MTH 620 Due Wednesday, Apr 17

Spring 2019

Homework 7

Let $H \leq G$ be a finite-index inclusion of groups. We defined a natural transformation

$$\mathrm{Cor}^i_{H \to G} : H^i(H, -) \to H^i(G, -)$$

of cohomological δ-functors ${}_G\mathrm{Mod} \to \mathrm{Ab}$ as the unique extension of the natural transformation

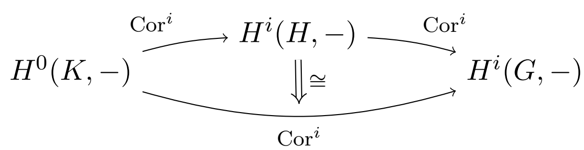
$$(-)^H = H^0(H, -) \to H^0(G, -) = (-)^G$$

defined, for a G-module M, by

$$M^H \ni m \mapsto \sum_i s_i m \in M^G$$

for a set $\{s_i\}_{i=1}^{[G:H]}$ of representatives for the left cosets G/H. The first problem is a version of [1, Problem 9.32].

Problem 1. *Let $K \leq H \leq G$ be group inclusions of finite index. Show that we have a natural isomorphism*



of cohomological δ-functors.

The analogue of $\mathrm{Cor}^i : H^i(H, -) \to H^i(G, -)$ for finite-index $H \leq G$ in homology was *restriction*:

$$\mathrm{Res}_i^{G \to H} : H_i(G, -) \to H_i(H, -).$$

For $i = 1$ we have

$$H_1(\Gamma, \mathbb{Z}) \cong \Gamma_{ab} := \Gamma/[\Gamma, \Gamma],$$

and we called the resulting morphism

$$V : G_{ab} \to H_{ab}$$

the *transfer morphism* (see [1, §9.6]). We also defined the transfer morphism directly, without appealing to homology. We'll recall that definition and give an alternative proof that it is indeed a morphism and independent of the various choices involved.

Let $n = [G : H]$ be the index. Recall that for a set $\{s_i\}$ of representatives for the left cosets G/H and every element $g \in G$ we have

$$g s_i = s_{\sigma i} h_i \tag{1}$$

for some permutation σ of $\{1, \cdots, n\}$ and some elements h_i. Then we said that V acts as

$$G_{ab} \ni \text{ class of g} \mapsto \text{ class of } \prod_i h_i \in H_{ab}.$$

The problem with this is that

- it's not immediately clear it is a definition (i.e. only depends on the class of g in G_{ab} rather than on g itself);
- it's not clear it is independent of the set $\{s_i\}$;
- it's not clear it is a morphism.

So in other words, the only thing wrong with the construction is everything. Here, we'll walk through one possible way of addressing all of the issues that bypasses homology.

Abusing notation slightly, we'll refer to the map $G \to H_{ab}$ sending g to the class of $\prod_i h_i$ by V again (the domain is now G rather than G_{ab}, so we at least know V is well defined).

Definition 1. A *character* of H is a group morphism $\chi : H \to \mathbb{S}^1$, where $\mathbb{S}^1$ denotes the multiplicative circle group of modulus-1 complex numbers.

Characters automatically factor through the abelianization $H \to H_{ab}$ (because their codomain $\mathbb{S}^1$ is abelian) and they form a group under pointwise multiplication.

If H is abelian we write $\widehat{H}$ for the *Pontryagin dual* (or just 'dual', for short) group of H. ♦

A character χ gives an action of H on $\mathbb{C}$ (with $h \in H$ acting scaling $\mathbb{C}$ by $\chi(h)$), so you can regard $\mathbb{C}$ as an H-module denoted $\mathbb{C}_\chi$.

We can then extend scalars to get a G-module

$$M_\chi := \mathbb{C}G \otimes_{\mathbb{C}H} \mathbb{C}_\chi$$

and we denote by

$$\rho_\chi : G \to Gl(M_\chi)$$

the resulting group morphism from G to the general linear group of the vector space M_χ. In other words, $\rho_\chi(g)$ is the operator g acting on the G-module M_χ.

Problem 2. *In the setup above, show that for every $\chi : H \to \mathbb{S}^1$ and every $g \in G$ we have*

$$\chi(V(g)) = \sigma(g) \det \rho_\chi(g) \tag{2}$$

where $\sigma(g)$ is the sign of the permutation implemented by left multiplication by g on the set G/H of left cosets.

(2) is kind of cool in its own right (if you like that kind of thing), but we'll actually use it to address the three bullet points in the above discussion. Specifically, note that (2) implies that for every character $\chi \in \widehat{H_{ab}}$

- $\chi(V(g))$ only depends on the class of g in G_{ab} (why?);
- $\chi(V(g))$ does not depend on the choice of coset representatives s_i;
- the map
$$G \ni g \mapsto \chi(V(g)) \in \mathbb{S}^1$$
is a morphism.

(these remarks precisely match the three bullet points listed before, in the same order).

Problem 3. *Conclude that the map $V : G \to H_{ab}$ defined by*

$$G \ni g \mapsto \text{ class of } \prod_i h_i \in H_{ab}$$

with h_i as in (1)

- *descends to a map $G_{ab} \to H_{ab}$;*
- *is independent of the coset representative set $\{s_i\}$;*
- *is a morphism $G_{ab} \to H_{ab}$.*

You already know from the discussion preceding the statement that all of these hold once you further compose V with arbitrary characters $\chi : H \to \mathbb{S}^1$. To conclude, you'll need to argue that characters *separate* the elements of H_{ab}: for every $x \neq y \in H_{ab}$ there is some character χ with $\chi(x) \neq \chi(y)$:

Problem 4. *Let H be an arbitrary abelian group. Show that the characters of H separate its elements in the sense of the preceding paragraph.*

Remark 2. A different way to phrase Problem 4 would have been: show that the canonical morphism $H \to \widehat{\widehat{H}}$ sending $h \in H$ to the morphism $\widehat{H} \to \mathbb{S}^1$ defined by

$$\chi \mapsto \chi(h)$$

is one-to-one. ♦

So I am suggesting you'll need Problem 4 to solve Problem 3.

Finally, I'd like to have at least one other quick application of transfer before moving on from the topic. First, just to make sure:

Definition 3. A (possibly non-abelian) group Γ is *torsion-free* if every $g \in \Gamma$ has infinite order. ♦

Problem 5. *Let Γ be a torsion-free group all of whose conjugacy classes are finite. Show that Γ is abelian.*

To get you started: you may as well assume that Γ is finitely generated by, say, a finite set $S \subset \Gamma$ (you'll have to argue *why* you can assume finite generation). The intersection

$$H := \bigcap_{s \in S} C_\Gamma(s)$$

of all of the centralizer subgroups

$$C_\Gamma(s) := \{g \in \Gamma \mid gsg^{-1} = s\}$$

is then a central finite-index subgroup of Γ (prove this!). You can now consider the transfer $V : G \to H$ (remember that the transfer to a *central* subgroup has a very special form).

References

[1] Joseph J. Rotman. *An introduction to homological algebra*. Universitext. Springer, New York, second edition, 2009.