MTH 620 Due Wednesday, Mar 6
Spring 2019

Homework 4

Recall that in general, for

- a ring R;
- a right R-module M and
- a left R-module N

we are writing $\mathrm{Tor}_i^R(M,N)$ for the i^{th} left derived functor of either of the two functors

- $M \otimes_R -$ from ${}_R\mathrm{Mod}$ to abelian groups;
- $- \otimes_R N$ from Mod_R to Ab

applied to N and M respectively.

The first few problems have us compute some Tor groups over $\mathbb{Z}$.

Problem 1. *Show that for any two abelian groups M and N and every positive integer $i \geq 2$ we have*

$$\mathrm{Tor}_i^{\mathbb{Z}}(M,N) = 0.$$

(Hint: there was something about *hereditary categories* in the previous assignment.)

Problem 2. *Let $n \geq 2$ be a positive integer. For each non-negative integer $i \geq 0$, describe the functor*

$$\mathrm{Tor}_i^{\mathbb{Z}}(\mathbb{Z}/n, -) : \mathrm{Ab} \to \mathrm{Ab}.$$

You can do this as follows.

We know from the preceding problem that only $i = 0, 1$ are interesting. Furthermore, we saw during the Wed, Feb 20 lecture that for any abelian group M, $\mathrm{Tor}_i(\mathbb{Z}/n, M)$ can be computed from a projective resolution of either M or $\mathbb{Z}/n$. Since M is generic here, it will be profitable to use a projective resolution of $\mathbb{Z}/n$ instead. See if you can use the projective resolution

$$0 \longrightarrow \mathbb{Z} \xrightarrow{\;n\;} \mathbb{Z} \longrightarrow \mathbb{Z}/n \longrightarrow 0.$$

We next turn to group homology, which I mentioned very briefly in class. Throughout, Γ denotes an arbitrary group and $\mathbb{Z}\Gamma$ its group algebra. Γ-*modules* are modules over the ring $\mathbb{Z}\Gamma$, i.e. simply abelian groups equipped with an action of Γ by automorphisms.

Definition 1. Let M be a Γ-module. The *group of coinvariants* of M (with respect to the action by Γ) is the quotient

$$M/(\gamma m - m,\ \gamma \in \Gamma,\ m \in M).$$

In other words, the quotient by the subgroup generated by all elements of the form $\gamma m - m$.

We write M_Γ for the group of coinvariants of M. It is easy to see that

$${}_{\mathbb{Z}\Gamma}\mathrm{Mod} \ni M \mapsto M_\Gamma \in \mathrm{Ab}$$

is an additive functor. ♦

Problem 3. *Let Γ be a group, and regard $\mathbb{Z}$ as a right Γ-module with the trivial action. Show that the functor*

$$(-)_\Gamma : {}_{\mathbb{Z}\Gamma}\mathrm{Mod} \to \mathrm{Ab}$$

introduced in Definition 1 is naturally isomorphic to the functor $\mathbb{Z} \otimes_{\mathbb{Z}\Gamma} -$.

In other words, the theory of left derived functors applies in its entirety to the coinvariant functor $(-)_\Gamma$. This affords

Definition 2. Let Γ be a group and M a Γ-module. The i^th *homology of* Γ *valued in* M is

$$H_i(\Gamma, M) := \mathrm{Tor}_i^{\mathbb{Z}\Gamma}(\mathbb{Z}, M),$$

where $\mathbb{Z}$ is regarded as a trivial Γ-module.

The i^{th} *group homology of* Γ is

$$H_i(\Gamma) := H_i(\Gamma, \mathbb{Z}).$$

♦

For the following problem let $\Gamma = \mathbb{Z}/n$ for some fixed positive integer n and denote by $t \in \mathbb{Z}/n$ a generator of that group; so if we write $\mathbb{Z}/n$ multiplicatively, we have

$$\mathbb{Z}/n = \{t^i,\ 0 \le i \le n-1\}.$$

We write $R = \mathbb{Z}\Gamma$ and

$$N : R \to R$$

for the map given by multiplication by the element $1 + t + \cdots + t^{n-1} \in R$. Finally,

$$\varepsilon : R \to \mathbb{Z}$$

is the abelian group morphism that sends all $\gamma \in \Gamma$ to $1 \in \mathbb{Z}$. We will work with the following complex:

$$\cdots \xrightarrow{N} R \xrightarrow{1-t} R \xrightarrow{N} R \xrightarrow{1-t} R \xrightarrow{\varepsilon} \mathbb{Z} \longrightarrow 0 \tag{1}$$

with $1-t$ and N alternating endlessly.

The next problem has you describe the homology groups $H_i(\Gamma, M)$ for all i and all Γ-modules M using this complex.

Problem 4. *Let $n \ge 2$ be a positive integer and $\Gamma = \mathbb{Z}/n$, as above.*

(a) Show that (1) is a projective resolution of the trivial Γ-module $\mathbb{Z}$.

(b) For an arbitrary Γ-module M and $i \ge 1$, conclude that we have

$$H_i(\Gamma, M) = \begin{cases} \ker(1-t)/\mathrm{im}\, N & \text{if } i \text{ is odd} \\ \ker N/\mathrm{im}(1-t) & \text{if } i \text{ is even} \end{cases}$$

where $1-t$ and $N = 1 + \cdots + t^{n-1}$ are regarded as endomorphisms of the abelian group M.

(c) As a consequence, show that

$$H_i(\mathbb{Z}/n) = \begin{cases} \mathbb{Z}/n & \text{if } i \text{ is odd} \\ 0 & \text{if } i \text{ is even.} \end{cases}$$