MTH 620: 2020-03-24 lecture

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1 δ-functors

We went over the topic briefly in class (Th, Mar 12 2020) right before Spring break. Recall:

Definition 1.1 A *(cohomological) δ-functor* $(F_i)_i : {}_R\mathrm{Mod} \to \mathrm{Ab}$ is a sequence of (additive, as always) functors $F_i : {}_R\mathrm{Mod} \to \mathrm{Ab}$, $i \in \mathbb{Z}_{\ge 0}$ which, for each short exact sequence

$$\mathcal{S} \quad = \quad 0 \to X \to Y \to Z \to 0$$

in the domain category ${}_R\mathrm{Mod}$, give rise to long exact sequences

$$\cdots \to F_n(X) \to F_n(Y) \to F_n(Z) \to F_{n+1}(X) \to F_{n+1}(Y) \to \cdots,$$

functorial in $\mathcal{S}$. ♦

See [2, Definition, p.359] for the analogous homological notion (or simply reverse arrows). [1, Chapter III, Section 1] is also an excellent (and, more importantly, short!) source for the material on δ-functors of interest to us.

Definition 1.2 A δ-functor (F_i) is *universal* if for every δ functor $(F'_i)_i$, every natural transformation $F_0 \to F'_0$ extends uniquely to a morphism $F_i \to F'_i$ of δ-functors in the sense of [2, Definition, p.359].

$(F_i)_i$ is *effaceable* if, for every object $X \in {}_R\mathrm{Mod}$ and every $i \ge 1$ there is an embedding $X \to I$ with $F_i(I) = 0$ for all $i \ge 1$.

Once more, there are homological versions of all of this (where you would require epimorphisms $P \to X$ rather than monomorphisms $X \to I$, etc.). ♦

See also [2, pp.358-359] (again, for the dual, homological version). Note that what Rotman calls *effaceable* there Hartshorne calls *coeffaceable* in [1, Definition, p.306]. Don't sweat it too much; it'll be clear from context whether you mean the homological or cohomological version, so I will use the single word 'effaceable'.

Effaceability is relevant for the following reason:

Theorem 1.3 *Effaceable (co)homological δ-functors are universal.* ■

In particular, let $F : {}_R\mathrm{Mod} \to \mathrm{Ab}$ be a left exact functor. We know that

- the derived functors $(R^iF)_{i\ge 0}$ constitute a cohomological δ-functor (indeed, Definition 1.1 is meant to abstract the long-exact-sequence construction for derived functors);
- every object X embeds into an injective;
- $R^iF(I) = 0$ for every $i \ge 1$ and every injective object I.

All in all, we have

Corollary 1.4 *For every left exact* $F : {}_R\mathrm{Mod} \to \mathrm{Ab}$*, the sequence* $(R^iF)_i$ *of corresponding derived functors constitutes an effaceable and hence universal cohomological* δ*-functor.*

The same goes for right exact functors and their left derived functors: in that case $(L_iF)_i$ *is an effaceable and hence universal homological* δ*-functor.* ■

2 Change of groups in (co)homology

We will consider group morphisms $H \to G$ and the sorts of structure they induce on group (co)homology. The reference for this (and whatever other related material future lectures might cover) is [2, §9.5, 9.6].

To fix ideas, and because the preceding section is biased towards cohomology, we focus on the cohomological setup.

2.1 Restriction and inflation (cohomology)

Let $H \to G$ be a group morphism. As noted in class, it induces a ring morphism $\mathbb{Z}H \to \mathbb{Z}G$ which in turn gives rise to a "scalar restriction" functor

$$ {}_G\mathrm{Mod} \to {}_H\mathrm{Mod}. $$

I will typically omit naming the functor explicitly, relying on context to distinguish between $M \in {}_G\mathrm{Mod}$ regarded as a G-module vs and H-module.

Now, for any G-module M, an element $m \in M^G$ (i.e. G-invariant) will also be H-invariant. This means that we have we have a natural transformation

$$ H^0(G,-) = (-)^G \to (-)^H = H^0(H,-) \tag{2-1} $$

between functors ${}_G\mathrm{Mod} \to \mathrm{Ab}$. Now, since $H^i(G,-)$ and $H^i(H,-)$ constitute cohomological δ-functors ${}_G\mathrm{Mod} \to \mathrm{Ab}$, the universality of the δ-functor $(H^i(G,-))_i$ (Corollary 1.4) ensures that the natural transformation (2-1) extends uniquely to a morphism of δ-functors. This justifies

Definition 2.1 Let $H \to G$ be a morphism of groups. We then have *restriction morphisms*

$$ \mathrm{res} : H^i(G,-) \to H^i(H,-) \tag{2-2} $$

of functors ${}_G\mathrm{Mod} \to \mathrm{Ab}$ that make up the δ-functor morphism uniquely extending (2-1). ♦

It can be shown that at the level of cocycles, restriction literally is restriction, as discussed in class: if $f : H \to G$ denotes the group morphism and $\psi : G^i \to M$ is an i-cocycle representing a cohomology class, then the image of that cohomology class through

$$ \mathrm{res} : H^i(G,M) \to H^i(H,M) $$

is represented by the cocycle

$$ \psi \circ f^i : H^i \to M, \quad (h_1, \cdots, h_i) \mapsto \psi(f(h_1), \cdots, f(h_i)) \in M. $$

Often, people call these natural transformations (2-2) 'restrictions' only when $H \to G$ is an embedding, but I will use the term more generally.

When $H \to G$ is *sur*jective, another phrase you'll see in the literature is *inflation*: in that case $G \cong H/N$ for some normal subgroup $N \trianglelefteq H$, and the inflation morphism is simply (2-2) adapted to this setting:

$$ \inf : H^i(H/N, M) \to H^i(H,M) $$

(see [2, §9.5.1]).

2.2 Corestriction (cohomology)

When $H \le G$ is a *finite-index* subgroup though, there's a "wrong-way" natural transformation too, called (unsurprisingly?) 'corestriction'. We need to elaborate a bit.

Throughout the remainder of this section, assume $H \le G$ is a subgroup with finite index d, say, and that s_i, $1 \le i \le d$ are representatives for the left cosets G/H. Now consider a G-module M (*not* an H-module!).

Problem 1 *Show that*

$$M^H \ni m \mapsto \sum_{i=1}^{d} s_i m$$

is an abelian group morphism from M^H to M^G (you pretty much just have to show the image is contained in M^G).

Clearly, the morphism $M^H \to M^G$ in Problem 1 (once you show is a morphism) will be functorial in $M \in {}_G\mathrm{Mod}$. This then gives you a natural transformation

$$\mathrm{cor} : H^0(H, -) \to H^0(G, -) \tag{2-3}$$

(*corestriction*, as the notation suggests) of functors ${}_G\mathrm{Mod} \to \mathrm{Ab}$. This would extend uniquely to a morphism of δ-functors

$$H^i(H, -) \to H^i(G, -) \tag{2-4}$$

(by Theorem 1.3) *if* we knew that the δ-functor $H^i(H, -)$ on ${}_G\mathrm{Mod}$ is effaceable.

Remark 2.2 Careful of the slight subtlety: we already know from Corollary 1.4 that

$$H^i(H, -) : {}_H\mathrm{Mod} \to \mathrm{Ab}$$

constitute an effaceable δ-functor. The problem here is different: we want these to make up an effaceable δ-functor on ${}_G\mathrm{Mod}$ instead! ♦

The remaining few problems will guide you through the effaceability of the δ-functor

$$H^i(H, -) : {}_G\mathrm{Mod} \to \mathrm{Ab}.$$

Taking that for granted for now, we have

Definition 2.3 Let $H \le G$ be a finite-index subgroup (of a possibly-infinite group G). For G-modules M, the *corestriction* morphisms

$$\mathrm{cor} : H^i(H, M) \to H^i(G, M)$$

are the components of the unique δ-functor morphism extending the natural transformation

$$H^0(H, -) \to H^0(G, -)$$

from Problem 1. ♦

2.3 An application

Suppose $H \le G$ is a finite-index subgroup, as in §2.2. We now have natural transformations

$$H^i(H,-) \underset{\mathrm{res}}{\overset{\mathrm{cores}}{\rightleftarrows}} H^i(G,-)$$

of δ-functors ${}_G\mathrm{Mod} \to \mathrm{Ab}$. it will be of some interest to see what happens if we compose them. Specifically, I want to see what

$$\mathrm{cores} \circ \mathrm{res} : H^i(G,-) \to H^i(G,-)$$

does. Well, we know what it does at $i = 0$: it is immediate, from the very definition(s) of the two natural transformations at level 0, that

$$\mathrm{cores} \circ \mathrm{res} : H^0(G,-) \to H^0(G,-)$$

is nothing but multiplication by the index $[G : H]$ (check this!). But now we're done in general: by the effaceability of $H^i(G,-)$ (which is still relegated to the problems attached below), $\mathrm{cores} \circ \mathrm{res}$ extends *uniquely* from degree 0 to a δ-functor morphism. Since clearly, multiplication by $[G : H]$ (on every $H^i(G,-)$) is such an extension, that must be it:

Theorem 2.4 *Let $H \le G$ be a finite-index subgroup. Then, for every G-module M, the composition*

$$\mathrm{cores} \circ \mathrm{res} : H^i(G,M) \to H^i(G,M)$$

is multiplication by the index $[G : H]$. ■

In particular, consider the case where G itself is finite and H is trivial.

Corollary 2.5 *For any finite group G, the higher cohomology groups $H^i(G,-)$, $i \ge 1$ are annihilated by the order $|G|$.*

Proof For every $i \ge 1$, $\mathrm{cores} \circ \mathrm{res}$ factors through the trivial abelian groups $H^i(H,-)$, and hence $\mathrm{cores} \circ \mathrm{res}$ vanishes. But according to Theorem 2.4, it is also multiplication by $[G : H] = |G|$. In conclusion, multiplication by $|G|$ annihilates every $H^i(G,-)$, $i \ge 1$. ■

2.4 Leftover problems

This section guides you through the claim (made and taken for granted in §2.2 above).

Problem 2 *Let $R \to S$ be a ring morphism, making S into a flat right R-module. Show that injective left S-modules are also injective when regarded as left R-modules.*

As a hint, consider the adjunction

$${}_R\mathrm{Mod} \underset{\text{scalar restriction}}{\overset{R\otimes_S -}{\rightleftarrows}} {}_S\mathrm{Mod} \qquad (\perp)$$

You're trying to show that scalar restriction, which is the *right* adjoint in this adjunction, preserves injectivity. Try to show that this follows from the fact that its *left* adjoint $R \otimes_S -$ is exact (and the definition of injectivity).

Finally, we can apply this to groups.

Problem 3 *Let $H \leq G$ be an inclusion of groups.*

(a) Show that injective G-modules are also injective over H.

(b) Conclude that the sequence of functors $H^i(H,-) : {}_G\mathrm{Mod} \to \mathrm{Ab}$, $i \in \mathbb{Z}_{\geq 0}$ is an effaceable δ-functor. In other words, every G-module X embeds into a G-module I whose higher cohomology groups $H^i(H, I)$ over H vanish.

References

[1] Robin Hartshorne. *Algebraic geometry.* Springer-Verlag, New York-Heidelberg, 1977. Graduate Texts in Mathematics, No. 52.

[2] Joseph J. Rotman. *An introduction to homological algebra.* Universitext. Springer, New York, second edition, 2009.

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