MTH 620 Due Wednesday, May 8
Spring 2019

Exam 2

Problem 1. *Show that a finite non-trivial group cannot act freely by homeomorphisms on a Euclidean space* $\mathbb{R}^n$*.*

A word on how I would approach this. Suppose some finite group Γ *does* act freely on $\mathbb{R}^n$. The quotient $X := \mathbb{R}^n/\Gamma$ would then be a topological n-dimensional manifold. How does the singular cohomology $H^*(X)$ relate to $H^*(\Gamma)$? What can you deduce from that? The phrase 'cohomological dimension' probably has something to do with this.

Problem 2. *Let* G *and* H *be finite groups of coprime orders. Describe the cohomology groups* $H^i(G \times H)$ *in terms of* $H^*(G)$ *and* $H^*(H)$*.*

You'll want to use the Lyndon-Hochschild-Serre spectral sequence

$$E_2^{p,q} = H^p(G, H^q(H)) \Rightarrow H^{p+q}(G \times H).$$

We proved a while back in class that for a finite group Γ the groups $H^i(\Gamma)$ are annihilated by $|\Gamma|$; this should help collapse that spectral sequence to something very computable.

For the next problem, let $H \trianglelefteq G$ be a normal subgroup of a group and M, N two G-modules. We discussed before how M^H is then a G/H-module and similarly for N^H. Furthermore, we have a morphism

$$M^H \otimes N^H \longrightarrow (M \otimes N)^H$$

of G/H-modules defined in the obvious fashion: for $m \in M^H$ and $n \in N^H$ just send $m \otimes n$ to $m \otimes n \in (M \otimes N)^H$! Applying the functor $H^*(G/H, -)$ to this morphism in ${}_{G/H}\mathrm{Mod}$ produces the lower left hand arrow in (1) below.

Problem 3. *Let* $H \trianglelefteq G$ *as above. Show that the inflation morphism*

$$\mathrm{Inf} : H^*(G/H, -^H) \to H^*(G, -)$$

respects cup products, in the sense that the diagram

$$\begin{array}{ccc} H^i(G/H, M^H) \otimes H^j(G/H, N^H) & \xrightarrow{\mathrm{Inf}\otimes\mathrm{Inf}} & H^i(G, M) \otimes H^j(G, N) \\ \downarrow \cup & & \downarrow \cup \\ H^{i+j}(G/H, M^H \otimes N^H) & & H^{i+j}(G, M \otimes N) \\ & \searrow \qquad \nearrow \mathrm{Inf} & \\ & H^{i+j}(G/H, (M \otimes N)^H) & \end{array} \tag{1}$$

commutes for any two G*-modules* M *and* N*.*

This should be fairly straightforward: you have very explicit formulas for everything here, both cup products and inflation, at the level of cocycles.

The last homework assignment had you work with profinite group cohomology. Perhaps we should compute some examples:

Problem 4. *Describe the cohomology groups $H^i(\mathbb{Z}_p)$ of the additive group of p-adic integers.*

Some comments on how to go about this follow.

First off, recall how this went: you write your group as a limit of finite quotients

$$\mathbb{Z}_p = \varprojlim_n \mathbb{Z}/p^n,$$

and the cohomology groups are then the resulting colimits

$$H^i(\mathbb{Z}_p) = \varinjlim_n H^i(\mathbb{Z}/p^n)$$

where the connecting maps

$$H^i(\mathbb{Z}/p^n) \to H^i(\mathbb{Z}/p^{n+1}) \tag{2}$$

being the inflation morphisms.

Now, you know the groups in (2), so it remains to understand the inflation maps. Try to show that these are one-to-one. There are several ways to do this, but perhaps the most direct is to go back to how we computed the cohomology of a finite cyclic group $\Gamma = \langle\sigma\rangle$ to begin with, via the projective resolution

$$\cdots \xrightarrow{\quad D \quad} \mathbb{Z}\Gamma \xrightarrow{\quad N \quad} \mathbb{Z}\Gamma \xrightarrow{\quad D \quad} \mathbb{Z}\Gamma \xrightarrow{\quad \varepsilon \quad} \mathbb{Z}$$

where

$$D = \sigma - 1 \text{ and } N = 1 + \cdots + \sigma^{\text{order of } \sigma - 1}.$$

Do this for $\Gamma = \mathbb{Z}/p^n$ and $\Gamma = \mathbb{Z}/p^{n+1}$, connect the two resolutions by maps induced by the quotient

$$\mathbb{Z}/p^{n+1} \to \mathbb{Z}/p^n,$$

apply the functor $\mathrm{Hom}(-, \mathbb{Z})$ to everything in sight and argue that this induces injections in cohomology.

Once you have injectivity for (2) you'll want to remember that the colimit

$$\varinjlim_n \mathbb{Z}/p^n$$

of the standard embeddings $\mathbb{Z}/p^n \to \mathbb{Z}/p^{n+1}$ is (isomorphic to) precisely what we've been calling $\mathbb{Z}_{p^\infty}$:

$$\mathbb{Z}_{p^\infty} := \{z \in \mathbb{C} \mid z^{p^t} = 1 \text{ for some } t \in \mathbb{Z}_{\geq 0}\}.$$