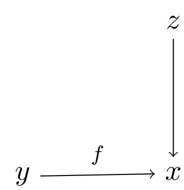
MTH 620 Due Wednesday, Apr 10
Spring 2019

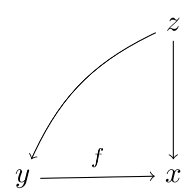
Homework 6

Recall that we have a notion of *projective object* that makes sense in any category whatsoever:

Definition 1. Let $\mathcal{C}$ be a category. An object $x \in \mathcal{C}$ is *projective* if every diagram



with epic f can be completed to a commutative diagram



♦

In particular, the concept makes sense in the category of groups (all of them, not just abelian ones). It turns out the projective objects therein are well known to you.

Problem 1. *Show that a group is projective in the category of groups if and only if it is free.*

You'll need a couple of things to prove this. First, the definition of projectivity makes essential use of epic morphisms, so we'd better understand what those are in the category of groups. This is settled in [1, Proposition 3]:

Theorem 2. *A morphism in the category of groups is epic if and only if it is surjective.*

Secondly, as you try to prove that projectivity implies freeness, you might find you can embed your projective group into a free one. This is sufficient by [2, Theorem 11.44] (aka the Nielsen-Schreier theorem):

Theorem 3. *A subgroup of a free group is free.*

Go right ahead and use those two results in your solution to Problem 1 (which I promise will tie in to cohomology over the course of this assignment).

We'll now discuss a cohomological measure of how "complicated" a group is.

Definition 4. Let Γ be a group. The *cohomological dimension* $cd(\Gamma)$ is the largest non-negative integer n such that

$$H^n(\Gamma, A) \neq 0 \text{ for some } \Gamma - \text{module } A.$$

We set $cd(\Gamma) = \infty$ if no such largest number exists. ♦

Problem 2. *Let Γ be a group and suppose $H^{n+1}(\Gamma, A) = 0$ for all Γ-modules A. Show that $cd(\Gamma) \le n$.*

You've seen many results of this flavor by now, some worked out in class and some in prior homework: the problem is asking you to show that if $H^{n+1}(\Gamma, -)$ is identically zero then so are $H^m(\Gamma, -)$ for all larger m (i.e. $m \ge n+1$).

Problem 3. *Show that a non-trivial free group has cohomological dimension 1.*

Showing that $H^1(\Gamma)$ is non-zero will not be difficult, so the crux of the matter is proving that $H^2(\Gamma, -)$ is the zero functor (this is sufficient, by Problem 2). For this, you can use the projectivity of free groups (Problem 1) and the interpretation of H^2 discussed in class, which we now recall here for convenience as Theorem 6 below.

Definition 5. Let M be a Γ-module. An *extension of Γ by M* is a short exact sequence

$$1 \longrightarrow M \longrightarrow G \xrightarrow{\ \pi\ } \Gamma \longrightarrow 1 \tag{1}$$

of (not necessarily abelian!) groups such that the induced action of Γ on M by conjugation coincides with the original action giving M its Γ-module structure.

Two extensions are *equivalent* if they fit into a commutative diagram of the form

$$\begin{array}{ccccccc} & & & & G & & & & \\ & & & \nearrow & \downarrow & \searrow & & & \\ 1 & \longrightarrow & M & & & & \Gamma & \longrightarrow & 1 \\ & & & \searrow & & \nearrow & & & \\ & & & & G' & & & & \end{array}$$

Finally, $e(\Gamma, M)$ denotes the set of equivalence classes of such extensions. ♦

We proved (or rather sketched) the following result in class (it's also [3, Theorem 9.13]):

Theorem 6. *For every group Γ and Γ-module M we have a bijection*

$$H^2(\Gamma, M) \cong e(\Gamma, M).$$

The bijection in Theorem 6 is such that the trivial element $0 \in H^2(\Gamma, M)$ corresponds to *split* extensions, i.e. those (1) for which π admits a right inverse group morphism $s : \Gamma \to G$ such that $\pi s = \mathrm{id}_\Gamma$.

Observe that cohomological dimension is non-decreasing with respect to group inclusions: if $H \le G$ then $cd(H) \le cd(G)$. This is a simple consequence of the fact that

$$H^i(H, M) \cong H^i(G, \mathrm{Ind}_G^H(M))$$

(as one of the problems in the prior homework assignment has you prove), so if the left hand side is non-zero so is the right hand side. See also [3, Proposition 9.111] for the same remark.

Using this (among other things) we'll prove

Problem 4. *$\mathbb{Z}$ is the only abelian group with cohomological dimension 1.*

You can proceed as follows:

Let Γ be an abelian group with $cd(\Gamma) = 1$. There is then a surjection $F \to \Gamma$ from some free abelian group F onto Γ. Use the vanishing of $H^2(\Gamma, -)$ to conclude that this surjection splits (i.e. has a right inverse), meaning that Γ embeds in F and is thus free abelian.

Then, assuming that as a free abelian group Γ has rank ≥ 2, it follows (why?) that

$$cd(\mathbb{Z}^2) \le cd(\Gamma). \tag{2}$$

Finally, you'll then have to prove that the left hand side in (2) is at least 2. This, in turn, is exactly what the next problem has you do. First, a definition.

Definition 7. Let $H_{\mathbb{Z}}$ be the *Heisenberg group* over the integers, i.e. the multiplicative group of integer-entry matrices of the form

$$\begin{pmatrix} 1 & x & z \\ 0 & 1 & y \\ 0 & 0 & 1 \end{pmatrix}.$$

♦

Problem 5. *Show that the integer Heisenberg group* $H_{\mathbb{Z}}$ *fits into a non-split exact sequence*

$$1 \longrightarrow \mathbb{Z} \longrightarrow H_{\mathbb{Z}} \xrightarrow{\pi} \mathbb{Z}^2 \longrightarrow 1.$$

Conclude that $H^2(\mathbb{Z}^2, \mathbb{Z}) \neq 0$ *(where* $\mathbb{Z}$ *is regarded as a* $\mathbb{Z}^2$*-module with the trivial action).*

A suggestion: define π as

$$H_{\mathbb{Z}} \ni \begin{pmatrix} 1 & x & z \\ 0 & 1 & y \\ 0 & 0 & 1 \end{pmatrix} \mapsto (x, y) \in \mathbb{Z}^2.$$

Problem 4 is a baby case of a theorem of Stallings classifying all groups of cohomological dimension one (not just abelian ones), at least in the finitely generated case:

According to [3, Corollary 9.115] a finitely generated group Γ has $cd(\Gamma) = 1$ if and only if it is free and non-trivial. The original source is [4, §6.8]; I don't know if the finite generation hypothesis can be dropped (it was certainly important for Stallings).

References

[1] G. A. Reid. Epimorphisms and surjectivity. *Invent. Math.*, 9:295–307, 1969/1970.

[2] Joseph J. Rotman. *An introduction to the theory of groups*, volume 148 of *Graduate Texts in Mathematics*. Springer-Verlag, New York, fourth edition, 1995.

[3] Joseph J. Rotman. *An introduction to homological algebra*. Universitext. Springer, New York, second edition, 2009.

[4] John R. Stallings. On torsion-free groups with infinitely many ends. *Ann. of Math. (2)*, 88:312–334, 1968.