MTH 620 Due Wednesday, Feb 27
Spring 2019

Homework 3

The first few problems will walk us through a few techniques for recognizing abelian categories of the form ${}_R\mathrm{Mod}$ (i.e. those equivalent to categories of modules).

Let $\iota : \mathcal{D} \to \mathcal{C}$ be a *full subcategory*. Recall what this means:

- the functor ι realizes the set $\mathrm{Ob}\,\mathcal{D}$ of objects of $\mathcal{D}$ as a subset of $\mathrm{Ob}\,\mathcal{C}$.
- hom spaces coincide:

$$\hom_{\mathcal{C}}(\iota d, \iota d') = \hom_{\mathcal{D}}(d, d'),\ \forall \text{ objects } d, d' \in \mathcal{D}.$$

Definition 1. A full subcategory $\iota : \mathcal{D} \to \mathcal{C}$ is *reflective* if ι has a left adjoint.

Dually, $\iota : \mathcal{D} \to \mathcal{C}$ is *coreflective* if ι has a right adjoint. ◆

Problem 1. *(a) Suppose $\iota : \mathcal{D} \to \mathcal{C}$ is a full reflective subcategory, with $\Psi : \mathcal{C} \to \mathcal{D}$ the left adjoint to ι.*

Let $F : \mathcal{E} \to \mathcal{D}$ be a functor, and suppose its colimit $\mathrm{colim}(\iota \circ F)$ exists. Show that

$$\Psi\left(\mathrm{colim}(\iota \circ F)\right)$$

is a colimit for F.

Conclude that if $\mathcal{C}$ is cocomplete then so is $\mathcal{D}$.

Dually, if $\iota : \mathcal{D} \to \mathcal{C}$ is coreflective and $\mathcal{C}$ is complete then so is $\mathcal{D}$.

Let Ab be the category of abelian groups and $\mathrm{Ab}_t \subset \mathrm{Ab}$ the full subcategory of torsion abelian groups.

Problem 2. *Show that the full subcategory $\iota : \mathrm{Ab}_t \to \mathrm{Ab}$ is coreflective (i.e. the inclusion functor has a right adjoint).*

Conclude from Problem 1 that Ab_t is complete and describe the limit of an arbitrary functor $F : \mathcal{E} \to \mathrm{Ab}_t$.

Problem 3. *Let R be a ring and consider a diagram*

$$\cdots \xrightarrow{\pi_{2,3}} M_2 \xrightarrow{\pi_{1,2}} M_1 \xrightarrow{\pi_{0,1}} M_0 \tag{1}$$

of modules over R. Its limit M in ${}_R\mathrm{Mod}$ comes equipped with a cone over the diagram, i.e. "projection maps" $\pi_i : M \to M_i$.

Show that if the connecting maps $\pi_{i-1,i} : M_i \to M_{i-1}$ are epimorphisms then so are π_i.

Problem 4. *Let p be a prime number and consider the diagram*

$$\cdots \xrightarrow{\pi_{2,3}} \mathbb{Z}/p^2 \xrightarrow{\pi_{1,2}} \mathbb{Z}/p \xrightarrow{\pi_{0,1}} 0$$

in the category Ab_t, where $\pi_{i-1,i} : \mathbb{Z}/p^i \to \mathbb{Z}/p^{i-1}$ are the canonical projections.

Prove using the first two problems that the limit of this diagram in Ab_t vanishes and conclude via Problem 3 that Ab_t is not equivalent to ${}_R\mathrm{Mod}$ for any ring R.

The next problem will run through yet another technique for distinguishing abelian categories in general from module categories. First, we need

Definition 2. An object $c \in \mathcal{C}$ in an abelian category is *small* if the functor

$$\hom_{\mathcal{C}}(c, -) : \mathcal{C} \to \mathrm{Ab}$$

preserves direct sums (or coproducts, if you prefer). ♦

Additionally:

Definition 3. An object $c \in \mathcal{C}$ is a *generator* if for every pair of distinct parallel morphisms

$$x \underset{g}{\overset{f}{\rightrightarrows}} y$$

in $\mathcal{C}$ there is a morphism $\pi : c \to x$ such that $\pi \circ f \neq \pi \circ g$.

In other words, we want the functor

$$\hom_{\mathcal{C}}(c, -) : \mathcal{C} \to \mathrm{Set}$$

to be *faithful*, i.e. one-to-one on hom spaces. ♦

Now fix a field F and consider the abelian category $\mathrm{Vect}^{\mathbb{Z}}$ of $\mathbb{Z}$-*graded* F-*vector spaces*: the objects are tuples

$$(V_n)_{n \in \mathbb{Z}},\ V_n \text{ a vector space}$$

and the morphisms

$$(V_n)_n \to (W_n)_n$$

are the tuples

$$(f_n)_n,\ f_n : V_n \to W_n \text{ is a linear map for each } n.$$

Problem 5. *(a) Show that for every ring R the category ${}_R\mathrm{Mod}$ has a small projective generator.*

(b) Show that an object $(V_n)_n$ of $\mathrm{Vect}^{\mathbb{Z}}$ is small if and only if

$$\sum_{n \in \mathbb{Z}} \dim V_n < \infty.$$

(c) On the other hand, show that an object $(V_n)_n$ of $\mathrm{Vect}^{\mathbb{Z}}$ is a generator if and only if

$$V_n \neq 0,\ \forall n \in \mathbb{Z}.$$

(d) Conclude that $\mathrm{Vect}^{\mathbb{Z}}$ is not equivalent to ${}_R\mathrm{Mod}$ for any ring R.

You don't have to prove this, but I mention it for edification: categories of the form ${}_R\mathrm{Mod}$ can in fact be characterized abstractly as follows:

Theorem 4. *An abelian category is of the form ${}_R\mathrm{Mod}$ if and only if it*

- *is cocomplete, and*
- *has a small projective generator.*

The next problem changes the subject somewhat; you might want to look over section 4.3 in Rotman on hereditary rings. Here, we extend that notion as follows

Definition 5. An abelian category $\mathcal{C}$ with enough projectives is *hereditary* if every subobject of a projective object is again projective. ♦

Problem 6. *Let $\mathcal{C}$ be a hereditary abelian category with enough projectives and $F : \mathcal{C} \to \mathrm{Ab}$ a right exact functor. Show that all left derived functors $L_i F$ of F with $i \geq 2$ vanish.*