathbf{D}\, \breve{\mathbf{J}} \tag{57.4}$$

$$\int_{\Omega_e} \delta \breve{J}_{ij} * B_{ijk}\ \breve{u}_k\ d\Omega = \delta \breve{\mathbf{J}}^T * \mathbf{B}\, \breve{\mathbf{u}} \tag{57.5}$$

$$\int_{\Omega_e} B_{ijk}\ \delta \breve{u}_k * \breve{J}_{ij}\ d\Omega = \delta \breve{\mathbf{u}}^T\ \mathbf{B}^T * \breve{\mathbf{J}} \tag{57.6}$$

For an isotropic solid, the matrices in Eqs. (57) become

$$\mathbf{M} = \rho_o\ A\, b\, \mathbf{I}_6 / 3 \tag{58.1}$$

$$\mathbf{C} = c\ A\, b\, \mathbf{I}_6 / 3 \tag{58.2}$$

$$\mathbf{B} = \frac{b}{2}\begin{bmatrix} y_2 - y_3 & 0 & y_3 - y_1 & 0 & y_1 - y_2 & 0 \\ 0 & x_3 - x_2 & 0 & x_1 - x_3 & 0 & x_2 - x_1 \\ x_3 - x_2 & y_2 - y_3 & x_1 - x_3 & y_3 - y_1 & x_2 - x_1 & y_1 - y_2 \end{bmatrix} \tag{58.3}$$

where x_q and y_q represent the coordinates of node q of the triangular element, c is the Kelvin-Voigt damping coefficient, A represents the area of the triangle, b is the thickness and $\mathbf{I}_p$ is the $p \times p$ identity matrix.

Furthermore, for plane strain, we may write

$$\mathbf{A} = \frac{1+\nu}{E} A\, b \begin{bmatrix} 1-\nu & -\nu & 0 \\ -\nu & 1-\nu & 0 \\ 0 & 0 & 2 \end{bmatrix} \tag{59.1}$$

while for plane stress

$$\mathbf{A} = \frac{1}{E} A\, b \begin{bmatrix} 1 & -\nu & 0 \\ -\nu & 1 & 0 \\ 0 & 0 & 2(1+\nu) \end{bmatrix} \tag{59.2}$$

with elastic modulus E and Poisson ratio ν. Meanwhile, for the Maxwell matrix, we use the simple form

$$\mathbf{D} = \frac{1}{t_M}\mathbf{A} \tag{60}$$

with relaxation time t_M.

In a similar manner, after spatial discretization, the body force contributions over an element become:

$$\int_{\Omega_e} \delta\breve{u}_k * \bar{\breve{j}}_k \; d\Omega = \delta\breve{\mathbf{u}}^T * \bar{\breve{\mathbf{j}}} \tag{61}$$

while the terms from the boundary conditions can be obtained by integration over an element edge, producing

$$\int_{\Gamma_t} \tfrac{1}{2}\left[\delta\breve{u}_k * \left(\bar{\breve{\tau}}_k + \breve{\tau}_k\right)\right] d\Gamma = \tfrac{1}{2}\delta\breve{\mathbf{u}}^T * \left(\bar{\breve{\boldsymbol{\tau}}} + \breve{\boldsymbol{\tau}}\right) \tag{62.1}$$

$$\int_{\Gamma_v} \tfrac{1}{2}\left[\delta\breve{\tau}_k * \left(\bar{\breve{u}}_k - \breve{u}_k\right)\right] d\Gamma = \tfrac{1}{2}\delta\breve{\boldsymbol{\tau}}^T * \left(\bar{\breve{\mathbf{u}}} - \breve{\mathbf{u}}\right) \tag{62.2}$$

With higher order elements, there are some interesting ways to accommodate the influences defined in Eqs. (62). Here we simply equate the unknown variables to the known values (i.e., $\breve{\boldsymbol{\tau}} = \bar{\breve{\boldsymbol{\tau}}}$, $\breve{\mathbf{u}} = \bar{\breve{\mathbf{u}}}$) on edges associated with Γ_t and Γ_v, such that the enforced tractions have a contribution defined by

$$\int_{\Gamma_t} \tfrac{1}{2}\left[\delta\breve{u}_k * \left(\bar{\breve{\tau}}_k + \breve{\tau}_k\right)\right] d\Gamma = \delta\breve{\mathbf{u}}^T * \bar{\breve{\boldsymbol{\tau}}} \tag{63.1}$$

while the enforced displacement integral has no explicit additional effect, because

$$\int_{\Gamma_v} \tfrac{1}{2}\left[\delta\breve{\tau}_k * \left(\bar{\breve{u}}_k - \breve{u}_k\right)\right] d\Gamma = 0 \tag{63.2}$$

Substituting Eqs. (57), (61) and (63) into Eq. (56) provides the spatially discretized mixed weak form for an element, which can be written:

$$\begin{aligned} &\delta\dot{\mathbf{u}}^T * \mathbf{M}\,\dot{\mathbf{u}} + \delta\breve{\mathbf{u}}^T * \mathbf{C}\,\breve{\mathbf{u}} - \delta\dot{\mathbf{J}}^T * \mathbf{A}\,\dot{\mathbf{J}} - \delta\breve{\mathbf{J}}^T * \mathbf{D}\,\breve{\mathbf{J}} \\ &\quad + \delta\breve{\mathbf{J}}^T * \mathbf{B}\,\breve{\mathbf{u}} + \delta\breve{\mathbf{u}}^T\,\mathbf{B}^T * \breve{\mathbf{J}} - \delta\breve{\mathbf{u}}^T * \bar{\breve{\mathbf{j}}} - \delta\breve{\mathbf{u}}^T * \bar{\breve{\boldsymbol{\tau}}} = 0 \end{aligned} \tag{64}$$

5.3 Temporal discretization of weak form

As mentioned previously, for the temporal representation of the displacement and stress impulse field variables, we use standard linear shape functions. Thus, over a time interval from $0 \le t \le \Delta t$, we write shape functions:

$$N_0(t) = 1 - \tfrac{t}{\Delta t}\,; \quad N_1(t) = \tfrac{t}{\Delta t} \tag{65.1,2}$$

such that

$$\mathbf{u}(t) = \mathbf{u}_0 N_0(t) + \mathbf{u}_1 N_1(t) \tag{66.1}$$

$$\mathbf{J}(t) = \mathbf{J}_0 N_0(t) + \mathbf{J}_1 N_1(t) \tag{66.2}$$

and similarly for the variations $\delta\mathbf{u}(t)$ and $\delta\mathbf{J}(t)$. For convenience, we may also assume

$$\overline{\mathbf{j}}(t) = \overline{\mathbf{j}}_0 N_0(t) + \overline{\mathbf{j}}_1 N_1(t) \tag{66.3}$$

$$\overline{\boldsymbol{\tau}}(t) = \overline{\boldsymbol{\tau}}_0 N_0(t) + \overline{\boldsymbol{\tau}}_1 N_1(t) \tag{66.4}$$

although one can instead just work directly with the given functions $\overline{\mathbf{j}}(t)$ and $\overline{\boldsymbol{\tau}}(t)$.

Next, we substitute Eqs. (66) into Eq. (64) and perform the necessary convolution integrals in closed form. Details for all of these temporal integrals are provided in Reference [28]. Then, after collecting terms, we may write the following discretized weak form:

$$\left\{ \delta\mathbf{u}_0^T \quad \delta\mathbf{u}_1^T \quad \delta\mathbf{J}_0^T \quad \delta\mathbf{J}_1^T \right\}$$
$$\left(\frac{1}{2\Delta t} \begin{bmatrix} 2\mathbf{M} - \Delta t\,\mathbf{C} & -2\mathbf{M} + \Delta t\,\mathbf{C} & -\Delta t\,\mathbf{B}^T & \Delta t\,\mathbf{B}^T \\ -2\mathbf{M} + \Delta t\,\mathbf{C} & 2\mathbf{M} + \Delta t\,\mathbf{C} & \Delta t\,\mathbf{B}^T & \Delta t\,\mathbf{B}^T \\ -\Delta t\,\mathbf{B} & \Delta t\,\mathbf{B} & -2\mathbf{A} + \Delta t\,\mathbf{D} & 2\mathbf{A} - \Delta t\,\mathbf{D} \\ \Delta t\,\mathbf{B} & \Delta t\,\mathbf{B} & 2\mathbf{A} - \Delta t\,\mathbf{D} & -2\mathbf{A} - \Delta t\,\mathbf{D} \end{bmatrix} \begin{Bmatrix} \mathbf{u}_0 \\ \mathbf{u}_1 \\ \mathbf{J}_0 \\ \mathbf{J}_1 \end{Bmatrix} - \begin{Bmatrix} \tfrac{1}{2}\left(\mathbf{j}_1 - \mathbf{j}_0\right) \\ \tfrac{1}{2}\left(\mathbf{j}_1 + \mathbf{j}_0\right) \\ \mathbf{0} \\ \mathbf{0} \end{Bmatrix} \right) = 0 \tag{67}$$

where now $\mathbf{j}_0$ and $\mathbf{j}_1$ are force impulses that collect all of the applied body force and surface traction contributions at times $t = 0$ and $t = \Delta t$, respectively.

For this time step from $t = 0$ to $t = \Delta t$, we assume that the field variables $\mathbf{u}_0$ and $\mathbf{J}_0$ are known and we seek to determine $\mathbf{u}_1$ and $\mathbf{J}_1$. Consequently, we must set the variations $\delta\mathbf{u}_0 = \mathbf{0}$ and

$\delta \mathbf{J}_0 = \mathbf{0}$, while allowing $\delta \mathbf{u}_1$ and $\delta \mathbf{J}_1$ to remain arbitrary. Satisfaction of Eq. (67) then requires that the following set of linear algebraic equations hold for each element:

$$\frac{4}{(\Delta t)^2} \begin{bmatrix} \mathbf{M}_1 & \frac{\Delta t}{2} \mathbf{B}^T \\ \frac{\Delta t}{2} \mathbf{B} & -\mathbf{A}_1 \end{bmatrix} \begin{Bmatrix} \mathbf{u}_1 \\ \mathbf{J}_1 \end{Bmatrix} = \frac{2}{\Delta t} \begin{Bmatrix} \mathbf{j}_1 + \mathbf{j}_0 \\ \mathbf{0} \end{Bmatrix} + \frac{4}{(\Delta t)^2} \begin{bmatrix} \mathbf{M}_0 & -\frac{\Delta t}{2} \mathbf{B}^T \\ -\frac{\Delta t}{2} \mathbf{B} & -\mathbf{A}_0 \end{bmatrix} \begin{Bmatrix} \mathbf{u}_0 \\ \mathbf{J}_0 \end{Bmatrix} \tag{68}$$

where

$$\mathbf{M}_0 = \mathbf{M} - \tfrac{\Delta t}{2} \mathbf{C}; \quad \mathbf{M}_1 = \mathbf{M} + \tfrac{\Delta t}{2} \mathbf{C} \tag{69.1,2}$$

$$\mathbf{A}_0 = \mathbf{A} - \tfrac{\Delta t}{2} \mathbf{D}; \quad \mathbf{A}_1 = \mathbf{A} + \tfrac{\Delta t}{2} \mathbf{D} \tag{69.3,4}$$

Recall that the stress impulses are interpolated element-by-element as C^{-1} functions. Consequently, the $\mathbf{J}_1$ variables may be condensed out of the equations at the element level. Solving the second set of Eqs. (68) for $\mathbf{J}_1$ produces the following:

$$\mathbf{J}_1 = \mathbf{A}_1^{-1} \left(\tfrac{\Delta t}{2} \mathbf{B}\, \mathbf{u}_1 + \tfrac{\Delta t}{2} \mathbf{B}\, \mathbf{u}_0 + \mathbf{A}_0 \mathbf{J}_0 \right) \tag{70}$$

After substituting Eq. (70) into the first set of Eqs. (68), we may write the condensed relations for an element as:

$$\mathbf{K}_1^e\, \mathbf{u}_1 = \mathbf{f}_1^e \tag{71}$$

where

$$\mathbf{K}_1^e = \mathbf{B}^T\, \mathbf{A}_1^{-1}\, \mathbf{B} + \tfrac{4}{(\Delta t)^2} \mathbf{M}_1 \tag{72.1}$$

$$\mathbf{K}_0^e = \mathbf{B}^T\, \mathbf{A}_1^{-1}\, \mathbf{B} - \tfrac{4}{(\Delta t)^2} \mathbf{M}_0 \tag{72.2}$$

$$\mathbf{B}_0^T = \tfrac{1}{2} \mathbf{B}^T \left(\mathbf{I}_3 + \mathbf{A}_1^{-1}\, \mathbf{A}_0 \right) \tag{72.3}$$

$$\mathbf{f}_1^e = \tfrac{2}{\Delta t} \left(\mathbf{j}_1 + \mathbf{j}_0 \right) - \mathbf{K}_0^e\, \mathbf{u}_0 - \tfrac{4}{\Delta t} \mathbf{B}_0^T\, \mathbf{J}_0 \tag{72.4}$$

with the superscript ^{e} as a reminder that all of these relations are written at the individual element level.

Generalizing this formulation for any step n in moving from a known solution at time t_{n-1} to the desired solution at time t_n with $\Delta t = t_n - t_{n-1}$, we obtain the following for each element:

$$\mathbf{K}_1^e \, \mathbf{u}_n = \mathbf{f}_n^e \tag{73}$$

where

$$\mathbf{f}_n^e = \tfrac{2}{\Delta t}\left(\mathbf{j}_n + \mathbf{j}_{n-1}\right) - \mathbf{K}_0^e \, \mathbf{u}_{n-1} - \tfrac{4}{\Delta t}\mathbf{B}_0^T \, \mathbf{J}_{n-1} \tag{74}$$

5.4 Global finite element system and comparisons

At this stage, the element relations presented in Eq. (73) are in the form of a standard displacement-based finite element method. Assembly can now proceed in the usual way to produce the overall system of linear algebraic equations, which may be written simply:

$$\mathbf{K}\,\mathbf{U}_n = \mathbf{F}_n \tag{75}$$

Of course, this simplicity tends to obscure the underlying fundamental differences, which in the present development arise from strict adherence to the variational principles presented in Sects. 3 and 4.

Interestingly, the coefficients appearing in the mixed convolved action finite element formulation defined above recall those associated with the well-known Newmark method for dynamic analysis [43, 44], which can be written:

$$\mathbf{K}^{eff} \, \mathbf{u}_n = \mathbf{f}_n^{eff} \tag{76}$$

where

$$\mathbf{K}^{eff} = \mathbf{K} + \tfrac{\gamma}{\beta\,\Delta t}\mathbf{C} + \tfrac{1}{\beta(\Delta t)^2}\mathbf{M} \tag{77.1}$$

$$\begin{aligned}\mathbf{f}_n^{eff} = \mathbf{f}_n^{ext} &+ \mathbf{M}\left(\tfrac{1}{\beta(\Delta t)^2}\mathbf{u}_{n-1} + \tfrac{1}{\beta\,\Delta t}\dot{\mathbf{u}}_{n-1} + \left(\tfrac{1}{2\beta} - 1\right)\ddot{\mathbf{u}}_{n-1}\right) \\ &+ \mathbf{C}\left(\tfrac{\gamma}{\beta\,\Delta t}\mathbf{u}_{n-1} + \left(\tfrac{\gamma}{\beta} - 1\right)\dot{\mathbf{u}}_{n-1} + \Delta t\left(\tfrac{\gamma}{2\beta} - 1\right)\ddot{\mathbf{u}}_{n-1}\right)\end{aligned} \tag{77.2}$$

with γ and β as the parameters. For the case without numerical dissipation, $\gamma = 1/2$ and $\beta = 1/4$. Then, the Newmark algorithm reduces to Eq. (76), along with the following:

$$\mathbf{K}^{eff} = \mathbf{K} + \tfrac{2}{\Delta t}\mathbf{C} + \tfrac{4}{(\Delta t)^2}\mathbf{M} \tag{78.1}$$

$$\mathbf{f}_n^{eff} = \mathbf{f}_n^{ext} + \mathbf{M}\left(\tfrac{4}{(\Delta t)^2}\mathbf{u}_{n-1} + \tfrac{4}{\Delta t}\dot{\mathbf{u}}_{n-1} + \ddot{\mathbf{u}}_{n-1}\right) + \mathbf{C}\left(\tfrac{2}{\Delta t}\mathbf{u}_{n-1} + \dot{\mathbf{u}}_{n-1}\right) \tag{78.2}$$

where $\mathbf{f}_n^{ext}$ represents the externally applied nodal forces at time t_n.

Notice that several familiar coefficients (e.g., $2/\Delta t$, $4/(\Delta t)^2$) now appear in Eqs. (78). Let us consider the case in which Maxwell damping is absent. Then, $\mathbf{A}_1 = \mathbf{A}$, $\mathbf{B}_0 = \mathbf{B}$ and from Eqs. (69.2) and (72.1),

$$\mathbf{K}_1^e = \mathbf{B}^T\ \mathbf{A}^{-1}\ \mathbf{B} + \tfrac{2}{\Delta t}\mathbf{C} + \tfrac{4}{(\Delta t)^2}\mathbf{M} \tag{79}$$

which is identical to the Newmark effective stiffness from Eq. (78.1). To compare $\mathbf{f}_n^e$ from Eq. (74) with $\mathbf{f}_n^{eff}$ from Eq. (78.2), several substitutions are needed. This includes the equation of motion in discretized form at time t_{n-1}, which within the Newmark formalism may be written

$$\mathbf{M}\ \ddot{\mathbf{u}}_{n-1} + \mathbf{C}\ \dot{\mathbf{u}}_{n-1} + \mathbf{K}\ \dot{\mathbf{u}}_{n-1} = \mathbf{f}_{n-1}^{ext} \tag{80}$$

and the corresponding discretized initial condition in terms of stress impulse that becomes

$$\mathbf{M}\ \dot{\mathbf{u}}_{n-1} + \mathbf{C}\ \mathbf{u}_{n-1} + \mathbf{B}^T\mathbf{J}_{n-1} = \mathbf{j}_{n-1} \tag{81}$$

Next, we may rearrange the terms in Eq. (78.2), as follows:

$$\mathbf{f}_n^{eff} = \mathbf{f}_n^{ext} + \left(\mathbf{M}\ddot{\mathbf{u}}_{n-1} + \mathbf{C}\dot{\mathbf{u}}_{n-1}\right) + \tfrac{4}{(\Delta t)^2}\left(\mathbf{M} - \tfrac{\Delta t}{2}\mathbf{C}\right)\mathbf{u}_{n-1} + \tfrac{4}{\Delta t}\left(\mathbf{M}\dot{\mathbf{u}}_{n-1} + \mathbf{C}\mathbf{u}_{n-1}\right) \tag{82}$$

Then, substituting Eqs. (80), (69.1) and (81), respectively, in the bracketed terms in Eq. (82), we obtain

$$\mathbf{f}_n^{eff} = \mathbf{f}_n^{ext} + \left(\mathbf{f}_{n-1}^{ext} - \mathbf{K}\mathbf{u}_{n-1}\right) + \tfrac{4}{(\Delta t)^2}\left(\mathbf{M}_0\right)\mathbf{u}_{n-1} + \tfrac{4}{\Delta t}\left(\mathbf{j}_{n-1} - \mathbf{B}^T\mathbf{J}_{n-1}\right) \tag{83}$$

or

$$\mathbf{f}_n^{eff} = \mathbf{f}_n^{ext} + \mathbf{f}_{n-1}^{ext} + \tfrac{4}{\Delta t}\mathbf{j}_{n-1} - \left(\mathbf{K} - \tfrac{4}{(\Delta t)^2}\mathbf{M}_0\right)\mathbf{u}_{n-1} - \tfrac{4}{\Delta t}\mathbf{B}^T\mathbf{J}_{n-1} \tag{84}$$

However, for linear variation of applied external force impulses over the time step,

$$\mathbf{j}_n = \mathbf{j}_{n-1} + \tfrac{\Delta t}{2}\left(\mathbf{f}_n^{ext} + \mathbf{f}_{n-1}^{ext}\right) \tag{85}$$

and Eq. (84) becomes finally

$$\mathbf{f}_n^{eff} = \tfrac{2}{\Delta t}\left(\mathbf{j}_n + \mathbf{j}_{n-1}\right) - \left(\mathbf{K} - \tfrac{4}{(\Delta t)^2}\mathbf{M}_0\right)\mathbf{u}_{n-1} - \tfrac{4}{\Delta t}\mathbf{B}^T\mathbf{J}_{n-1} \tag{86}$$

Notice that the right hand side of Eq. (86) is now identical to that of Eq. (74) and, therefore, we have

$$\mathbf{f}_n^{eff} = \mathbf{f}_n^{e} \tag{87}$$

Thus, the Newmark algorithm for a linear dynamic system with viscous damping is equivalent to the simplest mixed convolved action variational formulation using linear temporal shape functions. Unlike the Newmark algorithm, the mixed convolved action finite element method has been developed here to provide a variational framework for problems involving Maxwell damping processes. We also should mention that Marsden and colleagues [38-40] previously identified the Newmark algorithm as a variational statement. However, in their analysis, only purely reversible processes could be considered, while remaining strictly consistent with variational calculus principles. Dissipation was considered only within the framework of the Rayleigh formalism. Furthermore, their approach represented a hybrid between finite element methods in space and discrete variational calculus in time, which sidesteps the endpoint constraint problem. On the other hand, here we have a pure finite element approach in both space and time, which extends readily to non-conservative problems and to higher order representations.

6 Computational examples

In this section, four computational examples are considered to verify the mixed convolved action (MCA) numerical implementation for two-dimensional continuum problems. The first example involves one-dimensional elastodynamic wave propagation in a square domain under plane strain conditions, where the mixed convolved action results are compared to the analytical solution and to the corresponding solutions from the ABAQUS commercial finite element package without artificial numerical dissipation. As expected from the analysis of Sect. 5, the MCA and ABAQUS solutions are essentially identical. For the second example, we study the dynamic response of a square plate due to Heaviside loading under plane stress conditions with the medium modeled as a Maxwell viscoelastic material. Interestingly, by using the most simplistic linear temporal shape functions within the mixed convolved action formalism, we are able to recover solutions consistent with the ABAQUS commercial software. Then, we shift to consider two illustrative examples of stress wave attenuation. The first of these involves shielding a portion of the boundary by using lower impedance inserts within a square plate subjected to a sudden sinusoidal pulse load on the opposite edge. Lastly, we investigate elastodynamic stress wave attenuation in a layered medium with uniform wave speed, but graded impedances. By proper design of the layer properties, significant attenuation can be achieved, as is shown here through MCA analyses of a pulse loaded square plate. All four problems use the same basic geometric setting of a square planar body with rollers on three sides, but are designed to explore different aspects of dynamic response.

6.1 Elastodynamic wave propagation along uniform bar

First we consider a uniform square elastic body under plane strain conditions subjected to a suddenly applied Heaviside step loading at one end, as shown in Fig. 1. The problem reduces to one-dimensional wave propagation with speed c_L, having a known analytical solution available for comparison. We also compare results with the commercial finite element software ABAQUS 6.9-1 [45], with the Hilber-Hughes-Taylor parameter $\alpha = 0$ to avoid numerical dissipation and to align with the present MCA formulation.

For the numerical analysis, we employ 512 uniformly sized three node triangle elements, as illustrated in Fig. 1, and take the following non-dimensional parameters:

$$E = 5/2, \quad \nu = 1/4, \quad \rho = 1, \quad L = 1, \quad p_0 = 1 \tag{88}$$

with a constant time step $\Delta t = 0.025$. Vertical displacements versus time at the upper edge and at mid-height are compared in Fig. 2. As expected from the analysis at the end of Section 4, the MCA and ABAQUS solutions are essentially identical. The deviation from the exact solution is due to the low order approximations used over space and time within this initial MCA numerical formulation. However, despite the errors, the formulations are energy preserving and unconditionally stable. Furthermore, the MCA formulation derives from a pure variational statement of elastodynamics that has as its Euler-Lagrange equations the partial differential equations, boundary conditions and initial conditions of the problem.

6.2 Dynamic response of Maxwell viscoelastic plate

Next, we examine the response of a thin viscoelastic plate under dynamic excitation. The geometry, constraints and loading conditions are identical to those from the previous problem displayed in Fig. 1. The non-dimensional problem parameters defined in Eq. (88) are again used, along with a Maxwell relaxation time $t_M = 4$. However, here we assume plane stress conditions. Fig. 3 provides the numerical solutions for the vertical displacement versus time at the loaded edge using ABAQUS and MCA, both with time steps of $\Delta t = 0.0125$. The parameters have been set to illustrate inertial and creep effects, which are clearly visible in Fig. 3. Also, as is apparent in the figure, the MCA and ABAQUS solutions are nearly identical. This is further established by examining solutions using the two approaches with a much larger time step of $\Delta t = 0.40$. Interestingly, while the displacements at this longer time step are clearly in error, the two approaches again provide essentially the same results.

At this point, we should emphasize that the MCA formalism provides a true variational statement for this dissipative viscoelastic system. The action functional encapsulates the governing partial differential equations, boundary conditions and initial conditions of the problem. The MCA numerical implementation employs finite element representations in both space and time. Remarkably, the simplest MCA formulation in time reproduces the results from a highly respected commercial finite element code. On the one hand, this means that the dynamic viscoelastic formulation in ABAQUS is consistent with a pure variational statement of the problem. On the other hand, this suggests that further improvements should be possible by exploiting extensions of the present MCA variational implementation.

6.3 Stress wave shielding with reduced impedance inserts

Geometric tailoring of material properties can be designed in many patterns to reduce stress wave amplitudes. With the maturation of additive manufacturing processes, such designs can be readily fabricated. For the next illustrative example, we study the effects of reduced impedance inserts to shield portions of the lower edge of an elastic square block, having the upper edge subjected to an applied half-sine pulse load, such that

$$S(t) = \sin(\pi t)\left[H(t) - H(t-1)\right] \tag{89}$$

with $H(t)$ representing the Heaviside step function.

We again use non-dimensional parameters with $L = 1$. Both the matrix and inserts have the same Poisson ratio $\nu = 1/4$ and wave speed c_L, but for the matrix the elastic modulus $E_M = 40$ and mass density $\rho_M = 16$, while for the inserts $E_I = 5/2$ and $\rho_I = 1$. Thus, the impedance ratio is $Z_I = Z_M / 16$. The composite block is examined under plane strain conditions for the sinusoidal pulse load, defined in Eq. (89), applied to the top edge. The problem definition and finite element mesh for two different insert patterns, one involving four uniform inserts and the other employing the same number and total volume of graded inserts, are shown in Fig. 4. Each mesh consists of 2048 equally sized triangular elements and is analyzed using MCA with a fixed non-dimensional time step $\Delta t = 0.0125$.

The maximum vertical normal stress wave amplitudes for the two patterns are plotted in Fig. 5 versus horizontal location along the bottom edge. Also shown is the MCA result for the monolithic matrix case, which as expected is very nearly two all along that edge due to the wave reflection when reaching the hard boundary. Notice both lower impedance insert patterns can provide significant shielding along the bottom edge, at the expense of increased stress wave amplitudes outside of this attenuation zone.

6.4 Stress wave attenuation in layered media

As a final example, we look at the problem of stress wave attenuation in layered media under impulse loading using the MCA formulation. The problem setup is similar to that of the previous examples, except that now waves propagate through a heterogeneous domain composed of eight equal depth horizontal layers. All layers have the same wave speed c_L, but with impedances graded from Z_1 at the bottom to Z_8 at the top, where $Z_i = \rho_i\, c_L$ for layer i. We assume non-dimensional parameters with $L = 1$, and base elastic properties $E_1 = 5/2$, $\nu = 1/4$ and $\rho_1 = 1$, under plane strain conditions for the sinusoidal pulse load defined in Eq. (89), using a fixed non-dimensional time step $\Delta t = 0.0125$. Fig. 6 provides more problem definition detail, including the baseline triangular finite element mesh.

For the first case, the impedance of all layers is set equal to that of the base, that is $Z_i = Z_1$ for $i = 2,\ldots,8$. The MCA longitudinal stress solutions are displayed in Fig. 7, which clearly illustrates the expected stress doubling at the rigid bottom edge. Next, graded impedances are defined, such that $Z_i = 2Z_{i-1}$ for $i = 2,\ldots,8$ and the corresponding stress results are presented in Fig. 8. Notice that the peak stresses are greatly diminished and also significantly delayed. Several related stress wave attenuator concepts can be found, for example, in Reference [46].

7 Conclusions

In this work, we applied the idea of mixed convolved action to linear problems of continuum solid mechanics, involving both conservative and dissipative phenomena, namely, elastodynamics and viscoelasticity, respectively. In each case, this led to the formulation in the time-domain of a new *Principle of Stationary Mixed Convolved Action*, which produced the governing partial differential equations, boundary conditions and initial conditions as its Euler-Lagrange equations. In addition, the variations are taken in a manner that is completely consistent with the specified boundary and initial conditions.

Remarkably, each of the weak forms defined in Eqs. (23), (36) and (46) has an elegant structure, featuring a balanced appearance of the primary variables and variations. Thus, all of these formulations can lead to interesting and novel algorithms for computational dynamics. Here we limited the numerical implementation to two-dimensions with three-node triangles and linear temporal shape functions employed within the time-space finite element formulation. Several elementary examples are studied to verify the MCA formulation and numerical implementation versus analytical solutions and commercial finite element software. Two illustrative applications of the concept of impedance tailoring for stress wave attenuation were also considered.

Future work will focus on higher order shape functions in space and time, three-dimensional problems, multiphysics formulations and extensions to address geometric and material non-linearity for both conservative and dissipative phenomena. MCA formulations for consistent size-dependent couple stress mechanics also are underway, as are extensions of the principle of virtual convolved action that was introduced here as a new and potentially powerful concept.

Appendix A: Fractional calculus overview

Fractional calculus has a long history, dating back nearly to the beginning of calculus itself. For example, the idea of generalizing derivatives to fractional order can be traced to l'Hôpital, Leibniz and their contemporaries. A thorough historical review, including this earliest work, is presented in Ross [47], while the monographs by Oldham and Spanier [48] and Samko et al. [49] provide comprehensive collections of known results. In this appendix, we will present only a few basic definitions and formulas that are needed for understanding and developing mixed convolved action principles.

There are a number of different definitions of fractional integrals and derivatives in the literature. However, here we will focus on the versions attributed to Riemann and Liouville. Following Samko et al. [49], let $u(t)$ represent an L_1 Lebesgue integrable function over an interval $\left(0,t_f\right)$. Then, for the fractional integral of order α, we may write

$$\left(\mathcal{J}_{0^+}^{\alpha}u\right)(t) \equiv \frac{1}{\Gamma(\alpha)}\int_0^t \frac{u(\tau)}{\left(t-\tau\right)^{1-\alpha}}d\tau \quad \text{for } t>0,\ \alpha>0 \qquad \text{(A1.1)}$$

$$\left(\mathcal{J}_{t_f^-}^{\alpha}u\right)(t) \equiv \frac{1}{\Gamma(\alpha)}\int_t^{t_f} \frac{u(\tau)}{\left(\tau-t\right)^{1-\alpha}}d\tau \quad \text{for } t<t_f,\ \alpha>0 \qquad \text{(A1.2)}$$

where $\Gamma(\cdot)$ denotes the Gamma function. Eqs. (A1.1) and (A1.2) represent the left and right Riemann-Liouville fractional integral of order α, respectively, where we have chosen the left fractional integral to operate over the interval from 0 to t, while the right fractional integral works over the range from t to t_f. Notice from these definitions with $0<\alpha<1$ that we may view fractional integration as a convolution of a function, say $u(\tau)$, with a weakly singular kernel $\tau^{\alpha-1}/\Gamma(\alpha)$.

Next, we consider the corresponding left and right Riemann-Liouville fractional derivatives of order α, which can be written, respectively, in the following form

$$\left(\mathscr{D}_{0^+}^{\alpha}u\right)(t) \equiv +\frac{1}{\Gamma(1-\alpha)}\frac{d}{dt}\int_0^t \frac{u(\tau)}{\left(t-\tau\right)^{\alpha}}d\tau \quad \text{for } 0 \le t \le t_f,\ 0<\alpha<1 \tag{A2.1}$$

$$\left(\mathscr{D}_{t_f^-}^{\alpha}u\right)(t) \equiv -\frac{1}{\Gamma(1-\alpha)}\frac{d}{dt}\int_t^{t_f} \frac{u(\tau)}{\left(\tau-t\right)^{\alpha}}d\tau \quad \text{for } 0 \le t \le t_f,\ 0<\alpha<1 \tag{A2.2}$$

These exist almost everywhere in the interval for absolutely continuous $u(\tau)$. Notice that with the Riemann-Liouville definition, a fractional derivative of order α is equivalent to the first derivative of a fractional integral of order $1-\alpha$.

While a broad range of fractional derivative orders could be used to establish convolution-based variational principles, we will focus initially on the Riemann-Liouville temporal semi-derivatives, which from Eq. (A2.1) and (A2.2) are defined as [48, 49]

$$\left(\mathscr{D}_{0^+}^{1/2}u\right)(t) \equiv \frac{1}{\Gamma(1/2)}\frac{d}{dt}\int_0^t \frac{u(\tau)}{\left(t-\tau\right)^{1/2}}d\tau \tag{A3.1}$$

$$\left(\mathscr{D}_{t_f^-}^{1/2}u\right)(t) \equiv -\frac{1}{\Gamma(1/2)}\frac{d}{dt}\int_t^{t_f} \frac{u(\tau)}{\left(\tau-t\right)^{1/2}}d\tau \tag{A3.2}$$

For convenience, we will use a superposed breve to denote the left Riemann-Liouville semi-derivative with respect to time. Thus, we let

$$\breve{u}(t) = \left(\mathscr{D}_{0^+}^{1/2}u\right)(t) \tag{A4.1}$$

While not of the same importance in the present work, the right Riemann-Liouville semi-derivative will be written in shorthand notation using a superposed inverted breve as follows:

$$\hat{u}(t) = \left(\mathscr{D}_{t_f^-}^{1/2}u\right)(t) \tag{A4.2}$$

We also will employ the usual notation with a superposed dot to represent a first order time derivative and let the symbol $*$ indicate the Riemann convolution of two functions over time, such that

$$\left(u * v\right)(t) = \int_0^t u(\tau)v(t-\tau)d\tau \tag{A5}$$

Notice the finite lower and upper limits of 0 and t, respectively, in the convolution, which may differ from other definitions, but are chosen here to coincide with the typical temporal support of the functions at time t in initial values problems.

Next, we consider several different integration-by-parts formulas that are central to the development of variational principles in mechanics. First, let us recall the classical integration-by-parts result for the inner product of two functions, where we attempt to shift a first derivative from one function to the other. Using our shorthand notation, we may write

$$\int_0^t \dot{u}(\tau)v(\tau)d\tau = -\int_0^t u(\tau)\dot{v}(\tau)d\tau + u(t)v(t) - u(0)v(0) \tag{A6}$$

Of course, the sign of the integral on the right hand side flips with the shifting of the derivative between the two functions and there are product terms evaluated at the end and beginning of the time interval. The sign change here is problematic for temporal response and precludes the possibility of using such terms to represent viscous dissipation. Furthermore, the released terms are not consistent with the needs of an initial value problem [27].

Now we can attempt the same type of operation, but for the convolution of two functions. The result becomes

$$\left(\dot{u} * v\right)(t) = \left(u * \dot{v}\right)(t) + u(t)v(0) - u(0)v(t) \tag{A7}$$

Notice that the sign of the convolution integral does not flip and each of the released terms involve products of the functions at the beginning and end of the time interval.

These characteristic differences between inner product and convolution prove to be fundamental in the development of variational methods for dynamical problems of mechanics, as Tonti [24, 25] emphasized in much of his work on the subject. However, in order to resolve all of the previous inconsistencies in the variational representations for such problems, we also must utilize the fractional calculus counterparts, which are developed in Reference [27], based on the work of Hardy and Littlewood [50] and Love and Young [51].

Consider first the general integration-by-parts relation for fractional derivatives [27]:

$$\int_0^t \left(\mathscr{D}_{0^+}^{\alpha} u\right)(\tau) w(\tau) d\tau = \int_0^t u(\tau) \left(\mathscr{D}_{t^-}^{\alpha} w\right)(\tau) d\tau + u(t)\left(\mathscr{I}_{t^-}^{1-\alpha} w\right)(t) - \left(\mathscr{I}_{0^+}^{1-\alpha} u\right)(0) w(0) \tag{A8}$$

where we begin with the inner product of the left fractional derivative of one function $u(\tau)$ with another function $w(\tau)$ and then end with the inner product of the former function with a right fractional derivative of the latter function, plus some boundary terms involving both right and left fractional integrals. Now, if we take $w(\tau) = \left(\mathscr{D}_{t^-}^{1/2} v\right)(\tau)$ and let $\alpha = 1/2$ to specialize for semi-derivatives, then this becomes in our shorthand notation

$$\int_0^t \breve{u}(\tau) \widehat{v}(\tau) d\tau = -\int_0^t u(\tau) \dot{v}(\tau) d\tau + u(t) v(t) \tag{A9.1}$$

Alternatively, if we reverse the roles of the two functions, we may write

$$\int_0^t \widehat{u}(\tau) \breve{v}(\tau) d\tau = \int_0^t u(\tau) \dot{v}(\tau) d\tau + u(0) v(0) \tag{A9.2}$$

Notice that in order to end with a first derivative inside the integral on the right hand side, these relationships must start with the product of left and right Riemann-Liouville semi-derivatives on the left hand side. Unfortunately, the sign of the left hand integral changes depending upon which semi-derivative is shifted and again the released endpoint terms are of an inconvenient form.

On the other hand, for the convolutional integration by parts involving a left Riemann-Liouville fractional derivative, we have the following general result [27]:

$$\int_0^t \left(\mathscr{D}_{0^+}^{\alpha} u\right)(t-\tau) w(\tau) d\tau = \int_0^t u(t-\tau) \left(\mathscr{D}_{0^+}^{\alpha} w\right)(\tau) d\tau + u(t)\left(\mathscr{I}_{0^+}^{1-\alpha} w\right)(0) - \left(\mathscr{I}_{0^+}^{1-\alpha} u\right)(0) w(t) \tag{A10}$$

Again, letting $w(\tau) = \left(\mathscr{D}_{t^-}^{1/2} v\right)(\tau)$ and $\alpha = 1/2$, we have the following interesting and useful result:

$$\left(\breve{u} * \breve{v}\right)(t) = \left(u * \dot{v}\right)(t) + u(t) v(0) \tag{A11}$$

In Eq. (A11), we work with only left semi-derivatives on the left hand side and shift the half derivative to the second function without a sign change. Furthermore, there is just a single released term, which involves the product of function evaluations at the beginning and end of the time interval. These are exactly the characteristics needed for the development of new variational principles for both conservative and dissipative dynamical systems. Thus, the key integration-by-parts relations will be Eq. (A7) for integer derivatives and Eq. (A11) for semi-derivatives.

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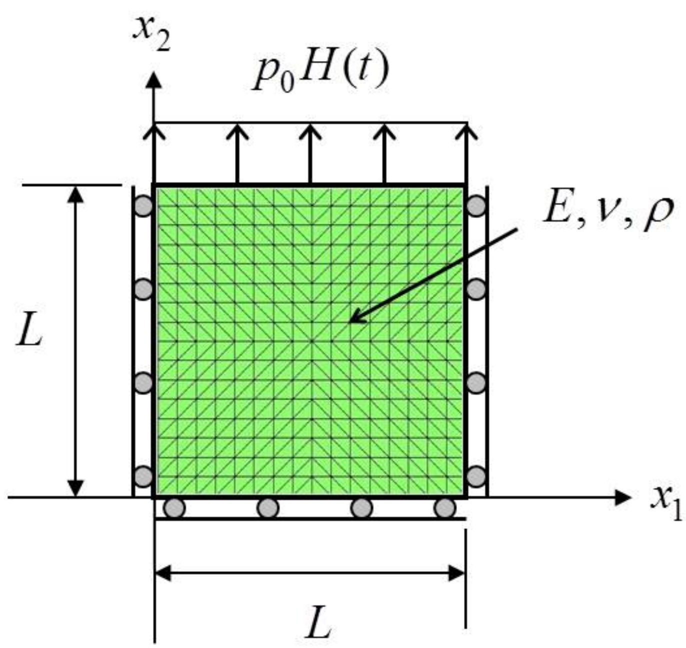

Fig. 1 Elastodynamic Wave Propagation along Uniform Bar – Problem Definition

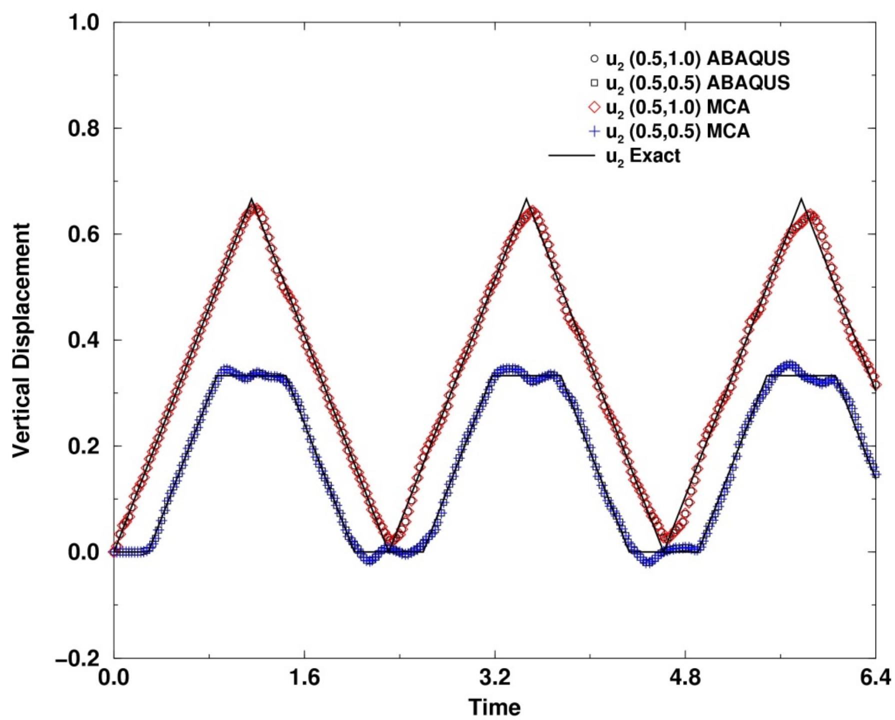

Fig. 2 Elastodynamic Wave Propagation along Uniform Bar – Displacement History

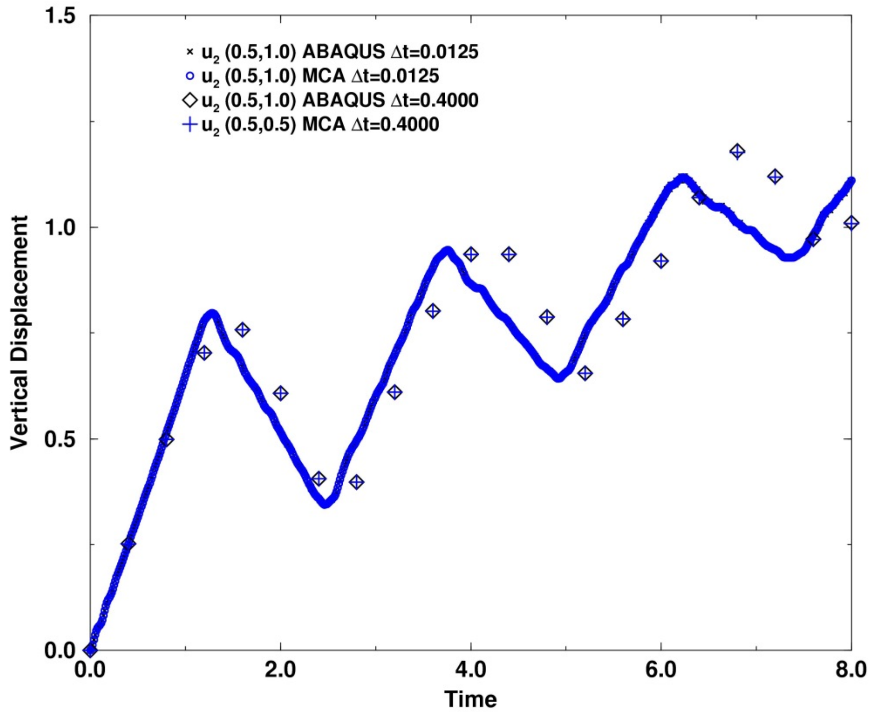

Fig. 3 Dynamic Response of Maxwell Viscoelastic Plate – Displacement History

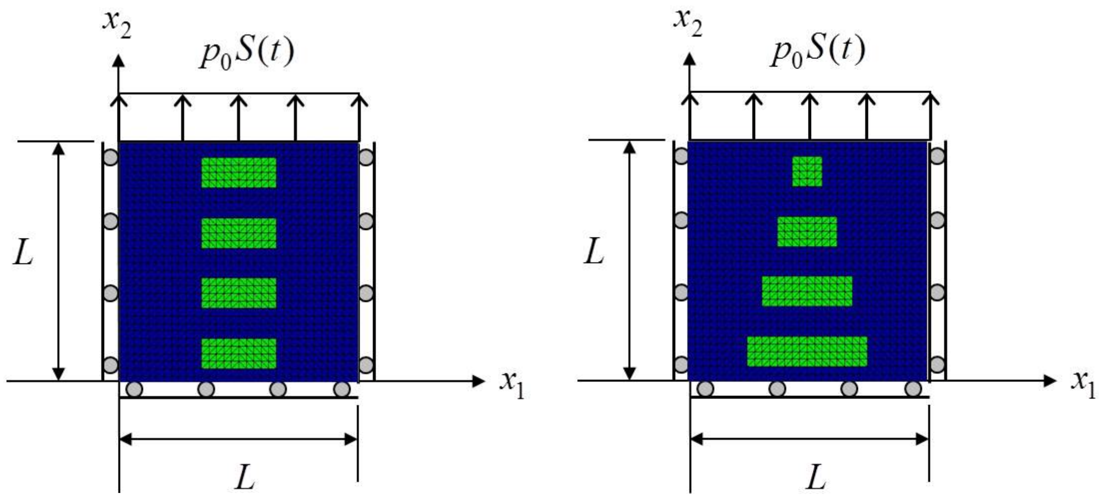

Fig. 4 Stress Wave Shielding with Reduced Impedance Inserts – Problem Definition (Left: Uniform insert pattern; Right: Graded insert pattern)

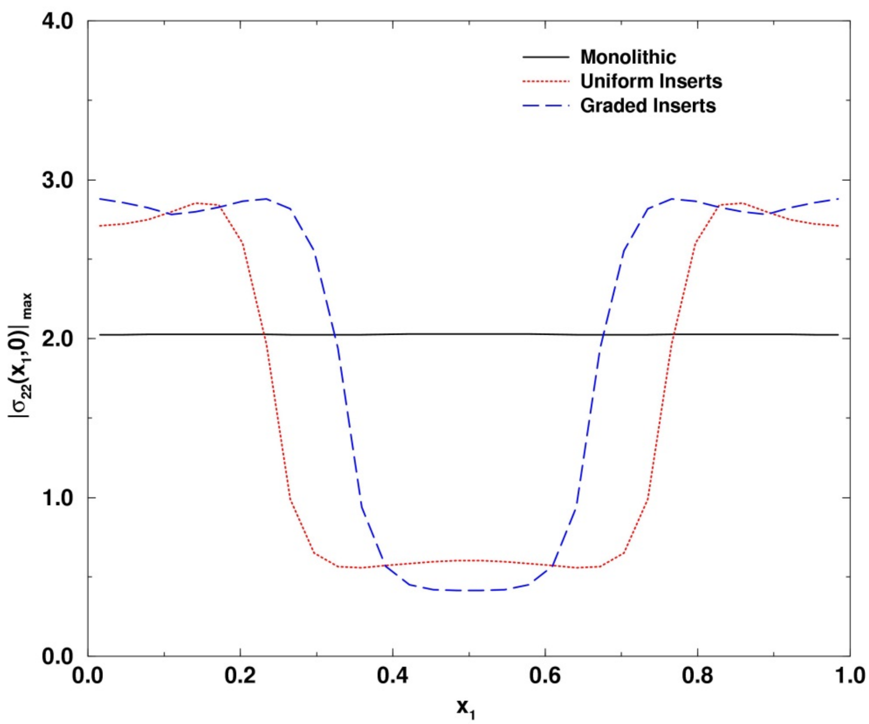

Fig. 5 Stress Wave Shielding with Reduced Impedance Inserts – Stress Amplitudes

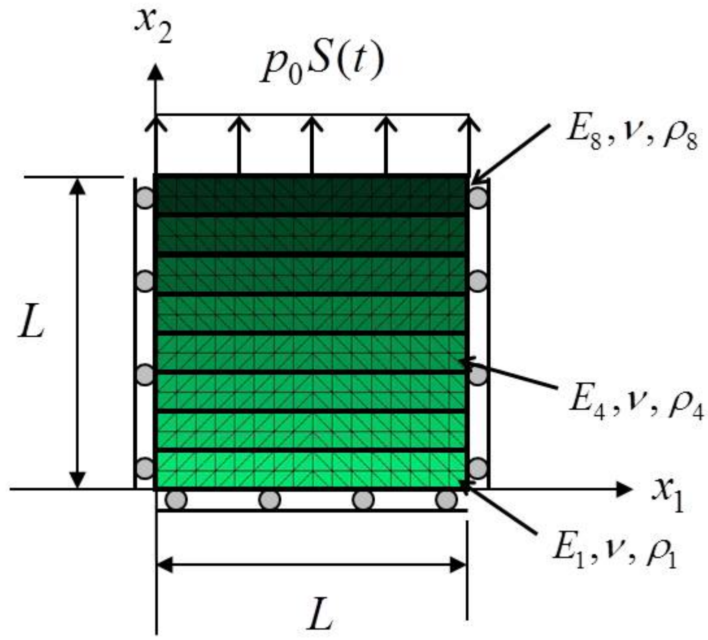

Fig. 6 Stress Wave Attenuation in Layered Medium – Problem Definition

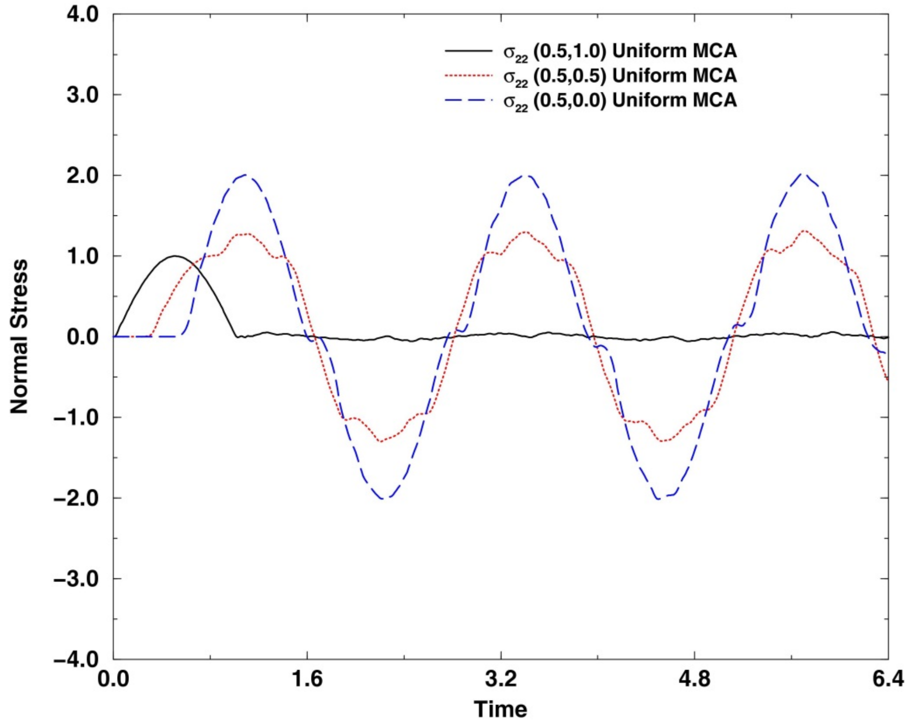

Fig. 7 Stress Impulse Response in Uniform Medium – Stress History

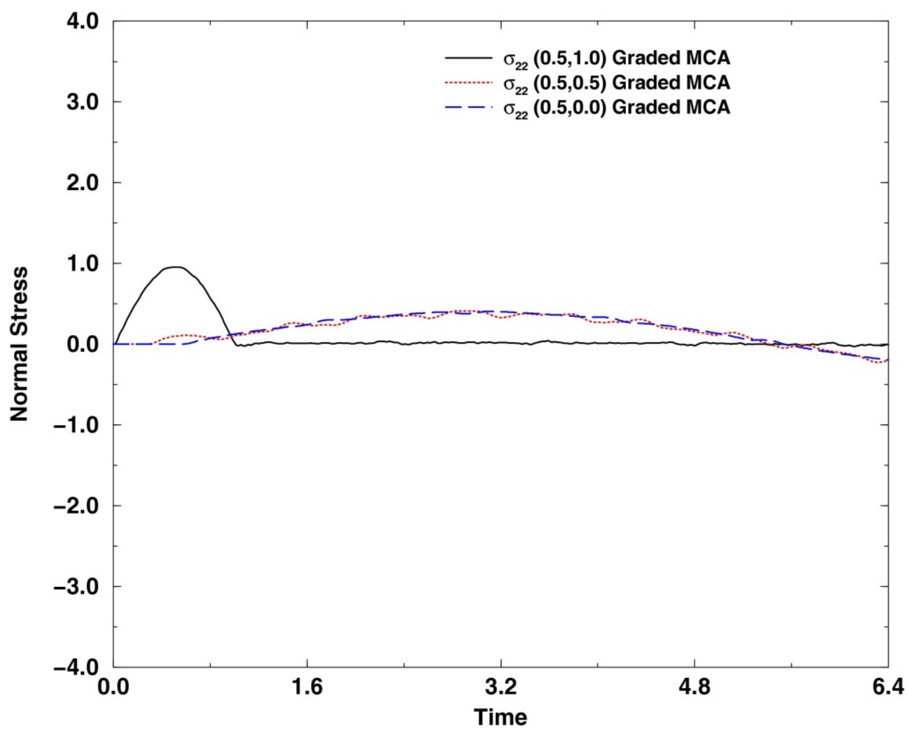

Fig. 8 Stress Impulse Response in Graded Medium – Stress History