MTH 620 Due Wednesday, Apr 24
Spring 2019

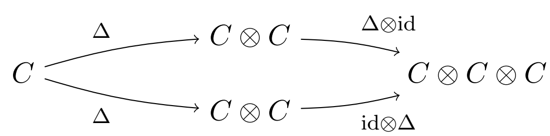
Homework 8

For the first problem we will be working over a commutative (and unital, as always) ring k; all objects pertaining to that problem are k-modules, '$\otimes$' means '$\otimes_k$', etc. You already know what a k-algebra A. To fix notation, I'll write $\mu : A \otimes A \to A$ and $\eta : k \to A$ for its multiplication and unit. Dually, we discussed these objects in class:

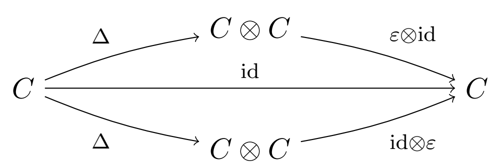
Definition 1. A(*coassociative, counital*) *k-coalgebra* is a k-module C equipped with morphisms

$$\Delta : C \to C \otimes C,\ \varepsilon : C \to k$$

called the *comultiplication* and *counit* respectively making the following diagrams commutative:



(encoding *coassociativity*) and



(meaning C is *counital*). ♦

The following observation came up in passing in class.

Problem 1. *Let B be a k-module equipped both with an algebra structure (μ, η) and a coalgebra structure (Δ, ε). Prove that the following conditions are equivalent:*

(a) Δ and ε are morphisms of unital algebras.
(b) μ and η are morphisms of counital coalgebras.

I have not defined morphisms of counital coalgebras formally, but I take it for granted you can fill in the blanks (just reverse the diagrams defining morphisms of unital algebras). Additionally, when I say that

$$\mu : B \otimes B \to B$$

is a morphism of coalgebras I am implicitly saying that there's an obvious coalgebra structure on $B \otimes B$; this is simply dual to the remark that an algebra structure on B gives you one on $B \otimes B$.

For completeness:

Definition 2. B equipped both with an algebra and a coalgebra structure satisfying the equivalent conditions of Problem 1 is called a *bialgebra*. ♦

The next few problems will have us go on a bit of a detour: we'll be discussing modules over *topological* groups, i.e. groups Γ equipped with a topology such that the multiplication $\Gamma \times \Gamma \to \Gamma$ and the inverse $-^{-1} : \Gamma \to \Gamma$ are continuous.

Definition 3. Let Γ be a topological group. A *discrete (left) Γ-module* is a Γ-module M with the property that the action structure map

$$\Gamma \times M \to M$$

is continuous when we equip M with the discrete topology.

Discrete Γ-modules form a category in the obvious fashion; we denote it by ${}^d_\Gamma\mathrm{Mod}$. ♦

Let's just take it for granted that ${}^d_\Gamma\mathrm{Mod}$ is an abelian category with the obvious abelian structure (kernels, cokernels, etc. are computed at the level of abelian groups). Just as for usual modules, there is a fixed-point functor

$${}^d_\Gamma\mathrm{Mod} \ni M \mapsto M^\Gamma \in \mathrm{Ab}.$$

to the category of abelian groups.

Problem 2. *Let Γ be a topological group. Show that the invariant functor $(-)^\Gamma : {}^d_\Gamma\mathrm{Mod} \to \mathrm{Ab}$ is a right adjoint.*

In particular, the functor is left exact. I now want to apply all of our derived functor machinery, but we know that in order for us to even get started I need enough injectives (since we're dealing with a *left* exact functor). We could give some ad-hoc proof that ${}^d_\Gamma\mathrm{Mod}$ has enough injectives, but this will be a good excuse to take a detour.

I'll be referring to [2], which is a good reference for abstract abelian category theory. We'll first need the following

Definition 4. A partially ordered set $(\mathcal{P}, \leq)$ is *filtered* or *directed* if every two elements $i, j \in \mathcal{P}$ are dominated by some third element $k \geq i, j$. ♦

See e.g. [1, §XI.1]. We'll think of posets $(\mathcal{P}, \leq)$ as categories with object set $\mathcal{P}$ and exactly one arrow $i \to j$ whenever $i \leq j$. In that context you can talk about a *filtered system or diagram* in a category $\mathcal{C}$, meaning a functor

$$F : \mathcal{P} \to \mathcal{C}$$

from a filtered poset $(\mathcal{P}, \leq)$. Similarly, we can talk about filtered colimits (meaning colimits of filtered diagrams), etc.

The following concept is pervasive in abelian category theory. See for instance [2, Theorem 8.6 in §2.8 and the discussion immediately preceding Corollary 8.9].

Definition 5. Let $\mathcal{C}$ be an abelian category. We say that it satisfies *Grothendieck's condition Ab5* (or that *it is Ab5*, for short) if the following conditions are satisfied:

- $\mathcal{C}$ is cocontinuous.
- for every object $c \in \mathcal{C}$, subobject $x \leq c$ and filtered system of subobjects $x_i \leq c$ the canonical map
$$\varinjlim_i (x \cap x_i) \to x \cap \varinjlim_i x_i$$
is an isomorphism.

♦

Finally,

Definition 6. An abelian category is *Grothendieck* if it is Ab5 and has a generator. ♦

We discussed generators in Hw3; looks like they're back in the picture. What does all of that buy you? Everything you wanted, it turns out! (see [2, Theorem 10.10 in §3.10])

Theorem 7. *A Grothendieck category has enough injectives.*

Checking that ${}_{\Gamma}^{d}\mathrm{Mod}$ is Ab5 is not very inspiring: module categories over rings are Ab5 and in particular Ab is, and subobjects, intersections, exact sequences, etc. in ${}_{\Gamma}^{d}\mathrm{Mod}$ are computed as in Ab. So you can go ahead and take it for granted that we have Ab5. The last ingredient might be slightly more interesting as it requires you cook something up on your own:

Problem 3. *Let Γ be a discrete group. Show that its abelian category of discrete modules has a generator.*

Hence it is Grothendieck and has enough injectives by Theorem 7. So the derived functors

$$R^i(-)^\Gamma : {}_{\Gamma}^{d}\mathrm{Mod} \to \mathrm{Ab}$$

are legal.

Now let Γ be a group and $(\mathcal{P}_\Gamma, \le)$ the filtered poset of finite-index normal subgroups of Γ ordered by *reverse* inclusion (so there's an arrow $L \to K$ whenever $K \le L$). We have a functor $Q : \mathcal{P}_\Gamma \to \mathrm{Grp}$ (category of groups) given by

$$Q : \mathcal{P}_\Gamma \ni K \mapsto \Gamma/K \in \mathrm{Grp}. \tag{1}$$

Grp being complete, Q has a limit. The following notions came up before, in 619, albeit phrased slightly differently.

Definition 8. Let Γ be a group. Its *profinite completion* $\widehat{\Gamma}$ is the limit of the functor Q from (1).

Γ is *profinite* if the canonical map

$$\Gamma \to \widehat{\Gamma}$$

is a group isomorphism.

We regard profinite groups as topological, with the inverse limit topology on

$$\Gamma \cong \widehat{\Gamma} = \varprojlim \Gamma/K$$

(K ranging over finite-index normal subgroups, i.e. over $\mathcal{P}_\Gamma$): this is the smallest topology that will make all maps $\Gamma \to \Gamma/K$ continuous when the finite group Γ/K is given the discrete topology. ♦

Now let Γ be a profinite group and $K \le L \le \Gamma$ two normal finite-index subgroups. Let also M be a discrete Γ-module. We have the quotient

$$\Gamma/K \to \Gamma/L$$

and hence the *inflation* morphisms [3, p.566]

$$H^i(\Gamma/L, M^L) \to H^i(\Gamma/K, M^K).$$

This is a functor $\mathcal{P}_\Gamma \to \mathrm{Ab}$: remember that we are ordering $\mathcal{P}_\Gamma$ by reverse inclusion, so in that category there's an arrow from $L \to K$. We can now take a colimit forming

$$Q^i(\Gamma, M) := \varinjlim_{K \in \mathcal{P}_\Gamma} H^i(\Gamma/K, M^K). \tag{2}$$

Clearly, the construction of $Q^i(\Gamma, M)$ is functorial in $M \in {}^d_\Gamma\mathrm{Mod}$.

The punchline is that you've just computed the derived functors of $(-)^\Gamma$:

Problem 4. *Let Γ be a profinite group and*

$$Q^i = Q^i(\Gamma, -) : {}^d_\Gamma\mathrm{Mod} \to \mathrm{Ab}$$

the functors constructed in (2). Show that we have natural isomorphisms

$$Q^i(\Gamma, -) \cong R^i(-)^\Gamma$$

to the right derived functors of the invariant functor $(-)^\Gamma : {}^d_\Gamma\mathrm{Mod} \to \mathrm{Ab}$.

You'll want to use the abstract characterization of right derived functors as cohomological ∂-functors discussed in class. The reference for that was [3, Theorem 6.51].

This wraps up the problems, but some comments follow. Specifically, you might ask: why? As in, why would one care about this profinite-group business?

The reason is that this framework is the right one to handle Galois groups: If $K \subseteq L$ is a (possibly infinite!) Galois field extension, like say $\mathbb{Q} \subset \overline{\mathbb{Q}}$, the Galois group $\mathrm{Gal}(L/K)$ of automorphisms of L fixing K pointwise is profinite: its finite quotients are the groups $\mathrm{Gal}(K'/K)$ for *finite* intermediate extensions

$$K \subseteq K' \subseteq L.$$

This observation together with the theory of profinite group cohomology sketched above in incipient form is the basis for what's known as *Galois cohomology*. I'll end this here, directing you to [4] for more (or [5] for a much shorter overview).

References

[1] Saunders Mac Lane. *Categories for the working mathematician*, volume 5 of *Graduate Texts in Mathematics*. Springer-Verlag, New York, second edition, 1998.

[2] N. Popescu. *Abelian categories with applications to rings and modules*. Academic Press, London-New York, 1973. London Mathematical Society Monographs, No. 3.

[3] Joseph J. Rotman. *An introduction to homological algebra*. Universitext. Springer, New York, second edition, 2009.

[4] Jean-Pierre Serre. *Galois cohomology*. Springer Monographs in Mathematics. Springer-Verlag, Berlin, english edition, 2002. Translated from the French by Patrick Ion and revised by the author.

[5] John Tate. Galois cohomology. In *Arithmetic algebraic geometry (Park City, UT, 1999)*, volume 9 of *IAS/Park City Math. Ser.*, pages 465–479. Amer. Math. Soc., Providence, RI, 2001.