MTH 620 Due Thursday, Feb 6
Spring 2020

Homework 1

I'll write ${}_R\text{Mod}$ for the category of left R-modules and MOd_R for the category of right modules, and simply Vect for the category of vector spaces when the field is understood.

Recall:

Definition 1. A ring R is *semisimple* if any of the following equivalent conditions holds:

- All R-modules are projective.
- All R-modules are injective.
- All R-module surjections split.
- All R-module injections split.
- All short exact sequences of R-modules split.

♦

It turns out that the concept is left-right symmetric, i.e. it doesn't matter if in the definition we use left or right modules. Furthermore, according to Proposition 4.5 in our textbook (which you are free to use) the property is also equivalent to any of the following:

- All (left or right) modules are *semisimple*, i.e. direct sums of *simple* modules, where

Definition 2. An R-module M is *simple* (or *irreducible*) if its only submodules are $\{0\}$ and M. ♦

If you need to remind yourselves of properties of modules covered in 619, such as being projective or injective or flat, Chapter 3 in Rotman's book is a good source for that. In particular:

Definition 3. A right R-module P is *flat* if the functor $P \otimes_R -$ from left R-modules to abelian groups is *exact*, i.e. for every short exact sequence

$$0 \to X \to Y \to Z \to 0 \tag{1}$$

of left R-modules the sequence

$$0 \to P \otimes_R X \to P \otimes_R Y \to P \otimes_R Z \to 0$$

is again exact.

Projective modules are automatically flat (see Proposition 3.46 in Rotman). ♦

The following problem guides you through one possible proof of a claim made in class, that matrix rings over fields are semisimple.

Problem 1. *Let F be a field, $n \geq 1$ a positive integer, and $R = M_n(F)$ the ring of $n \times n$ matrices with entries in F.*

(a) Let P be the right R-module of length-n row vectors on which matrices act by right multiplication. Show that the functor $F := P \otimes_R -$ defined by

$${}_R\text{Mod} \quad \ni \quad M \mapsto P \otimes_R M \quad \in \quad \text{Vect} = {}_F\text{Mod}$$

is an equivalence between ${}_R\text{Mod}$ and Vect *(vector spaces over F).*

(**Hint**: *If* Q *denotes the vector space of length-*n **column** *vectors instead, on which the ring* R *acts by* **left** *multiplication, show that the functor* $G := Q \otimes_F -$ *defined by*

$$\mathrm{Vect} \quad \ni \quad V \mapsto Q \otimes_F V \quad \in \quad {}_R\mathrm{Mod}$$

is inverse to F *in the sense that both* $F \circ G$ *and* $G \circ F$ *are functors naturally isomorphic to the identity.)*

(b) Show that P *is a projective right* R*-module, meaning, according to Definition 3, that it's also flat. Conclude that* F *is an exact functor.*

(c) Conclude that R *is semisimple.*

(**Hint**: *Start with a short exact sequence (1) in* ${}_R\mathrm{Mod}$*, turn it into an exact sequence of vector spaces by applying* F *and using part (b), split that exact sequence in* Vect*, and then pull back the splitting to the equivalent category* ${}_R\mathrm{Mod}$ *with the inverse functor* G*).*

The sketch above is one possible approach, but feel free to prove the claim that R is semisimple in some other fashion if you like; I'll still count it as a valid solution if the proof achieves that goal. For instance, another possibility would be to try to show directly that

- Every right R-module is a direct sum of copies of P;
- P is projective and hence every module, being a direct sum of projective modules, is projective.

Keeping to the same topic:

Problem 2. *Prove that the ring* $\mathbb{Z}$ *is* **not** *semisimple.*