MTH 620 Due Wednesday, Mar 27
Spring 2019

Homework 5

Let $R \to S$ be a morphism of (unital, as always) rings. Remember that in general, we have two adjunctions

$$ {}_R\mathrm{Mod} \underset{\text{restriction}}{\overset{S\otimes_R -}{\underset{\perp}{\rightleftarrows}}} {}_S\mathrm{Mod} \tag{1} $$

and

$$ {}_R\mathrm{Mod} \underset{\text{restriction}}{\overset{\mathrm{Hom}_R(S,-)}{\underset{\top}{\rightleftarrows}}} {}_S\mathrm{Mod} \tag{2} $$

where the tail of the turnstile symbol '⊣' points towards the left adjoint. We will now soup up these adjunctions to Ext functors.

Problem 1. *Let $R \to S$ be a ring morphism, as above.*

(a) Suppose S is projective as a right R-module. Then, for every $i \geq 0$ and every (left) S-module N and R-module M we have an isomorphism

$$ \mathrm{Ext}^i_R(M, N) \cong \mathrm{Ext}^i_S(S \otimes_R M, N), \tag{3} $$

natural in both $M \in {}_R\mathrm{Mod}$ and $N \in {}_S\mathrm{Mod}$, where on the left hand side we regard N as an R-module via the scalar restriction functor from (1).

(b) Similarly, suppose S is projective as a left R-module instead. Then, for every $i \geq 0$ and every (left) R-module N and S-module M we have an isomorphism

$$ \mathrm{Ext}^i_R(M, N) \cong \mathrm{Ext}^i_S(M, \mathrm{Hom}_R(S, N)), $$

natural in both $M \in {}_S\mathrm{Mod}$ and $N \in {}_R\mathrm{Mod}$, where as before, on the left hand side we regard M as an R-module via the scalar restriction functor from (2).

Some remarks on how you might approach this:

For (a), compute the left hand side of (3) using a projective resolution

$$ P_\bullet = \cdots \to P_1 \to P_0 \to M \to 0 $$

of M in ${}_R\mathrm{Mod}$ and show that

$$ S \otimes_R P_\bullet = \cdots \to S \otimes_R P_1 \to S \otimes_R P_0 \to S \otimes_R M \to 0 $$

is a projective resolution in ${}_S\mathrm{Mod}$.

For (b) argue analogously, using an *injective* resolution of N in ${}_R\mathrm{Mod}$ instead.

You'll have to argue that if $P \in {}_R\mathrm{Mod}$ is projective then so is $S \otimes_R P \in {}_S\mathrm{Mod}$ and dually, if $J \in {}_R\mathrm{Mod}$ is injective then so is $\mathrm{Hom}_R(S, J) \in {}_S\mathrm{Mod}$. The former is easy to do directly using the characterization of projectives (summands of free modules), but the following problem gives you the abstract framework for arguing both points.

Problem 2. *Let $F : \mathcal{C} \to \mathcal{D}$ be an exact functor between abelian categories, and*

$$F \dashv F_r, \quad F_l \dashv F$$

adjunctions for functors $F_l, F_r : \mathcal{D} \to \mathcal{C}$.

(a) Prove that if $P \in \mathcal{D}$ is projective then so is $F_l(P) \in \mathcal{C}$.

(b) Dually, prove that if $J \in \mathcal{D}$ is injective then so is $F_r(J) \in \mathcal{C}$.

Recall that as always, the tail of '$\dashv$' points to the left adjoint. In motto form, the problem says that:

- Functors admitting an exact right adjoint preserve projectivity.
- Functors admitting an exact left adjoint preserve injectivity.

The relevance to Problem 1 should be clear: take $F : \mathcal{C} \to \mathcal{D}$ to be the restriction functor from (1) and (2) (which is obviously exact). The left and right adjoints F_l and F_r of F will then be the functors $S \otimes_R -$ and $\mathrm{Hom}_R(S, -)$ from (1) and (2) respectively.

Now let $H \to G$ be a group inclusion.

Definition 1. Let $H \leq G$ be an inclusion of groups and M and H-module. The *induced module* $\mathrm{Ind}_H^G M$ is defined by

$$\mathrm{Ind}_H^G(M) := \{f\ G \to M \mid f(hg) = hf(g),\ \forall g \in G,\ h \in H\},$$

equipped with the G-action given by

$$gf(g') = f(gg'),\ \forall f \in \mathrm{Ind}_H^G(M),\ \forall g, g' \in G.$$

♦

Problem 3. *Let $H \leq G$ be a subgroup and M an H-module. Show that for every $i \geq 0$ we have an isomorphism*

$$H^i(H, M) \cong H^i(G, \mathrm{Ind}_H^G(M)),$$

natural in $M \in {}_H\mathrm{Mod}$.

Problem 4. *Let Γ be a finite group and $\mathbb{Q}$ the Γ-module equipped with the trivial action. Prove that $\mathbb{Q}$ is injective as a Γ-module and conclude that*

$$H^i(\Gamma, \mathbb{Q}) = 0,\ \forall i \geq 1.$$

One way to attack this would be to collate a number of ideas appearing above. For instance, note first that you have an embedding

$$\mathbb{Q} \subset \mathrm{Ind}_{\{1\}}^\Gamma(\mathbb{Q}) = \text{functions}(\Gamma \to \mathbb{Q}), \tag{4}$$

where the left hand $\mathbb{Q}$ is identified with the constant functions $\Gamma \to \mathbb{Q}$ (and of course $\{1\}$ denotes the trivial subgroup of Γ).

Now, $\mathbb{Q}$ is injective over $\mathbb{Z}$ (why?), and hence $\mathrm{Ind}_{\{1\}}^\Gamma(\mathbb{Q})$ is injective over $\mathbb{Z}\Gamma$ (by one of the previous problems). It then remains to show that the inclusion (4) splits in ${}_\Gamma\mathrm{Mod}$, and hence $\mathbb{Q}$ is a direct summand of the injective Γ-module $\mathrm{Ind}_{\{1\}}^\Gamma(\mathbb{Q})$ (so is itself injective).

Finding a splitting

$$\text{functions}(\Gamma \to \mathbb{Q}) \to \mathbb{Q}$$

of (4) (in the category of Γ-modules) is likely the hardest part of this. For this, remember that Γ is finite; that's relevant.