Using Geo-statistical Methods to Decide an Additional Facility Location

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Abstract

In this paper, we propose an approach for locating an additional facility—to add to a chain of existing facilities—based on geo-statistical methods. This approach does not make assumptions on customer demand behavior with respect to distance from the nearest facility (as is commonly made in location models) but rather implicitly takes into account intrinsic factors. The goal is to maximize the probability of a randomly chosen customer visiting the chain. Our model is derived from the adaptive spatial sampling problem in geo-statistics. We present a case study for a cellular application to illustrate the behavior of the non-linear objective function. Two versions, based respectively on discrete and continuous optimization techniques, are presented. A Simulated Annealing heuristic is developed to solve the discrete version whereas a special heuristic called Geometric Search is proposed to solve the continuous version. Results show that both of these heuristics perform remarkably well considering the fact that the objective function is highly non-linear and complex (non-differentiable; discontinuous). Geometric Search reports optimal solutions in all the instances tested. Our empirical investigation reveals that the continuous version cannot be solved efficiently by using a discrete modeling approximation.
Subject Classifications: Location Models, Correlation, Estimation, Sampling.

1. Introduction

Facility location is a critical aspect of strategic planning for a broad spectrum of public and private firms. Geographic location choice can determine the success or failure of firms, communities, and even nations. With the rapid increase in demand for products and services and with increasing competition, firms are faced with the task of expanding their existing facilities or networks. Expanding a network might imply adding one or more facilities or stores to its existing set. The problem of locating a single additional facility becomes equally important when the cost of such an installation runs into millions of dollars. Retail stores like Wal-Mart and Target are typical examples. Hence a substantial amount of planning, research and computations go into facility location problems.

Whilst researchers to date have considered modeling facility location problems with the primary variable as distance or cost (which is again a direct function of distance) to a facility, very few papers consider other aspects that influence locations. Though we are aware of several factors, it is highly difficult if not impossible to model these dependencies and develop a closed form expression for the objective function. Certain stochastic facility location problems do

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incorporate demand and distance uncertainty in them in order to take these parameters into account. But these are entirely dependent on the distribution functions chosen to represent uncertainty.

Consider the case of a restaurant franchise such as Burger King (BK). Suppose that they have seven restaurants in a region and wish to open another one. They want to do this in a way that maximizes the total demand generated from the region. What determines whether a consumer chooses to visit a BK is not just distance, but several other factors, which include, but are not limited to, waiting time at the restaurant, competitor locations, and traffic conditions en route. Building a model that takes all of these factors into account is difficult to calibrate. The approach used commonly in location theory is to isolate one factor and to study its effect in depth. By far the most studied of these factors is distance, where the objective is to minimize the demand weighted travel distance (*p*-median), or minimize the maximum distance of a customer to a BK center (*p*-center). Another factor that has been modeled is the influence of competitors, often using a game theoretic framework (see, e.g. Ghosh and Craig 1983, DePalma *et al.* 1989). Yet another factor that has been studied is the effect of waiting time, which usually uses basic formulas for waiting time from queueing theory and focuses not just on travel time but on a combination of travel time and queueing delay, e.g. see Brandeau and Chiu 1994, Brandeau *et al.* (1995), and Huff (1963). Our method directly seeks to use data on the probability of a customer visiting the existing chain at certain discrete locations. Inherent in this data is the effect of non-quantifiable parameters such as competitor locations, waiting times, and traffic conditions en route. Spatial interpolation using kriging (see Section 3) does exactly this. One use of spatial interpolation is in the field of adaptive sampling, where one is interested in determining where additional readings should be taken so as to reduce the uncertainty of the quantity being estimated (see Thompson and Seber 1996 for a basic reference on adaptive sampling). What we are doing is to use the same technique for locating additional facilities. The technique we develop can locate additional facilities sequentially.

Another area of application of our model is that of a cellular provider who is faced with the problem of locating an additional cell tower so as to improve, as best as possible, cellular coverage in a region. A commonly accepted measure of the strength of the bond between a cell phone and a cell tower is the Relative Signal Strength Indicator (RSSI)—see Akella *et al.* (2003) for a discussion. Given the location of existing cell towers, one can use RSSI propagation models to predict the values at specific points in the region. These estimates ignore signal attenuation due to the presence of buildings, foliage, season, etc. To use our approach we would need only to collect RSSI data at a suitable set of discrete locations. Then our geo-statistical model can be used to determine the location of an additional cell tower, with the goal of maximizing the completion probability of a randomly placed cell call in the region.

The paper is organized as follows: Section 1 presents a brief introduction of facility location and the need for efficient planning methods. The justification for using a geo-statistical technique on existing data to locate an additional facility is also provided. In section 2, we present some background work on facility location and adaptive spatial sampling. We define the adaptive spatial sampling in section 3 and relate it to additional facility location problems in section 4. Section 5 introduces the discrete optimization version followed by computational results of proposed heuristics. The continuous optimization version is presented in section 6 along with a Geometric Search heuristic to solve this problem. Finally we perform a comparison of the discrete and continuous optimization versions in section 7. We assume the reader has prior understanding of some of the basic geo-statistical interpolation techniques.

2. Literature Review

Based on our survey, we did not find any previous paper that studied the additional facility location problem from the perspective of adaptive spatial sampling. In this section, we review some fundamental problems in facility location and then proceed to present some closely related problems in the literature. We also review the relevant adaptive sampling literature.

Hakimi (1964) introduced the p-median problem. The problem is stated as follows: Find the location of p facilities so as to minimize the total demand-weighted travel distance between demands and facilities. ReVelle (1986) present a modified version of the p-median problem for locating retail facilities in the presence of competing firms. The objective in this retail environment is to locate facilities to maximize the number of new customers captured or to maximize the retailer's added market share. This modification illustrates how the p-median problem can be applied in a strategic decision making context.

The p-center problem is also known as the *minimax* problem, since we seek to minimize the maximum distance between any demand and its nearest facility. If facility locations are restricted to the nodes of the network, the problem is a *vertex center problem*. Center problems that allow facilities to be located anywhere on the network are *absolute center problems*. Texts by Daskin (1995), Drezner (1995), and Drezner and Hamacher (2002) provide surveys of location problems and solution algorithms—we refer the reader to these for recent advances in the field.

The p-median and p-center problems are design problems, in that no facilities are assumed to presently exist. The problem of augmenting an existing set of facilities has been more recently studied. Berman and Simchi-Levi (1990) and Drezner (1995) considered the problem of adding some new facilities. Chhajed and Lowe (1992) studied the problem of adding *m* new facilities on a tree, given that there are *n* pre-existing facilities. They found an $O(mn)$ algorithm for this problem. Savage *et al.* (Associated Press 1998) studied the relocation of ATMs for Wells Fargo Bank. Complementary to the problem of augmenting facilities is the problem of deleting facilities in the least harmful way, which was studied by Dell (1998) in the context of US Army installations. More recently, Wang *et al.* (2003) consider a more general model that allows for both addition and deletion of facilities.

Adaptive sampling was first introduced under the concept of progressive sampling (Makarovic 1973). It provides an objective and automatic method for sampling terrain of varying complexity, characterized by great altitude variation. The size of the matrix with altitude information increases progressively as the complexity increases.

Ayeni (1982) conducted a similar study to determine the optimum number and spacing of terrain elevation data points to produce a Digital Elevation Model (DEM). Ayeni stressed the importance of evaluating an adequate number of data points, as well as the appropriate sampling distribution of such points, which in turn constitutes a good match to characterize a given terrain.

The ideas suggested in progressive sampling by Makarovic and Ayeni were later carried over to the field of adaptive sampling. Adaptive sampling uses Bayesian theory, based on conditional probability to guide subsequent sampling search. A major difference with conventional designs lies in the selection of additional samples in adaptive designs—here iterations depends upon the new sample value observed in the field. In other words, the procedure for selecting additional samples is a function of the outcome of the variable of interest, as observed during the survey of an initial sampling phase of the survey.

3. Adaptive Spatial Sampling Problem

There are different techniques to spatially interpolate a variable and predict its value at unknown locations. The common nearest neighbors and inverse-distance-weighted methods are distance-based approaches. They account for the fact that nearby sample observations should be given more weight in determining the value at an unknown location. However, these techniques are of a purely deterministic nature and do not account for the presence of spatial autocorrelation among sample observations (Burrough 1986). The French mathematician Matheron (Cressie 1990) developed a stochastic interpolation method, naming it kriging after a South-African miner (D.G. Krige). Central to his theory is the covariogram $\hat{C}(h)$, an empirical statistical function modeling the spatial covariance among pairs of data points given their separating distance h. A covariogram model $C(h)$ is then fitted to the empirical covariogram $\hat{C}(h)$. Kriging is an optimal technique in the sense that its estimates are not only unbiased but the estimation variance is also kept to a minimum. We introduce the following notation:

- h: distance of separation
- $n(h)$: number of data points separated by a distance of h
- z: variable under study—customer probability
- $\overline{z}(h)$: average value of z over all data points that are separated by a distance h
- s_i: i^{th} sample site
- $z(s_i)$: observed value of the variable at sample site s_i

The covariogram function is then given by:

$$\hat{C}(h) = \frac{1}{n(h)} \sum_{s_i - s_j = h} (z(s_i) - \overline{z}(h))(z(s_j) - \overline{z}(h)) \qquad (1)$$

This defines the spatial correlation between any two points. The covariogram is a maximum for values of separation close to 0 and then gradually drops as the distance increases. After a certain distance called the range r, it almost falls to 0. This is the distance beyond which a sample point does not have any effect on the predicted values. There are several mathematical models that approximate the covariogram, e.g. exponential, spherical, Gaussian etc.

Depending upon the assumptions of the phenomenon under study, different types of kriging can be used. In this paper, we focus on a method derived from simple kriging, which is a weighted linear interpolation technique that estimates the value of a variable at unknown locations based on observed values at pre-specified locations. The first-order component of the data is constant and known, and can be subtracted from the original sample to provide a set of residuals (Bailey and Gatrell 1995). The main reason behind using simple kriging in this paper is the ease of computations. We introduce the following additional notation:

- $\boldsymbol{M}$: set of initial sample observations, with cardinality m
- s: location at which to predict
- $z(s)$: predicted value at location s
- $\lambda(s_i)$: weight at location s_i

Then the predicted value at a location s is given by:

$$z(s) = \sum_{i \in M} \lambda(s_i) z(s_i) \qquad (2)$$

This is a linear estimator that predicts z at a location s as a linear combination of the values at observed points. The weights are chosen such that the *expected mean square error across all points* (also called the *kriging variance*) is minimized (Cressie 1993). The optimal weights for the minimum kriging variance are:

$$\lambda(s) = \boldsymbol{C}^{-1}\boldsymbol{c}(s) \tag{3}$$

where $\boldsymbol{C}$ is the matrix of covariances among the original sample points, and $\boldsymbol{c}$ is the column vector of covariances of location s with the original sample points. From equations (2) and (3) it follows that:

$$z(s) = \boldsymbol{\lambda}^T(s)\boldsymbol{z} = \boldsymbol{c}^T(s)\boldsymbol{C}^{-1}\boldsymbol{z} \tag{4}$$

where $\boldsymbol{z}$ represents the vector of $z(s_i)$ values. The goal of adaptive sampling is to optimally locate additional samples that minimize the uncertainty in the estimates. In simple kriging, the variance at a point g is defined as:

$$\sigma^2(g) = \sigma^2 - \boldsymbol{c}^T(g)\boldsymbol{C}^{-1}\boldsymbol{c}(g) \tag{5}$$

where $\boldsymbol{c}^T(g)$ is a (1 x m) vector and $\boldsymbol{C}$ is of dimension (m x m). Here σ^2 denotes the *sill*, which is the semivariance value corresponding to the *range r*.

4. Implication for Additional Facility Location

In this section, we develop an analogy between the adaptive spatial sampling problem and the additional facility location problem. The customer probabilities are assumed given as input. Locating an additional sample point in the variance minimization problem is equivalent to locating an additional facility. The only difference is that the objective in the sampling problem is to minimize the variance of an estimate whereas here we maximize the estimate itself. Thus if we let $z(s_i)$ denote the customer probability at sample site s_i, $z(g)$ denote the predicted probability at location g, where g is the set of grid points where the probability has to be maximized, we can use Equation (4) for a grid point g to get:

$$z_s(g) = \boldsymbol{\lambda}_s^T(g)\boldsymbol{z}_s = \boldsymbol{c}_s^T(g)\boldsymbol{C}_s^{-1}\boldsymbol{z}_s \tag{9}$$

The subscript s denotes that a new point has been added to the sample set whose optimal location has to be determined. z_s is the vector of probability values with the last entry being equal to the probability at the new point added. This would correspond to the location of the new facility. Since z_s requires the probability information at the added sample point (the last entry in the column vector z_s), we assume this value to be equal to one since the probability of a customer (located at a facility) visiting the chain is one. The objective in the additional facility location problem would then be:

$$s^{opt} = \underset{s}{Max} \sum_{g \in G} w(g) z_s(g) = \sum_{g \in G} w(g)\boldsymbol{\lambda}_s^T(g)\boldsymbol{z}_s = \sum_{g \in G} w(g)\boldsymbol{c}_s^T(g)\boldsymbol{C}_s^{-1}\boldsymbol{z}_s \tag{10}$$

Here *w(g)* represents the weight of grid point g. Theoretically, it is the probability of selecting grid point g. This can be computed for instance by taking the ratio of the population at g

to the total population. The single additional facility location we consider is the problem of allocating one additional facility to maximize the probability that a customer randomly chosen from a region specified in the set ***G*** visits the chain.

On the contrary, consider an indoor microcellular environment in cell tower location. In such a region, coverage might be desired over the whole area and just not at discretely spaced points. It might be argued that a discrete problem with a large number of equally spaced grid points spread throughout the region might approximate the continuous version. As will be shown later, the number of grid points greatly influences the computational complexity—hence the need for a study of the continuous version.

We consider both the discrete and continuous versions of the problem. To evaluate the heuristics for the discrete version we also solve the problem instances by using a discrete enumeration algorithm. A common assumption to both the discrete and continuous approaches is that of a negative exponential covariogram model.

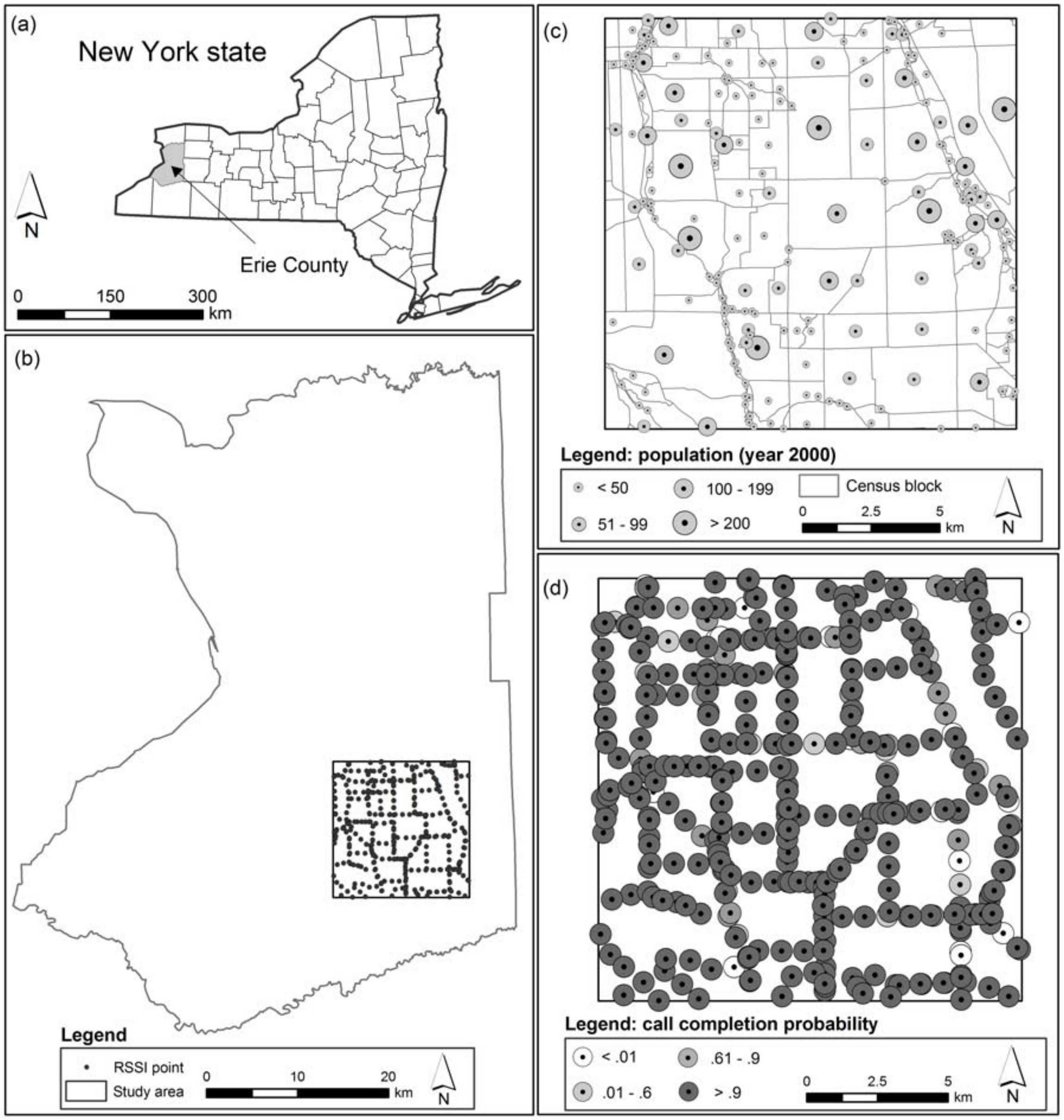
5. Discrete Version

We start by presenting a case study for a cellular application and study the output surface that is generated for the location of up to three additional cell towers. This is followed by two heuristics—one based on a simulated annealing heuristic and the other based on the Nelder Mead simplex method. Computational results for both heuristics are then reported.

5.1. Case Study from a Cellular Application

Here we try to maximize the completion probability for a randomly chosen call (a call is a surrogate for a customer). The input data is the cell phone RSSI values over the study area. This was measured using a modified Automated Crash Notification (ACN) device. We then mapped these signal strengths to call completion probability values from the results of Akella *et al.* (2003). The study area is located in the southern part of Erie County (see Figure 1b), within western New York State (Figure 1a). It is a 15 km by 15 km area, characterized by large altitude variation. The grid points (demand points) are the centroids of actual census blocks (Figure 1c) for the study area. Each of the 166 grid points is characterized by a weight associated with it, which is the population of the census block for the year 2000, which ranges from 1 to 300. There are 380 RSSI data points (Figure 1d) in this region.

The first step in the analysis was the development of the covariogram. We based this on the initial RSSI data points collected within the study region. This covariogram captures the spatial interaction between the data points — it has a range of 850m — this value for the range is used throughout our computational runs. The next step is to draw a bounding rectangle based upon the coordinates of the grid points. Note here that the optimal location of the cell tower does not go beyond the bounding rectangle. The rectangular area is divided into 75 horizontal and vertical lines with 200 meter spacing. The intersection of these lines forms the discretized search space. The choice of 75 and 200 is arbitrary here, since we are presenting this for the purpose of illustration. The possibility of locating a cell tower at each of these potential points is evaluated by measuring the call completion probability value. A graph of call completion probability versus the cell tower location is developed and the optimal point (that potential point at which the probability is maximum) is observed. All of the output figures have been developed using MATLAB version 6.5.



1 additional cell tower: The problem is to determine the location of one additional cell tower. Figure 2 shows that the total call completion probability values are low in most of the regions, except at one location where it peaks to 0.868. This is the optimal location of the cell tower since it maximizes the call completion probability of a random call over the grid points. Development of this graph is computationally expensive (approx 50 minutes on a high-end PC) though a clear picture of the call completion probability surface is obtained.

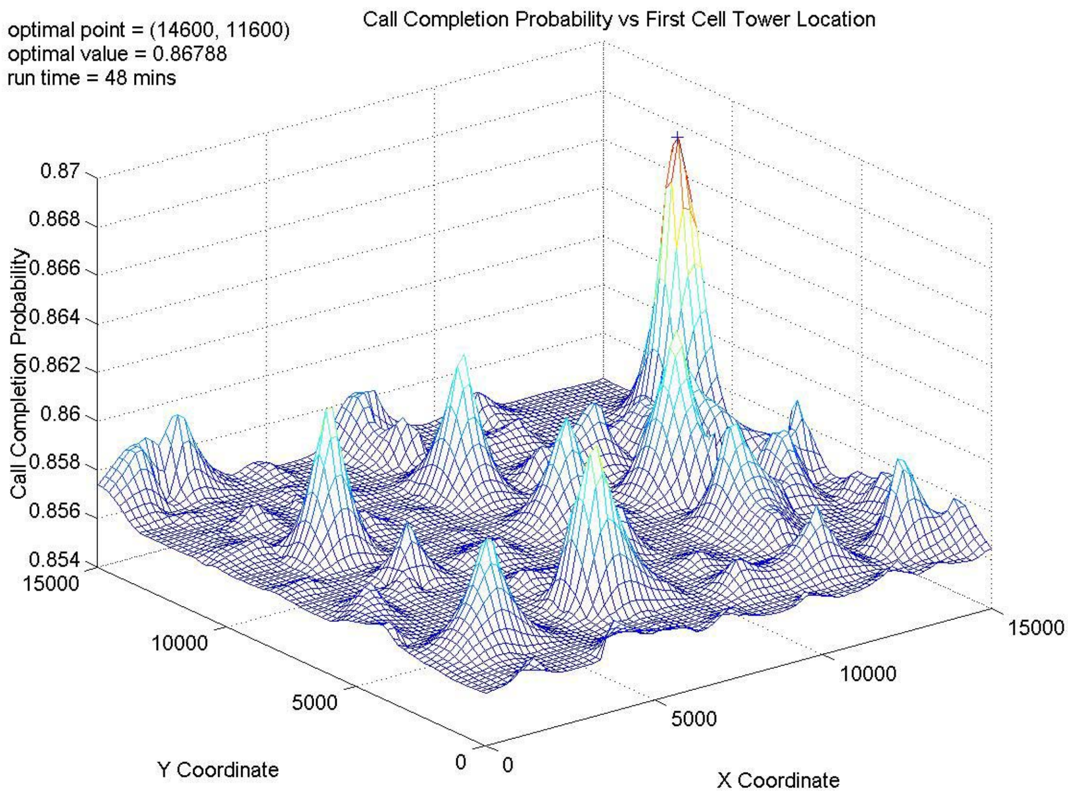


Figure 2: Location of first tower

2 additional cell towers: The problem is to determine the optimal locations of two towers that maximize the call completion probability. In this paper, we employ a greedy heuristic technique to obtain the locations. We use a myopic approach where we evaluate the location of each tower. So we first search for one tower, fix its location and search for the second tower using the same technique. The solution obtained may be suboptimal. This method helps us in understanding the variation in the surface with every new tower added to the existing set. The solution obtained for the 1-tower case would then be one of the towers in the final solution. This location is added to the existing sample set with a call completion probability value of 1. The search for the second tower is carried out in the same manner as before. The graph obtained is shown in Figure 3 with the optimal location marked by an asterisk. Note that the running time now includes the time needed for locating the first tower. A look at the graph shows that multiple peaks exist with

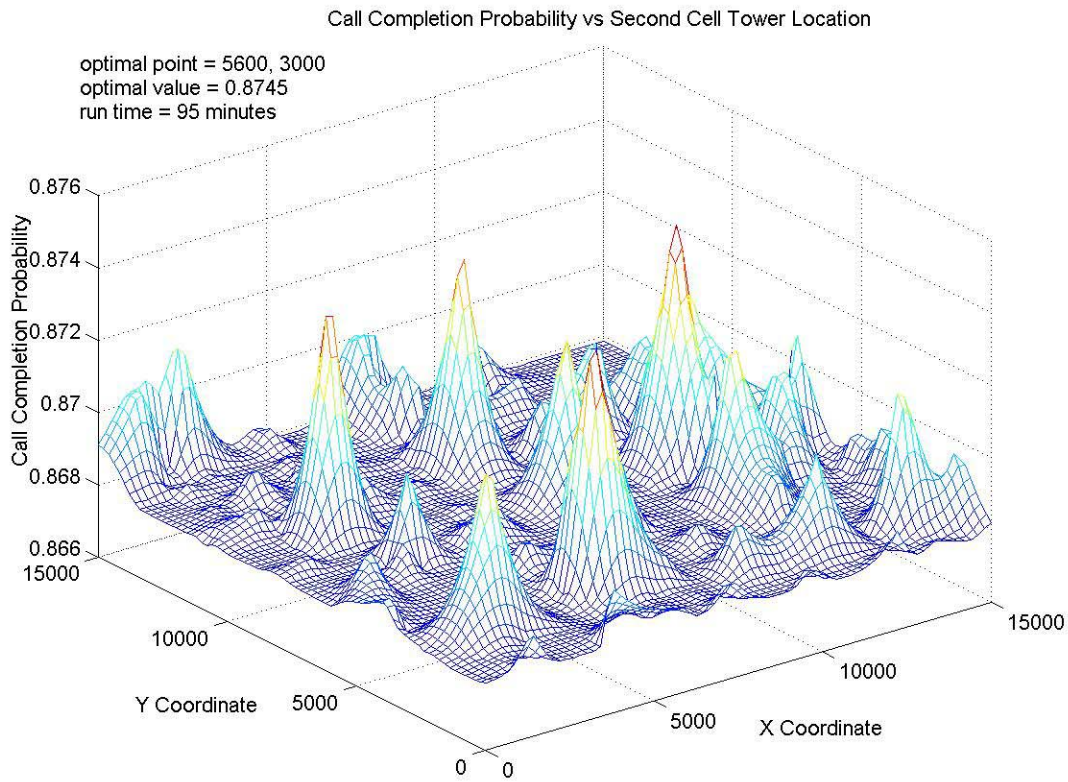


Figure 3: Location of second tower

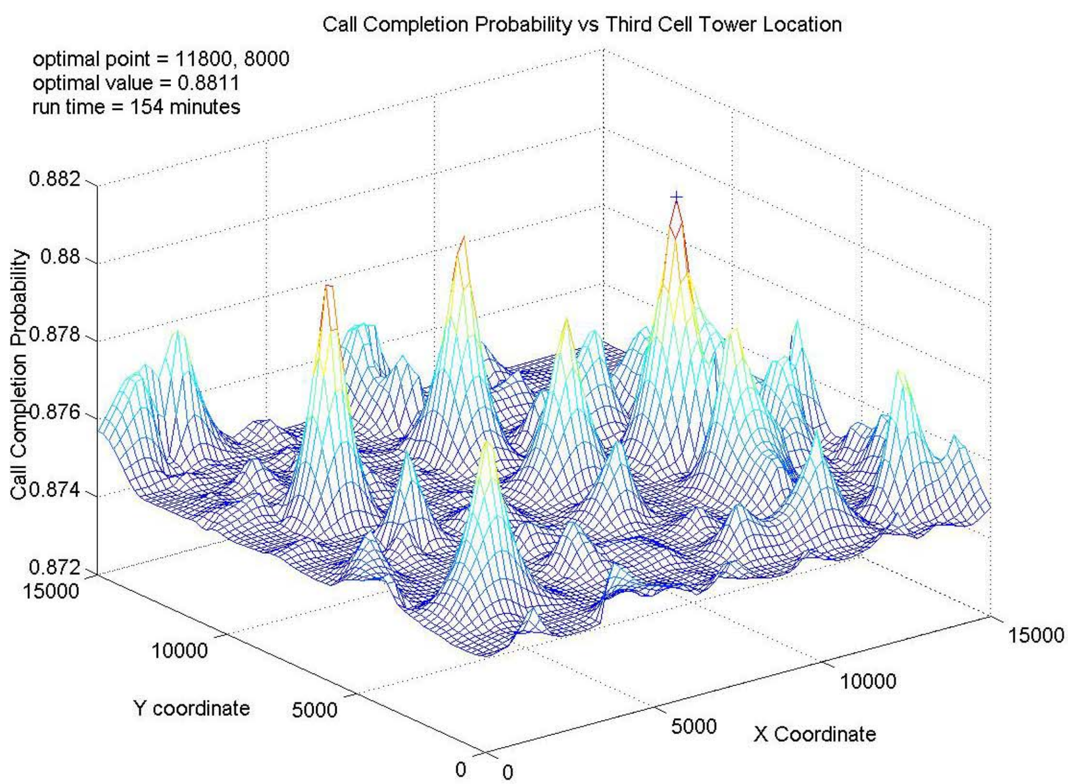


Figure 4: Location of third tower

approximately similar values. The optimal location now moves far from the location of the first tower.

3 additional cell towers: A similar approach is used to locate the third cell tower — see Figure 4. The running time is approximately 3 hours, but this is justified considering the fact that we use a total enumeration scheme to locate the towers.

In this section, we have developed several numerical examples along with a case study to illustrate the behavior of the objective function. It should be noted that this behavior could be extended to any other standard additional facility location problems since the only change would be the values of probabilities and the range of the covariogram. Hence the example of cell tower location provides an important base to understand the additional facility location problem.

5.2. Simulated Annealing

The algorithm is based upon that of Metropolis *et al.* (1958), which was originally proposed as a means of finding the equilibrium configuration of a collection of atoms at a given temperature. Pincus (1970) was the first to note the connection between this algorithm and mathematical minimization, but it was Kirkpatrick, Gelatt and Vecchi (1983) who proposed that it forms the basis of an optimization technique for combinatorial (and other) problems.

SA's major advantage over other methods is an ability to avoid becoming trapped at local minima. The algorithm employs a random search that not only accepts changes that decrease the objective function *f*, but also some changes that increase it. The latter are accepted with a probability

$$p = e^{-\frac{\delta f}{t}} \tag{11}$$

where δf is the increase in f and T is a control parameter, which by analogy with the original application is known as the system 'temperature' irrespective of the objective function involved. The algorithm is run for a fixed number of iterations and may terminate before if the termination criteria are met.

Some of the output figures shown in the previous example indicate clearly that the surface generated is highly uneven and any search technique has the drawback of getting stuck at local optima. Since simulated annealing has the inherent property of jumping out of local optima we employ this technique for finding the optimal solution. In most of our implementations of SA, we start with a temperature of 7000K and cool it by a factor of 0.8 at the end of a fixed number of iterations. The algorithm proceeds until the temperature drops to 1K. The neighborhood is selected in the following manner. We first pick a random direction in the interval (0, 2π). Then the neighbor of the current solution will be the new point obtained by moving a distance of step size in the chosen direction. The step size is also reduced by a factor after a fixed number of iterations. This new solution is accepted if it is better. If the solution obtained is poor then it is accepted with a probability as described above. This procedure is repeated until the temperature falls to 1K. The best solution obtained is reported.

5.3. Nelder Mead Simplex Algorithm

The simplex method is one of many direct search techniques used in optimizing a nonlinear function over a finite region. Parenthetically, it should be mentioned that the simplex method of unconstrained minimization should not be confused with the simplex method in linear programming, although the origin of the name is the same for both. A simplex is the convex hull of $n + 1$ points in R^n – for example, a line segment in R, a triangle in R^2 and so forth. The simplex method of unconstrained minimization was devised by Spendley, Hext and Himsworth (1962) and later improved by Nelder and Mead (1965).

Consider a minimization problem. The simplex algorithm considers a set of points that form a simplex. From among the set of vertices, replace the vertex with the poorest function value by a new point. The replacement of this point involves three types of steps: reflection, contraction or expansion. For a detailed description of this algorithm see Avriel (1976).

The original simplex method of Spendley, Hext, and Himsworth, based on regular simplexes without expansion and contraction steps, as well as the Nelder and Mead (NM) version just described, have been successfully tested on many problems, but they were found to be considerably affected by the scale and orientation chosen for the first simplex. The NM version, which is usually superior to the original method, was reported to be quite inefficient for problems with a large number of variables say $n \geq 10$ (Pierre 1969).

5.4. Computational Results

We now present performance results for the three heuristics, discrete enumeration (DE), SA, and NM. DE provides a benchmark to assess the quality of solutions obtained using other heuristic techniques. Note that the degree of optimality of DE is based on the extent to which the search space is discretized. All problems were programmed in MATLAB 6.1 and run on a Pentium III, 800 MHz processor and 768MB RAM. The *fminsearch* function available in the MATLAB optimization library was used for the NM method. The starting point for SA is chosen by picking the best grid point solution, whereas NM is run with randomly chosen and multiple starting points (due to its computational efficiency).

There are three sets of problems on which the three methods are tested: discrete small (DS), discrete medium (DM) and discrete large (DL). Table 1 lists the problem library with the 3 types of data sets and a brief description of each type. The interval for discrete search during enumeration was set to 0.5.

5.4.1. Discrete Small

The results for small size problems are shown in Table 2a, in which *CP* refers to customer probability of a randomly chosen customer visiting the chain. We can conclude the following from this table:

- The NM algorithm performed best both in terms of time and quality of solution reported.
- For DE we note a decrease in solution quality due to a sparser measurement point set.
- SA has intermediate performance and its running time is slightly less than DE. For the problem instance DS4, note that the simplex algorithm reports a suboptimal solution. On the other hand SA reports the optimal solution in all instances.
- NM performs better due to its feature of quickly converging to a local optimum rather than trying to diversify the search to seek the global optimum.

Table 2b shows the improvement in coverage obtained by locating an additional tower. The percentage improvement in coverage is defined as

$$\%\ improvement\ in\ coverage = \left(\frac{final\ CP - initial\ CP}{initial\ CP}\right) * 100 \tag{12}$$

Table 1: Problem library (discrete)

Problem Library (discrete)						
		Working area	Initial No of sample points	No of grid points	Wt of grid points	Covariogram range
DISCRETE SMALL(DS)	DS1	10 by 10	5	50	random(1,10)	6
	DS2	10 by 10	5	50	random(1,10)	6
	DS3	10 by 10	5	50	random(1,10)	6
	DS4	10 by 10	5	50	random(1,10)	6
	DS5	10 by 10	5	50	random(1,10)	6
DISCRETE MEDIUM (DM)	DM6	50 by 50	50	100	random(1,50)	10
	DM7	50 by 50	50	100	random(1,50)	10
	DM8	50 by 50	50	100	random(1,50)	10
	DM9	50 by 50	50	100	random(1,50)	10
	DM10	50 by 50	50	100	random(1,50)	10
DISCRETE LARGE (DL) - ERIE COUNTY CASE STUDY	DL11	15000 by 15000	380	166	census block population	5000
	DL12	15000 by 15000	381	166	census block population	5000
	DL13	15000 by 15000	382	166	census block population	5000

Table 2a: Discrete small results

	Discrete Enumeration			Simulated Annealing			Nelder Mead Simplex		
	CP	time (sec)	optimal point	CP	time (sec)	optimal point	CP	time (sec)	optimal point
DS1	0.3217	26.01	(6.8, 4.6)	0.3219	23.67	(6.81, 4.52)	0.3219	2.802	(6.83, 4.49)
DS2	0.2277	24.56	(7.4, 4.8)	0.2285	24.18	(7.32, 4.91)	0.2287	3.59	(7.31, 4.92)
DS3	0.2907	24.84	(6.8, 4.4)	0.2912	23.86	(6.85, 4.32)	0.2911	3.03	(6.43, 4.66)
DS4	0.2911	25.83	(4.6, 7)	0.2932	24.15	(4.71, 6.95)	0.2855	2.83	(2.98, 6.35)
DS5	0.2024	24.83	(6.8, 4.4)	0.2039	23.7	(6.84, 4.33)	0.2048	3.23	(6.86, 4.29)

Table 2b: Percentage improvement in coverage

	% improvement in coverage			
	Initial Coverage	% imp of DE	% imp of SA	% imp of NM
DS1	0.235	37.09	37.18	37.18
DS2	0.119	91.34	92.02	92.18
DS3	0.195	48.97	49.23	49.17
DS4	0.215	35.39	36.37	32.78
DS5	0.102	98.86	100.33	101.22

5.4.2. Discrete Medium

Here we notice a difference in the running times and the SA heuristic outperforms both DE and NM. The improvement over the initial solution obtained by the SA heuristic is 1.2% over DE and 7.2% over NM. NM fails to report a good solution owing to its localized search. SA takes just over 2 minutes whereas DE takes more than 3 minutes to solve this problem. Still, the time difference is not much to suggest that SA is the recommended technique to solve the additional facility location problem.

Table 3a: Discrete medium results

	Discrete Enumeration			Simulated Annealing			Nelder Mead Simplex		
	CP	time (sec)	optimal point	CP	time (sec)	optimal point	CP	time (sec)	optimal point
DM6	0.1669	198.84	(29, 45.5)	0.1694	132.78	(28.91, 45.28)	0.1698	14.15	(28.97, 45.20)
DM7	0.1578	199.49	(29, 45.5)	0.1599	147.64	(28.81, 45.36)	0.1416	15.03	(21.03, 48.21)
DM8	0.1746	201.97	(29, 45.5)	0.1768	134.16	(28.99, 45.21)	0.1531	13.06	(29.95, 35.81)
DM9	0.1787	198	(29, 45.5)	0.18	131.07	(22.28, 6.89)	0.1666	13.7	(47.45, 21.64)
DM10	0.1616	198.54	(29, 45.5)	0.1635	134.52	(28.90, 45.30)	0.1637	15.11	(41.41, 49.53)

Table 3b: Percentage improvement in coverage

	% improvement in coverage			
	Initial Coverage	% imp of DE	% imp of SA	% imp of NM
DM6	0.12253	36.21	38.25	38.58
DM7	0.12105	30.36	32.09	16.98
DM8	0.13305	31.23	32.88	15.07
DM9	0.14555	22.78	23.67	14.46
DM10	0.12122	33.31	34.88	35.04

Another point worth clarifying about the results pertains to the fact that the grid locations for all of the problems of DM are the same. An explanation for this is as follows: We only generate the locations of the sample point data. Due to a greater concentration of grid points around the coordinates (29, 45.5) the optimal coordinates remain unchanged with change in the sample locations. Table 3a and 3b list the results.

5.4.3. Discrete Large (Case Study for Cellular Application)

This data was collected from a section of rural Erie County for the cell tower location problem. Due to the large area for the case study we use a smaller search interval of 50 units as opposed to 200 from the example. Due to this there is a change in the optimal location of the second tower from the one shown in the example. SA performs remarkably well for this size of problem and gives a better solution than the one obtained using DE. Though NM takes the least amount of time to solve this class of problems, it fails in terms of the quality of the solution. Whereas DE takes nearly three hours to optimally locate the first tower, SA takes just over four minutes. This is a significant difference in computation time and it makes sense to rely on SA for solving real life additional facility location problems using spatial interpolation. The three problems represent sequential addition of facilities. So the optimal location obtained in DL11 is added to the sample data set in DL12 and the location obtained in DL12 is added to the sample data set in DL13 to determine the location of the third facility. Results are shown in Tables 4a and 4b. Note that the improvement obtained in *CP* declines with the addition of each facility. It makes sense in this case to add just one facility because the improvement obtained in coverage by adding the second and third facility is less than 1%.

Table 4a: Discrete large results

	Discrete Enumeration			Simulated Annealing			Nelder Mead Simplex		
	CP	time (min)	optimal point	CP	time (min)	optimal point	CP	time (min)	optimal point
DL11	0.868	180	(14600, 11644)	0.8692	4.15	(14493, 11672)	0.8625	1.36	(7725, 10997)
DL12	0.8765	183.42	(11750, 7950)	0.8765	4.18	(11755, 7951)	0.874	1.39	(11467, 5496)
DL13	0.8836	185.29	(5500, 2950)	0.8838	4.2	(5483.9, 2961)	0.8813	1.45	(8389.1, 7860.2)

Table 4b: Percentage improvement in coverage

	% improvement in coverage			
	Initial Coverage	% imp of DE	% imp of SA	% imp of NM
DL11	0.86	1.38	1.52	0.74
DL12	0.87	0.84	0.84	0.55
DL13	0.88	0.81	0.83	0.55

An overall conclusion is that SA performs remarkably well for discrete problems. The largest problem size was solved in just above 4 minutes. This clearly shows that SA can solve much larger problems if allowed to run for more than an hour. Since most network design problems need a one-time solution, this heuristic can be used to solve large problems.

6. Continuous Version

To develop a solution approach for this case we rewrite equation (10) for the continuous case by replacing the summations by integrals, to get:

$$s^{opt} = \underset{s}{Max} \int_{g \in G} z_s(g) = \underset{s}{Max} \int_{g \in G} \boldsymbol{\lambda}_s^T(g)\boldsymbol{z}_s = \underset{s}{Max} \int_{g \in G} \left[\boldsymbol{c}_s^T(g)\boldsymbol{C}_s^{-1}\boldsymbol{z}_s \right] \tag{13}$$

Notice that although the last term of the above expression is a product of three matrices, the integration is performed only on the first term. This is because the matrix $\boldsymbol{C}$ or the vector $\boldsymbol{z}$ does not depend on the grid points. Hence (13) can be rewritten as

$$\begin{gathered} s^{opt} = \underset{s}{Max} \int_{g \in G} \left[\boldsymbol{c}_s^T(g) \right] \boldsymbol{C}_s^{-1}\boldsymbol{z}_s = \underset{s}{Max}\ \boldsymbol{h}_s^T \boldsymbol{C}_s^{-1}\boldsymbol{z}_s \\ \text{where } \boldsymbol{h}_s^T = \int_{g \in G} \boldsymbol{c}_s^T(g) \end{gathered} \tag{14}$$

6.1. Numerical Examples

We start by presenting some simple examples to better understand the behavior of the objective function with varying locations of the new facility. Consider the problem in one dimension: Given a set of sample observations on a line of length L, what is the optimal location of a facility on the line that maximizes the customer probability integrated on this line? The objective function for this one-dimensional case is

$$f(x) = \boldsymbol{h}^T(x)\boldsymbol{C}^{-1}(x)\boldsymbol{z} \tag{15}$$

where x is the location of the new facility—note that the vector $\boldsymbol{z}$ is independent of x, and that the entries of the vector $\boldsymbol{z}$ vector are the customer probability values measured at the sample locations and this vector has a value of one for its last entry (which corresponds to the measurement taken at the new facility location).

We now examine the vector $\boldsymbol{h}$

$$h^T(x) = \int_0^L c_x^T(g)\,dg$$

$$where\ \ c_x^T(g) = \begin{bmatrix} c_{gs_1} & c_{gs_2} & c_{gs_3} & \ldots & c_{gs_4} \end{bmatrix} \tag{16}$$

s_i, $i = 1, 2, \ldots n$ represents the sample points

c_{gs_1} represents the covariance betweeen a grid point g and sample point s_i

With the assumption of a negative exponential model for the covariogram (assuming R = 1) we have:

$$\int_0^L c_{s_i}(g)\,dg = \int_0^L e^{-\frac{\|y-x_i\|}{h}}\,dy = \int_0^{x_i} e^{-\frac{(x_i-y)}{h}}\,dy + \int_{x_i}^L e^{-\frac{(y-x_i)}{h}}\,dy$$

$$= h\left(1-e^{-\frac{x_i}{h}}\right) - h\left(e^{-\frac{(L-x_i)}{h}} - 1\right) \tag{17}$$

$$= 2h - h\left(e^{-\frac{x_i}{h}} + e^{-\frac{(L-x_i)}{h}}\right)$$

Thus, for the one-dimensional case, each term of the $\boldsymbol{h}$ vector is represented by equation 17. This expression appears to be manageable. But when the problem is extended to two dimensions the expression becomes highly cumbersome. Hence we perform theoretical analysis for the one dimension case and then extrapolate our observations to two dimensions. This analysis has been done mainly to get some insight into the structure of the function.

6.2. Analysis

To establish an efficient solution procedure we investigate some theoretical properties of the objective function.

Theorem: For the simplest case with one sample observation on a line, the customer probability graph is concave with the location of the second sample point (new facility).

Proof: Consider a line of length 1 unit. Assign values to the parameters: h = 1, R = 1 without loss of generality. We will show that the function is piecewise concave with the location of the second sample point (new facility).

$$f(x) = \boldsymbol{h}^T(x)\boldsymbol{C}^{-1}(x)\boldsymbol{z}$$
$$= \begin{bmatrix} h_{s_1} & h_{s_2} \end{bmatrix} \boldsymbol{C}^{-1}(x)\boldsymbol{z}, \quad where$$
$$h_{s_1} = 2 - e^{-x_1} - e^{x_1 - 1}$$
$$h_{s_2} = 2 - e^{-x} - e^{x-1}$$
$$\boldsymbol{C}(x) = \begin{bmatrix} 1 & e^{(x-x_1)} \\ e^{(x-x_1)} & 1 \end{bmatrix} \text{assuming that we are working in the region } x > x_1$$

$$Let\, t = x - x_1$$
$$\boldsymbol{C}^{-1}(t) = \frac{1}{1-e^{-2t}} \begin{bmatrix} 1 & -e^{-t} \\ -e^{-t} & 1 \end{bmatrix}$$
$$\boldsymbol{h}^T(t)\boldsymbol{C}^{-1}(t) = \begin{bmatrix} 2 - e^{-x_1} - e^{x_1-1} & 2 - e^{-t-x_1} - e^{t+x_1-1} \end{bmatrix} . \frac{1}{1-e^{-2t}} \begin{bmatrix} 1 & -e^{-t} \\ -e^{-t} & 1 \end{bmatrix}$$

The above equation when simplified reduces to

$$\boldsymbol{h}^T(t)\boldsymbol{C}^{-1}(t) = \begin{bmatrix} \frac{2}{1+e^{-t}} - e^{-x_1} & \frac{2}{1+e^{-t}} - e^{-t+x_1-1} \end{bmatrix}$$

Each of the two terms in the above vector can be shown to be concave by twice differentiation. Since ***z*** is a vector of positive numbers, a positive linear combination of concave functions is also concave. Hence we have proved that *f(x)* is concave for $x > x_1$. Similarly we can prove the concavity for the case when $x < x_1$. The theorem follows. □

We have tried several generalizations of the above result, e.g. several sample points in one/two dimensions. In all cases we tested we found that the result holds but we are unable to theoretically establish it. However, we proceeded to develop a heuristic based on the following hypothesis in two dimensions.

Hypothesis: The customer probability is concave in the convex polygon formed with the sample points as vertices such that there is no sample point within any polyogon.

The choice of convex polygons in the hypothesis is explained as follows: The customer probability is undefined at the sample locations and in one dimension it is concave in any chosen convex region. Hence one way to choose a convex region in two dimensions is to consider a convex polygon with the sample points as vertices and not including the sample points in the polygon.

6.3. Geometric Search (GS)

We propose a geometric search (GS) heuristic based on the hypothesis presented in section 6.2. We use the Delaunay triangulation routine (a polynomial time algorithm) available in MATLAB to construct the triangles with the sample points as vertices.

procedure GEOMETRIC SEARCH

begin

input sample point coordinates with call completion probabilities, *samplepoint*

numtriangles = delaunay(*samplepoint*) // this is a routine in matlab that constructs a Delaunay triangulation with the sample points as vertices

for $i = 1, \ldots,$ *numtriangles*

begin // perform a steepest ascent search within each triangle to approximately determine the optimal objective function value within that triangle

select i^{th} triangle, *triangle*(i)

opt(i), *tccp*(i) = *steepestsearch*(*triangle*(i))

end

*ccp**, *index* = max(*tccp*(i)) // find the maximum value among the optimal points within each triangle

*opt** = opt(*index*)

end

The procedure begins with input of the sample point coordinates, with the corresponding customer probability values. We then use the routine to construct the triangulation of the sample points. Note that to cover the entire rectangular region we add the coordinates of the four corner points to the sample set. This partitions the region into mutually exclusive and collectively exhaustive triangles, and allows us to get the optimum solution by searching over all of these triangles.

A steepest descent search within each triangle is implemented as follows: We measure the *CP* values at the midpoints of each side and choose the maximum. A neighborhood of a point is defined as the six points generated by moving parallel to each side by a distance equal to 1/12th (the choice of 1/12 is empirically determined as the tradeoff between quality of the solution and computational complexity) the length of the corresponding side. This in a way divides the triangle into grid lines parallel to each one of the three sides. Now the best point (in terms of the objective function) is chosen from among its neighbors to be the new point. If the value of the function at all of its neighbors is less than the value at the current point, then the search stops and steepest descent returns the current point as the optimal solution for that triangle. If not, the best neighbor is updated to be the current point and search proceeds in a similar manner.

6.4. Results

All problems have been programmed in MATLAB 6.1. The problems were run on a Pentium III, 800 MHz processor and 768MB RAM. There are two sets of problems on which the three methods are tested: continuous small (CS) and continuous medium (CM) and continuous large (CL). We use the same problem library used for the discrete case. The only modification in that problem set is the elimination of grid point data since we are maximizing customer probability over a continuous region. Figure 5 demonstrates the geometric search heuristic. It first constructs the triangulation and then searches within each triangle. Circles show the search within each triangle, with the optimal location shown in bold.

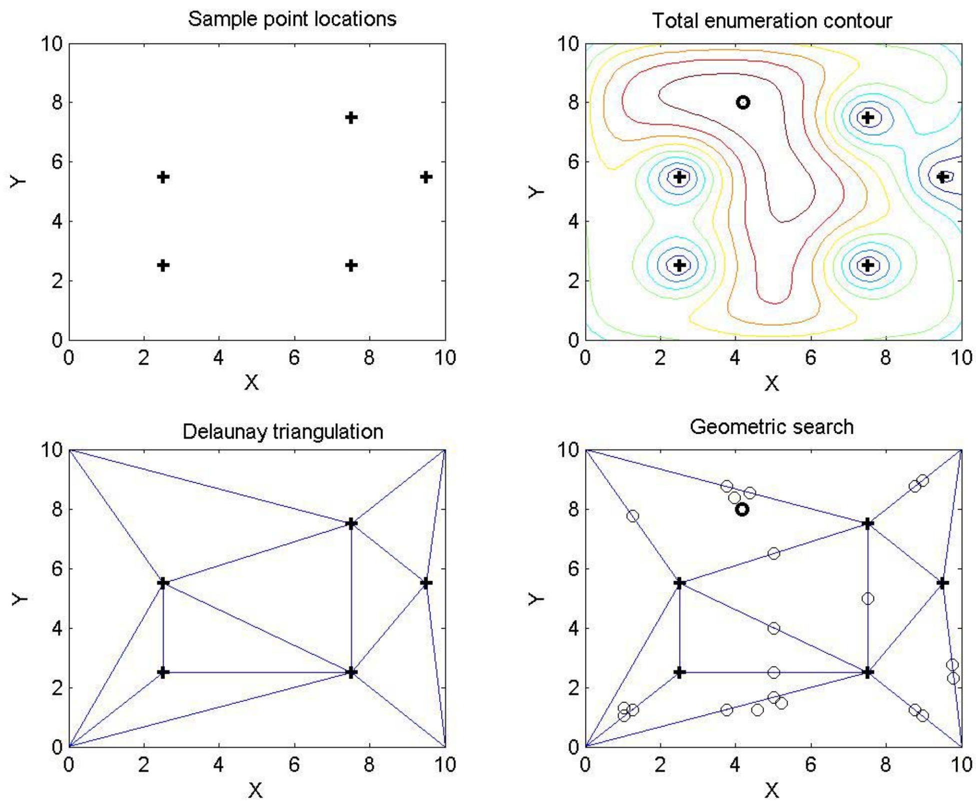


Figure 5: Geometric Search heuristic

6.4.1. Continuous Small

The results for small size problems are shown in Table 5. We note that there is a substantial increase in running time when compared to the discrete case for the same size problems. In the continuous version, we have to resort to numerical methods to evaluate the integral as mentioned before and this takes a considerable portion of the running time. DE takes 35 minutes on average to solve small size problems, whereas GS takes 1.6 minutes. There is a considerable reduction in the running time by the use of GS heuristic. Both heuristics report the same solution in all 5 cases. Also note that there was no major change in the location of the optimal solution with different sample inputs. This emphasizes the fact that the location of sample points did not have much effect on the location of the new facility and we would get maximum improvement in coverage by locating the facility at the center.

Table 5a: Continuous small results

	Discrete Enumeration			Geometric Search		
	CP	time (min)	optimal point	CP	time (min)	optimal point
CS1	0.2936	34.84	(4.2, 8)	0.2936	1.59	(4.16, 8)
CS2	0.1788	34.11	(5.4, 2.2)	0.1788	1.59	(5.75, 2.3)
CS3	0.2052	34.08	(5.2, 3.8)	0.2052	1.47	(5.11, 3.73)
CS4	0.209	34	(2.6, 4.4)	0.209	1.63	(2.6, 4.3)
CS5	0.1584	34.06	(3.8, 6.6)	0.1584	1.59	(3.57, 6.64)

Table 5b: Percentage improvement in coverage

	% improvement in coverage		
	Initial Coverage	% imp of DE	% imp of GS
CS1	0.2472	18.77	18.77
CS2	0.1286	39.04	39.04
CS3	0.1532	33.94	33.94
CS4	0.1551	34.75	34.75
CS5	0.1015	56.06	56.06

6.4.2. Continuous Medium

In continuous medium size problems, we notice a large difference in the running times of the GS and DE heuristics. And both report exactly the same solution in all five problems tested. While DE takes nearly six hours to solve each problem GS heuristic takes around 30 minutes. This is a significant improvement considering the fact that the problem size is considerably big. Here we notice a shift in the optimal location for each new problem set. This means that the sample locations influenced the location of the new tower unlike the small size problem set. Results are shown in Table 6.

Notice that we get an improvement of 5 to 6 % in coverage with the introduction of a new facility. A 5% improvement when translated to number of additional customers served during a day and consequently over a period of time would be very significant and might lead to substantial profits for a firm.

Table 6a: Continuous medium results

	Discrete Enumeration			Geometric Search		
	CP	time (min)	optimal point	CP	time (min)	optimal point
CM6	0.133	343.66	(27, 38.5)	0.133	32.58	(26.47, 39.12)
CM7	0.1392	341.46	(42.5, 12.5)	0.1392	31.65	(42.86, 12.17)
CM8	0.1531	342.71	(23, 26)	0.1531	30.92	(22.56, 26.02)
CM9	0.1634	343.96	(42, 18.5)	0.1634	34.51	(41.7, 19)
CM10	0.1416	342.06	(32, 8)	0.1416	32.38	(31.3, 8.89)

Table 6b: Percentage improvement in coverage

	% improvement in coverage		
	Initial Coverage	% imp of DE	% imp of GS
CM6	0.1253	6.15	6.15
CM7	0.1315	5.86	5.86
CM8	0.1454	5.30	5.30
CM9	0.1555	5.08	5.08
CM10	0.1339	5.75	5.75

From the results shown above for the three problem sizes, we can conclude that Geometric Search heuristic performs significantly better than DE for continuous problems. As mentioned before since network design problems usually need a one-time solution, GS heuristic can be used to solve huge problems. The quality of the solution obtained is better than the best heuristic (discrete enumeration) so far.

7. Which should we use: Continuous or Discrete?

We now compare the SA heuristic for the discrete problem with the GS heuristic for the continuous problem. In order to solve the continuous version of the problem using discrete methods, we break the region into finely spaced grid points. The customer probability is summed over all these grid points.

7.1. Results

7.1.1. Continuous Vs Discrete Small

Results of Table 7 show that although there is not much difference in the running time of GS and SA, the quality of solution obtained using GS is much better than that of SA. On average, GS improves the initial coverage 7% more than the improvement obtained using SA. The continuous surface was discretized into equally spaced grid points using a gap of 0.5 between grid lines. We might obtain a better solution with even finer grid spacing but that is at the expense of the computation time. Note that the optimal locations in the two heuristics are quite close and the solution improves by nearly 6% in such a small distance.

7.1.2. Continuous Vs Discrete Medium

Table 8 tabulates results for continuous vs. discrete medium. Again we see that GS outperforms SA both in terms of solution time and objective function value. We have used a discretization interval of 0.5 again for this although this is not a fine enough resolution. From our experience we observed that increasing the resolution of discretization greatly increases the computation time. Hence we chose this interval for our computational study. Where SA takes 48 minutes on average to solve, GS solves the problem in 32 minutes. In terms of the solution, GS provides a better solution (1% more than SA) than SA for all five of the problem instances. Note that in order to get the same quality of solution as GS using SA the region has to be finely discretized and this made the problem run for several hours.

Table 7a: Continuous vs Discrete small results

	Simulated Annealing			Geometric Search		
	CP	time (min)	optimal point	CP	time (min)	optimal point
CS1	0.277	2.12	(4, 8.05)	0.294	1.59	(4.16, 8)
CS2	0.17	2.12	(5.03, 2)	0.179	1.59	(5.75, 2.3)
CS3	0.196	2.14	(5.03, 3.5)	0.205	1.47	(5.11, 3.73)
CS4	0.198	2.13	(2.48, 4.48)	0.209	1.63	(2.6, 4.3)
CS5	0.15	2.13	(4, 4.49)	0.158	1.59	(3.57, 6.64)

Table 7b: Percentage improvement in coverage

	% improvement in coverage		
	Initial Coverage	% imp of TE	% imp of GS
CS1	0.2472	12.10	18.77
CS2	0.1286	32.50	39.04
CS3	0.1532	28.00	33.94
CS4	0.1551	27.34	34.75
CS5	0.1015	48.08	56.06

Table 8a: Continuous vs Discrete medium results

	Simulated Annealing			Geometric Search		
	CP	time (min)	optimal point	CP	time (min)	optimal point
CM6	0.132	48.55	(26.34, 39.45)	0.133	32.58	(26.47, 39.12)
CM7	0.138	48.14	(41.19, 10.68)	0.139	31.65	(42.86, 12.17)
CM8	0.151	48.3	(23.17, 26)	0.153	30.92	(22.56, 26.02)
CM9	0.162	48.24	(42.51, 18.96)	0.163	34.51	(41.7, 19)
CM10	0.14	48.34	(32.06, 7.48)	0.142	32.38	(31.3, 8.89)

Table 8b: Percentage improvement in coverage

	% improvement in coverage		
	Initial Coverage	% imp of TE	% imp of GS
CM6	0.1253	5.43	6.15
CM7	0.1315	4.94	5.86
CM8	0.1454	4.06	5.30
CM9	0.1555	4.24	5.08
CM10	0.1339	4.63	5.75

The results for the continuous versus discrete version show that SA cannot be used efficiently to solve this type of problem and new heuristics have to be developed. The GS heuristic gives excellent results and can be used to efficiently solve large problems.

8. Conclusions and Future Research

In this paper, we have addressed the adaptive spatial sampling problem and its relation to the additional facility location problem. The problem of determining an optimal additional sample point has been shown to be analogous to that of determining the optimal additional facility. The numerical expression for the optimization of both functions has been explained. Several numerical examples have been developed to illustrate the behavior of this optimization function with a cell tower location context. We also developed a case study to locate one, two and three additional cell towers in a rural section of Erie County, New York.

For the discrete case, we developed a SA heuristic and obtained good quality solutions—SA was empirically demonstrated to be superior to the NM and DE methods. For the continuous case, we analyzed the behavior of the objective function. This led us to a hypothesis of piecewise concavity of the objective function. We proposed a GS heuristic based on this observation and found through empirical tests that it performed extremely well. We also performed a comparison between the discrete and continuous modeling approaches. Our empirical results show that the continuous modeling method using the GS heuristic is the best algorithmic choice.

There are several opportunities for future research in this area:

1. Establishing the concavity property would be an important step towards the development of an optimal algorithm for additional facility location using spatial interpolation.
2. Our paper used simple kriging to develop the optimization routine. Use of ordinary kriging (as opposed to simple kriging) would be useful since ordinary kriging assumes a spatially variant mean.
3. To make this approach more practically applicable, research needs to be done to extend this model to multiple facility locations.

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