MTH 620: 2020-04-21 lecture

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1 Cup products realized via cocycles

We revisit the cup products introduced in the previous lecture [2]. If you recall [2, Definition 2.1], the construction was rather abstract and maybe difficult to work with directly. To mitigate that, we start with a concrete description of the cup product

$$\cup : H^p(G, M) \otimes H^q(G, N) \to H^{p+q}(G, M \otimes N)$$

at the level of cocycles.

To do that, we'll review very briefly the construction of cohomology $H^*(G, M)$ via cochains. The relevant cochain complex was

$$0 \to C^0(G, M) \to C^1(G, M) \to \cdots ,$$

where

- $C^i(G, M)$ means functions $G^i \to M$ (where that means simply M itself when $i = 0$);
- the differential $d : C^n(G, M) \to C^{n+1}(G, M)$ was defined as

$$\begin{aligned}(d\varphi)(g_1, \cdots, g_{n+1}) = g_1\varphi(g_2, \cdots, g_{n+1}) - \varphi(g_1g_2, g_3, \cdots, g_{n+1}) + \cdots \\ + (-1)^n\varphi(g_1, \cdots, g_{n-1}, g_ng_{n+1}) + (-1)^{n+1}\varphi(g_1, \cdots, g_n).\end{aligned} \tag{1-1}$$

Now let $\varphi \in C^p(G, M)$ and $\psi \in C^q(G, N)$ be two cocycles (i.e. cochains on which the differential vanishes) representing two cohomology classes

$$x \in H^p(G, M) \text{ and } y \in H^q(G, N)$$

respectively. Then, we'll take the following for granted.

Proposition 1.1 *The cohomology class $x \cup y \in H^{p+q}(G, M \otimes N)$ is represented by the $(p+q)$-cocycle*

$$\varphi \cup \psi \in C^{p+q}(G, M \otimes N)$$

defined by

$$(g_1, \cdots, g_p, h_1, \cdots, h_q) \mapsto (-1)^{pq}\varphi(g_1, \cdots, g_p) \otimes g_1 \cdots g_p\psi(h_1, \cdots, h_q),$$

where in the rightmost tensorand the product $g_1 \cdots g_p \in G$ acts on the element $\psi(h_1, \cdots, h_q) \in N$. ■

[1, §V.3] is another good source for cup products; there, they're constructed exactly as in Proposition 1.1. This is perhaps a good opportunity for a first problem.

Problem 1 *Let $H \leq G$ be a subgroup and $A \in {}_G\mathrm{Mod}$ a G-algebra in the sense of [2, Definition 2.2].*

(a) Show that the restriction map

$$\mathrm{res} : H^*(G, A) \to H^*(H, A)$$

is a a morphism of unital rings (i.e. it respects cup products and units).

(b) If H has finite index in G, is the corestriction map

$$\mathrm{cor} : H^*(H, A) \to H^*(G, A)$$

a morphism of unital rings?

As a hint for (b), you know something (from [3], say) about the composition

$$\mathrm{cor} \circ \mathrm{res} : H^*(G, A) \to H^*(G, A).$$

2 Remarks on finite cyclic groups

Throughout this section, let $G = \mathbb{Z}/n$ be a finite cyclic group for some positive integer $n \geq 2$. I'll remind you that we computed the cohomology $H^*(G)$ in class:

$$H^n(G) = H^n(G, \mathbb{Z}) = \begin{cases} \mathbb{Z} & \text{if } n = 0 \\ \mathbb{Z}/n & \text{for even } n > 0 \\ 0 & \text{otherwise} \end{cases}$$

We'll take the following result for granted (for now, at least); there are many approaches to proving it, including, for instance, [1, Section V.4, Exercise 2].

Proposition 2.1 *Let $u \in H^2(G) \cong \mathbb{Z}/n$ be a generator. For every $n \geq 2$, the multiplication map*

$$u \cdot : H^n(G) \to H^{n+2}(G)$$

is an isomorphism. ■

As a consequence, we have

Corollary 2.2 *For $G = \mathbb{Z}/n$ the cohomology ring $H^*(G)$ is isomorphic to the graded commutative ring*

$$\mathbb{Z}[x]/(nx) \cong \mathbb{Z} \oplus \mathbb{Z}/n \oplus \mathbb{Z}/n \oplus \cdots ,$$

where x has degree 2.

I'll now propose [1, Section V.4, Problem 4] as an excuse for us to go over some auxiliary material of independent interest. You don't need to go look it up, since I am expanding on the statement here (see the following discussion and finally the statement, in Problem 2 below).

Remember that we've taken our G to be the additive group $\mathbb{Z}/n$. The latter is in fact a ring, so multiplication by a positive integer m is an endomorphism $\alpha = \alpha_m$ of G (I'll usually suppress the 'm' subscript).

Now, moving briefly to a more general picture, consider a group Γ (to keep it separate from G) and an endomorphism $\alpha : G \to G$ (so just an endomorphism of groups). Then, α induces endomorphisms

$$\alpha^* = \alpha^{*n} : H^n(\Gamma) \to H^n(\Gamma) \tag{2-1}$$

for all n via the material on restriction in [4, §2.1]: (2-1) is simply the restriction morphism

$$\mathrm{res} : H^n(\Gamma, \mathbb{Z}) \to H^n(\Gamma, \mathbb{Z})$$

induced by the group morphism $\alpha : G \to G$. In [4] we have the even more general setup of a group morphism $H \to G$; there is no reason why H couldn't itself be G. Furthermore, by Problem 1 the α^{*n} for varying n assemble together into a (graded, unital) *ring* morphism

$$\alpha^* : H^*(\Gamma) \to H^*(\Gamma).$$

Applying this to our G and to our endomorphism $\alpha = \alpha_m$ (given by multiplication by m), we obtain an endomorphism α_m^* of the ring $\mathbb{Z}[x]/(nx)$ from Corollary 2.2. Our problem asks:

Problem 2 *Describe the endomorphism α_m^* of $\mathbb{Z}[x]/(nx) \cong H^*(\mathbb{Z}/n)$ explicitly.*

Some remarks and supplementary material follow, some of it taking us on detours (and suggesting one more problem).

First, since the ring $H^*(\mathbb{Z}/n)$ in Problem 2 is generated (as a ring) by x, it's enough, really, to determine what α^* does to x itself, i.e. to the generator of $H^2(G) = H^2(\mathbb{Z}/n) \cong \mathbb{Z}/n$. The following section might come in handy in your attempts to work with this cohomology group $H^2(G)$ directly.

3 Torsion-free modules and connecting maps in cohomology

The notation $G = \mathbb{Z}/n$ is still in effect.

In order to get a more concrete description of $H^2(G)$, we'll do the following. First, consider the exact sequence

$$0 \to \mathbb{Z} \to \mathbb{Q} \to \mathbb{Q}/\mathbb{Z} \to 0$$

of abelian groups equipped with trivial G-actions (so that it's now an exact sequence in ${}_G\mathrm{Mod}$). It gives rise to a long exact cohomology sequence, a fragment of which looks like this:

$$\cdots \to H^1(G, \mathbb{Q}) \to H^1(G, \mathbb{Q}/\mathbb{Z}) \to H^2(G, \mathbb{Z}) \to H^2(G, \mathbb{Q}) \to \cdots \tag{3-1}$$

The first item on the agenda is to note that the two extreme terms vanish:

Problem 3 *Let Γ be a finite group. Then, if $\mathbb{Q}$ is equipped with the trivial Γ-action, we have $H^i(\Gamma, \mathbb{Q}) = 0$ for all $i \geq 1$.*

As a hint, you know from [4, Corollary 2.5] something about what the map

$$(\text{multiplication by } |\Gamma|) : H^i(\Gamma, \mathbb{Q}) \to H^i(\Gamma, \mathbb{Q})$$

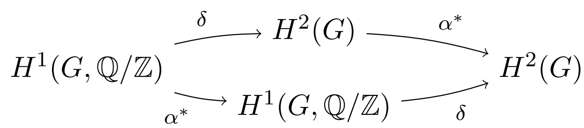
looks like. On the other hand though, that map must be an isomorphism because multiplication by $|\Gamma|$ is an isomorphism of $(\mathbb{Q}, +)$. Try to conclude from this.

Anyway, once Problem 3 is in place, you'll know that the middle map in (3-1) is an isomorphism:

$$\delta : H^1(G, \mathbb{Q}/\mathbb{Z}) \cong H^2(G, \mathbb{Z}). \tag{3-2}$$

The "connecting map" from the title of the present section is precisely (3-2), so called because in the long exact cohomology sequence (3-1) it connects lower cohomology to higher cohomology (like in [5, Definition, p.333] for instance). Now, my proposal for how to use all of this to determine what α^* does to $H^2(G) \cong \mathbb{Z}/n$ is as follows:

- transport the problem to $H^1(G, \mathbb{Q}/\mathbb{Z})$ via (3-2). Feel free to use the fact that restriction morphisms (like α^*) are compatible with connecting morphisms, in the sense that



  commutes. This is essentially the naturality of connecting maps, as in [5, Theorem 6.13], say.

- Use the fact that, as observed in class, the fact that $\mathbb{Q}/\mathbb{Z}$ is equipped with the *trivial* G-action means that

$$H^1(G, \mathbb{Q}/\mathbb{Z}) \cong \mathrm{Hom}(G, \mathbb{Q}/\mathbb{Z}) \cong \mathbb{Z}/n,$$

  where 'Hom' means morphisms of abelian groups.

This should make it all fairly explicit.

3.1 More on connecting maps

I will take the opportunity now to discuss what connecting maps

$$\delta : H^n(\Gamma, Z) \to H^{n+1}(\Gamma, X) \tag{3-3}$$

look like concretely, at cocycle level.

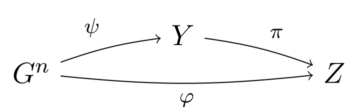
So consider an exact sequence

$$0 \to X \longrightarrow Y \xrightarrow{\pi} Z \to 0$$

of Γ-modules for a group Γ, and let

$$\varphi : G^n \to Z$$

be an n-cocycle representing a cohomology class in $H^n(\Gamma, Z)$. Now, because $Y \to Z$ is a surjection, we can lift φ to a map $\psi : G^n \to Y$:



Now, in general, ψ is just a map; it need *not* be a cocycle! It's only a cocycle once you push it back down to Z by composing with π. In other "words", if

$$\partial : C^{n+1}(\Gamma, Y) \to C^{n+1}(\Gamma, Y)$$

is the usual differential for cochains (as in (1-1), say), then

$$C^{n+1}(\Gamma, Y) \ni \partial\psi \neq 0 \text{ in general },$$

but

$$\partial\varphi = \partial(\pi \circ \psi) = 0.$$

This latter equation implies that in fact

$$\partial\psi : \Gamma^{n+1} \to Y$$

takes values in X (because it vanishes once you pass to $Z \cong Y/X$). So in fact, we can regard $\partial\psi$ as an element of $C^{n+1}(\Gamma, X)$. Because ∂^2 vanishes we have

$$\partial(\partial\psi) = 0,$$

so in fact $\partial\psi \in C^{n+1}(\Gamma, X)$ is a *cocycle* (not just a cochain), and hence represents a cohomology class in $H^{n+1}(\Gamma, X)$. That class is nothing but the image of (the class of) φ through the connecting map (3-3). To summarize:

- consider a cohomology class $x \in H^n(\Gamma, Z)$, represented by a cocycle $\varphi : \Gamma^n \to Z$;
- lift φ to a map $\psi : \Gamma^n \to Y$;
- $\partial\psi : \Gamma^{n+1} \to Y$ in fact takes values in X, and is a cocycle;
- the cohomology class $y \in H^{n+1}(\Gamma, X)$ represented by $\partial\psi$ is precisely δx, where

  $$\delta : H^n(\Gamma, Z) \to H^{n+1}(\Gamma, X)$$

  is the connecting map.
- in particular, it can be shown that the cohomology class y does not depend on the lift $\psi : \Gamma^n \to Y$ of $\varphi : \Gamma^n \to Z$, so this is a well-defined map on cohomology.

We can now apply this to the particular situation considered above, where

- $G = \mathbb{Z}/n$;
- the exact sequence is

  $$0 \to \mathbb{Z} \to \mathbb{Q} \to \mathbb{Q}/\mathbb{Z} \to 0$$

  with trivial G-actions.

We then saw above that the connecting map

$$\delta : \mathrm{Hom}(G, \mathbb{Q}/\mathbb{Z}) \cong H^1(G, \mathbb{Q}/\mathbb{Z}) \to H^2(G)$$

is an isomorphism. We can now describe explicitly the image through this isomorphism of a generator

$$\chi : G \to \mathbb{Q}/\mathbb{Z}$$

of the group $\mathrm{Hom}(G, \mathbb{Q}/\mathbb{Z}) \cong \mathbb{Z}/n$. We can choose χ so that it sends a generator σ of

$$G = \mathbb{Z}/n$$

to the image of $\frac{1}{n} \in \mathbb{Q}$ in $\mathbb{Q}/\mathbb{Z}$:

$$\chi(\sigma^m) = \text{ image of } \frac{m}{n} \quad \text{for} \quad m = 0, 1, \cdots, n-1.$$

Following the recipe outlined above, we can now lift this to a map $\psi : G \to \mathbb{Q}$ described by the same formula:

$$\psi(\sigma^m) = \frac{m}{n} \in \mathbb{Q} \quad \text{for} \quad m = 0, 1, \cdots, n-1. \tag{3-4}$$

The preceding discussion now tells us that the cohomology class in $H^2(G)$ of $\delta\chi$ is represented by the cocycle

$$\partial\psi : G \times G \to \mathbb{Z}.$$

In turn, the very definition of the differential ∂ tells us that this is

$$\partial\psi(a, b) = \psi(b) - \psi(ab) + \psi(a),\ \ a, b \in G,$$

where we're using multiplicative notation for G (to be consistent with the notation σ^m in (3-4)). Explicitly, then, this says that for $m, m' = 0, 1, \cdots, n-1$ we have

$$\partial\psi(\sigma^m, \sigma^{m'}) = \begin{cases} 1 & \text{if } m + m' \geq n \\ 0 & \text{otherwise.} \end{cases}$$

This is a concrete 2-cocycle representing a generator of $H^2(G) \cong \mathbb{Z}/n$.

References

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