MTH 620 Due Wednesday, Feb 20
Spring 2019

Homework 2

Recall:

Definition 1. A category $\mathcal{C}$ is *complete* (respectively *cocomplete*) if every functor $F : \mathcal{D} \to \mathcal{C}$ from a small category $\mathcal{D}$ has a limit (respectively colimit). ♦

On the other hand, we can require less:

Definition 2. A category $\mathcal{C}$ *has (co)products* if arbitrary (small) families of objects in $\mathcal{C}$ admit a (co)product in $\mathcal{C}$. Equivalently, every functor $F : \mathcal{D} \to \mathcal{C}$ for discrete small $\mathcal{D}$ admits a (co)limit.

Similarly, $\mathcal{C}$ *has (co)equalizers* if every functor $F : \mathcal{D} \to \mathcal{C}$ from the category

$$\mathcal{D} \;=\; \bullet \rightrightarrows \bullet$$

has a (co)limit. ♦

Problem 1. *Show that a category is (co)complete if and only if it has (co)products and (co)equalizers.*

Feel free to prove only the product/equalizer version, as the other result is entirely parallel by simply reversing the errors. One possible course of action I recommend:

Start with an arbitrary functor $F : \mathcal{D} \to \mathcal{C}$ from a small category $\mathcal{D}$ and form the products (which you know exist by the hypothesis)

$$P_o := \prod_{\text{objects } d \in D} Fd$$

and

$$P_a := \prod_{\text{arrows } f:d\to d' \text{ in } \mathcal{D}} x_f \text{ where } x_f = Fd'.$$

Then consider the two parallel morphisms

$$P_o \underset{\psi_r}{\overset{\psi_l}{\rightrightarrows}} P_a$$

defined as follows :

For ψ_r and an arrow $f : d \to d'$ in $\mathcal{D}$, the component $P_o \to x_f$ of the morphism into the product $P_a = \prod_f x_f$ is the composition

$$P_o \xrightarrow{\pi_{d'}} Fd' \xrightarrow{\text{id}} x_f = Fd'$$

where

$$\pi_{d'} : P_o = \prod_d Fd \to Fd'$$

is the structure morphism coming from the product structure.

On the other hand, for ψ_l and an arrow $f : d \to d'$ the component $P_o \to x_f$ is the composition

$$P_o \xrightarrow{\pi_d} Fd \xrightarrow{Ff} x_f = Fd'.$$

By assumption, the morphisms ψ_l and ψ_r have an equalizer

$$\bullet \xrightarrow{\iota} P_o \underset{\psi_r}{\overset{\psi_l}{\rightrightarrows}} P_a$$

by assumption. Try to show that the morphisms

$$\bullet \xrightarrow{\iota} P_o \xrightarrow{\pi_d} Fd$$

for $d \in \mathcal{D}$ constitute a cone from $\bullet$ to F, and prove that his cone is universal (i.e. a limit for F).

As a consequence of Problem 1, prove:

Problem 2. *Let $F : \mathcal{C} \to \mathcal{C}'$ be a functor between complete categories. Show that F is continuous if and only if it preserves products and equalizers.*

Here's a remark I made in passing at the end of the Feb 6 lecture, which one of you pointed out we hadn't gone over before; I'm making it into a problem:

Problem 3. *Let R and S be two rings and $F : {}_R\mathrm{Mod} \to {}_S\mathrm{Mod}$ an additive functor. Prove that F is left exact if and only if it preserves equalizers (i.e. turns equalizers into equalizers).*

As an immediate consequence, continuous functors (such as right adjoints, say) are left exact.

Problem 4. *5.17 from Rotman.*

Problem 5. *5.32 from Rotman.*

Problem 6. *5.34 from Rotman.*