MTH 620: 2020-04-28 lecture

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Introduction

In this (last) lecture we'll mix it up a little and do some topology along with the homological algebra. I wanted to bring up the cohomology of profinite groups if only briefly, because we discussed these in MTH619 in the context of Galois theory. Consider this as us connecting back to that material.

1 Topological and profinite groups

This is about the cohomology of profinite groups; we discussed these briefly in MTH619 during Fall 2019, so this will be a quick recollection. First things first though:

Definition 1.1 A *topological group* is a group G equipped with a topology, such that both the multiplication $G \times G \to G$ and the inverse map $g \mapsto g^{-1}$ are continuous.

Unless specified otherwise, all of our topological groups will be assumed separated (i.e. Hausdorff) as topological spaces. ♦

Ordinary, plain old groups can be regarded as topological groups equipped with the discrete topology (i.e. so that all subsets are open). When I want to be clear we're considering plain groups I might emphasize that by referring to them as 'discrete groups'.

The main concepts we will work with are covered by the following two definitions (or so).

Definition 1.2 A *profinite group* is a compact (and as always for us, Hausdorff) topological group satisfying any of the following equivalent conditions:

- G embeds as a closed subgroup into a product $\prod_{i\in I} G_i$ of finite groups G_i, equipped with the usual product topology;
- For every open neighborhood U of the identity $1 \in G$ there is a normal, open subgroup $N \trianglelefteq G$ contained in U. ♦

[4] is an excellent resource for the subject in general, by the way. We will see other characterizations of profinite groups in just a bit. For now, some examples.

Example 1.3 The obvious, unenlightening ones first: every finite group is profinite, when equipped with its discrete topology. ♦

Example 1.4 In MTH619 we discussed (at some length) the additive group $\mathbb{Z}_p$ of *p-adic integers* for a prime p. Remember that this was defined as the inverse limit of all $\mathbb{Z}/p^n$ along the canonical surjections

$$\cdots \to \mathbb{Z}/p^{n+1} \to \mathbb{Z}/p^n \to \cdots \to \mathbb{Z}/p. \tag{1-1}$$

Here, the "inverse limit" meant the set

$$\{(a_n)_{n\geq 1} \mid a_n \in \mathbb{Z}/p^n \text{ and for all } n \text{ we have } \pi(a_{n+1}) = a_n\},$$

where, by a slight abuse of notation, π denotes the various projections $\mathbb{Z}/p^{n+1} \to \mathbb{Z}/p^n$.

So it's like the direct product, except the elements in the various groups $\mathbb{Z}/p^n$ are chosen so as to be compatible with the connecting maps in (1-1). ♦

More generally, there's a sense in which *all* profinite groups arise as in Example 1.4, as "inverse limits" of finite groups. I will elaborate.

Fix a topological group Γ, and let $(\mathcal{P}, \leq)$ be the poset of open, normal, finite-index subgroups $H \trianglelefteq \Gamma$ ordered by inclusion (so all quotients Γ/H, $H \in \mathcal{P}$ are finite groups). Then, for $N \leq H$ in $\mathcal{P}$, we have a canonical surjection

$$\pi_{H,N} : \Gamma/N \to \Gamma/H \tag{1-2}$$

We can then form the *limit* $\varprojlim_{H\in\mathcal{P}} \Gamma/H$ along the connecting maps (1-2), in the same sense as before: tuples of elements, one in each quotient Γ/H, compatible with the surjections (1-2). In short,

$$\varprojlim_{H\in\mathcal{P}} \Gamma/H := \{(a_H)_{H\in\mathcal{P}} \mid a_H \in \Gamma/H \text{ and } \pi_{H,N}(a_N) = a_H,\ \forall N \leq H \in \mathcal{P}\}.$$

The resulting set embeds naturally in the direct product

$$\prod_{H\in\mathcal{P}} \Gamma/H$$

and inherits a group structure via the embedding. It turns out it's also a *closed* subset of that product, so it's a profinite group by Definition 1.2. We refer to it as the *profinite completion of* Γ (notation: $\widehat{\Gamma}$).

Note that there's always a canonical group morphism

$$\Gamma \to \widehat{\Gamma},$$

sending $\gamma \in \Gamma$ to the tuple of images of γ in the various quotients Γ/H, $H \in \mathcal{P}$.

Finally, to finish this train of thought, it turns out that profinite groups can be characterized as precisely those that arise as profinite completions:

Proposition 1.5 *A topological group G is profinite if and only if the canonical map $G \to \widehat{G}$ to its profinite completion is an isomorphism.* ■

So these are exactly the groups that arise as inverse limits of finite groups, in the sense of the above discussion.

To get to our first problem, recall that for the topological group Γ in the discussion above we *defined* the poset $\mathcal{P}$ to be that of open, normal, finite-index subgroups. For *profinite* groups though, the finite-index assumption is unnecessary. This follows from the fact that by definition, profinite groups are compact.

Problem 1 *Let G be a compact Hausdorff topological group and $H \leq G$ an open subgroup. Show that*

(a) H is also closed;

(b) the set G/H of left cosets is finite (i.e. H has finite index).

This should be a fairly straightforward application of compactness; little more than the definition should be necessary.

2 Modules and cohomology for profinite groups

Definition 2.1 Let Γ be a topological group. A *discrete (left) Γ-module* is a Γ-module M with the property that the action structure map

$$\Gamma \times M \to M$$

is continuous when we equip M with the discrete topology.

Discrete Γ-modules form a category in the obvious fashion; we denote it by ${}^d_\Gamma\mathrm{Mod}$. ◆

The definition makes sense for arbitrary topological groups Γ, but we will usually specialize to profinite G (not always though; not in Problem 2 below for instance).

Now, just as for plain (discrete) groups, for an arbitrary topological group Γ we have a fixed-points functor

$$ {}^d_\Gamma\mathrm{Mod} \ni M \mapsto M^\Gamma \in \mathrm{Ab}. \tag{2-1}$$

to the category of abelian groups.

Problem 2 *Let Γ be a topological group. Show that the invariant functor $(-)^\Gamma : {}^d_\Gamma\mathrm{Mod} \to \mathrm{Ab}$ is a right adjoint.*

How did we prove this for the ordinary category ${}_G\mathrm{Mod}$ of modules over a discrete group? Do the same thing here, and note that the left adjoint

$$\mathrm{Ab} \to {}_G\mathrm{Mod}$$

to $(-)^G$ actually lands in the category of discrete modules (i.e. its image consists of G-modules M for which the module map $G \times M \to M$ is continuous when M is equipped with the discrete topology).

The reason why, in homological algebra, we tend to work with *abelian categories* rather than just categories of modules is the desire to cover important examples such as ${}^d_G\mathrm{Mod}$: when G is profinite and infinite this category is *not* of the form ${}_R\mathrm{Mod}$ for any ring R! It is always, however, an abelian category in the sense of [5, Definition, p.307]. We discussed the notion briefly in MTH619, but you might want to brush up on it for this.

In general, ${}^d_G\mathrm{Mod}$ will not have enough projectives (this is one way to see it's not a module category). In fact, more is true:

Theorem 2.2 *For a profinite group G, the category ${}^d_G\mathrm{Mod}$ has no non-zero projective objects unless G is finite.* ■

This is a particular case of [3, Theorem 3.1]. Sorry for the self-citation, but it's topical here: that result grew out of a natural question that occurred to me while teaching MTH620 last year.

Nevertheless, ${}^d_G\mathrm{Mod}$ always has enough *injectives*. This is one reason why I said in passing, during one of the in-person lectures, that in practice injectives tend to show up more than projectives.

Remark 2.3 Though abelian categories are a useful abstraction allowing us to gather ${}_R\mathrm{Mod}$, ${}^d_G\mathrm{Mod}$ and many other categories of interest under the same umbrella, in the case of ${}^d_G\mathrm{Mod}$ for a profinite group G the abelian category structure is very transparent: kernels are just the usual kernels you know from abelian group theory, ditto for cokernels, ditto for direct sums. ◆

In any case, given Problem 2, we have a left exact invariant functor (2-1) whose right derived functors we can compute using injectives. This allows us to make sense of

Definition 2.4 Let G be a profinite group. For a non-negative integer i, the i^{th} *cohomology functor*

$$H^i(G,-) : {}^d_G\mathrm{Mod} \to \mathrm{Ab}$$

is defined as the i^{th} derived functor of the invariant functor (2-1). ♦

2.1 Cohomology via inflation

Henceforth, unless specified otherwise, G denotes a profinite group. If $M \in {}^d_G\mathrm{Mod}$ is a discrete G-module, there are alternative ways to recover the cohomology $H^*(G,M)$ (alternative to Definition 2.4, that is).

Consider an open normal subgroup $H \trianglelefteq G$ (so that by Problem 1 the group G/H is finite). Because $H \trianglelefteq G$ is normal, G acts on M^H (you might want to check this, if it's not clear). Since H itself acts trivially on M^H, this gives us an action of the finite group G/H on M^H.

Now consider an inclusion $N \le H$ in $\mathcal{P}$ (the poset of open, normal subgroups of G, as in the discussion above). Then, H/N is a normal subgroup of the finite group G/N, and applying the discussion in the preceding paragraph to that normal subgroup we get an action of

$$(G/N)/(H/N) \cong G/H$$

on $(M^N)^{H/N} = M^H$. Then, from the normal inclusion

$$H/N \trianglelefteq G/N,$$

we get an *inflation morphism*

$$\inf_{N,H} : H^i(G/H, M^H) \to H^i(G/N, M^N) \tag{2-2}$$

discussed before in these lectures in a somewhat restricted setup [1, §2.1]. You can consult [5, p.566] for a full discussion, but the morphism is definable as plainly as you could possibly hope for at cocycle level: given an i-cocycle

$$\varphi : (G/H)^i \to M^H$$

representing a cohomology class α in the domain of (2-2), you can first

- restrict it to $(G/N)^i$ via the canonical surjection

$$(G/N)^i \to (G/H)^i$$

- corestrict it to M^N via the canonical inclusion

$$M^H = (M^N)^{H/N} \subset M^N.$$

This will give you an i-cocycle $(G/N)^i \to M^N$ that represents the image of α through (2-2).

Now think of our poset $(\mathcal{P}, \le)$ as a *category* (rather than just a poset): its objects are its elements, and there's a morphism $N \to H$ (and only one such morphism) exactly when $N \le H$. With this perspective, the inflation maps $\inf_{N,H}$ from (2-2) give you a functor

$$F^i : \mathcal{P}^\circ \to \mathrm{Ab}$$

from the *opposite* poset $\mathcal{P}^\circ$ to $\mathcal{P}$ (i.e. the poset of open normal subgroups ordered by *reversed* inclusion).

As with any functor, we can now define the *colimit* (or *direct limit*) $\varinjlim_{\mathcal{P}^\circ} F^i$. There's a general, purely category-theoretic definition, but in our context we can give a more concrete description: $\varinjlim F^i$ is obtained by

- first forming the direct sum

$$\bigoplus_{H\in\mathcal{P}} F^i(H) = \bigoplus_{H\in\mathcal{P}} H^i(G/H, M^H);$$

- then identifying the elements of that direct sum connected by the inflation morphisms (2-2). In other words, we mod out (in the direct sum above) relations of the form

$$x - \inf_{N,H}(x) \in H^i(G/H, M^H) \oplus H^i(G/N, M^N)$$

for various $x \in H^i(G/H, M^H)$ and inclusions $N \le H$ in $\mathcal{P}$.

The main punchline of the present subsection §2.1 is

Proposition 2.5 *Let G be a profinite group and i a non-negative integer. We have a natural isomorphism*

$$\varinjlim_{\mathcal{P}^\circ} F^i \cong H^i(G, -)$$

of functors ${}_G^d\mathrm{Mod} \to \mathrm{Ab}$. ■

The rest of this subsection outlines how you might go about proving this, taking the opportunity to assign some more homework along the way.

The high-level idea is to use the machinery of universal δ-functors, as discussed in [1, 2] and the references therein. You'll want to do two things:

- show that the sequence

$$(\varinjlim_{\mathcal{P}^\circ} F^i)_{i\ge 0} \tag{2-3}$$

constitutes an effaceable δ-functor ${}_G^d\mathrm{Mod} \to \mathrm{Ab}$ in the sense of [1, Definition 1.2];

- show that that effaceable δ-functor agrees with the other one, namely

$$(H^i(G, -))_{i\ge 0},$$

at $i = 0$.

Let's start with that last part, as it's easier:

Problem 3 *Prove Proposition 2.5 for $i = 0$.*

In other words, show that for a profinite group G and a discrete G-module M, you can recover M as the union of its abelian subgroups M^H as H ranges over the open normal subgroups of G.

So that leaves us having to show that (2-3) is an effaceable δ-functor. Being a δ-functor says something about long exact sequences and so on. I will not unpack this in any substantive detail, but constructing the long exact sequences hinges on the following discussion (which itself is a bit of a detour).

Definition 2.6 A partially ordered set $(\mathcal{J}, \le)$ is *filtered* or *directed* if every two elements $i, j \in \mathcal{J}$ are dominated by some third element $k \ge i, j$. ♦

Note that the opposite poset $\mathcal{P}^\circ$ is filtered: for every two open normal subgroups $H, N \trianglelefteq G$, their intersection $H \cap N$ is again open and normal. The point of introducing Definition 2.6 is

Proposition 2.7 *Let $(\mathcal{J}, \leq)$ be a filtered poset. Then, colimits over $\mathcal{J}$ are exact; in other words, the functor*

$$\varinjlim_{\mathcal{J}} : (\textit{category of functors } \mathcal{J} \to \mathrm{Ab}) \to \mathrm{Ab}$$

that sends $F : \mathcal{J} \to \mathrm{Ab}$ to $\varinjlim_{\mathcal{J}} F$ is exact. ■

It should be believable now that Proposition 2.7, applied to the filtered poset $\mathcal{P}^\circ$ of open normal subgroups $H \trianglelefteq G$ ordered by reverse-inclusion, will give rise to long exact sequences by taking colimits over $H \in \mathcal{P}^\circ$ of the long exact sequences from the "usual" cohomology of the finite groups G/H; I will not elaborate.

Now, there's also the issue of being effaceable. This means, in the present context, that for every discrete G-module M there is an embedding into a discrete G-module I with

$$\varinjlim_{\mathcal{P}^\circ} F^i(I) = \varinjlim_{H \in \mathcal{P}} H^i(G/H, I^H) = 0, \ \forall i \geq 1. \tag{2-4}$$

This is easy though, if you take for granted the assertion made above that ${}^d_G\mathrm{Mod}$ has enough injectives:

- M embeds into an injective object I in the category of discrete G-modules;
- For every open normal subgroup $H \trianglelefteq G$, the image I^H of I through the functor

  $$(-)^H : {}^d_G\mathrm{Mod} \to {}_{G/H}\mathrm{Mod}$$

  is again injective, because this functor is right adjoint to the exact "scalar restriction" functor

  $${}_{G/H}\mathrm{Mod} \ni X \mapsto X \in {}^d_G\mathrm{Mod};$$

- and hence it follows that every cohomology group $H^i(G/H, I^H)$ in (2-4) vanishes, along with the resulting colimit ranging over $H \in \mathcal{P}$.

This wraps up the problems, but some comments follow. Specifically, you might ask: why? As in, why would one care about this profinite-group business?

The reason is that this framework is the right one to handle Galois groups: If $K \subseteq L$ is a (possibly infinite!) Galois field extension, like say $\mathbb{Q} \subset \widehat{\mathbb{Q}}$, the Galois group $\mathrm{Gal}(L/K)$ of automorphisms of L fixing K pointwise is profinite: its finite quotients are the groups $\mathrm{Gal}(K'/K)$ for *finite* intermediate extensions

$$K \subseteq K' \subseteq L.$$

This observation together with the theory of profinite group cohomology sketched above in incipient form is the basis for what's known as *Galois cohomology*. I'll end this here, directing you to [6] for more (or [7] for a much shorter overview).

References

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