MTH 620 Due Wednesday, Mar 13
Spring 2019

Exam 1

Let Γ be a group and $\mathbb{Z}\Gamma$ its group ring. As before (in the previous homework assignment, for instance) we sometimes refer to $\mathbb{Z}\Gamma$-modules as Γ-*modules*.

Definition 1. Let A be a Γ-module and equip $\mathbb{Z}$ with the trivial Γ-module structure (i.e. Γ acts on it trivially).

The *A-valued cohomology groups of* Γ are the groups

$$H^i(\Gamma, A) := \mathrm{Ext}^i(\mathbb{Z}, A).$$

The i^{th} *cohomology group of* Γ is $H^i(\Gamma) := H^i(\Gamma, \mathbb{Z})$. ♦

The first problem requires the following concept.

Definition 2. Let Γ be a group and A a Γ-module. The *subgroup of invariants* of A is

$$A^\Gamma := \{a \in A \mid \gamma a = a, \ \forall \gamma \in \Gamma\}.$$

In other words, it is the largest subgroup of A on which Γ acts trivially. ♦

Problem 1. *Let Γ be a group. Show that the invariant functor $(-)^\Gamma$ defined as above by*

$${}_{\mathbb{Z}\Gamma}\mathrm{Mod} \ni A \mapsto A^\Gamma \in \mathrm{Ab}$$

is left exact and that for every Γ-module A the cohomology group $H^i(\Gamma, A)$ is precisely the image of A through the right derived functor

$$R^i(-)^\Gamma$$

of the invariant functor.

Problem 2. *Let $\Gamma = \mathbb{Z}/n$ for some positive integer $n \geq 2$ and A a Γ-module. Describe the cohomology groups $H^i(\Gamma, A)$ for an arbitrary Γ-module A.*

(Hint: The previous assignment laid out a projective resolution of the trivial Γ-module $\mathbb{Z}$ and had you use that to compute *homology* instead. Use that resolution similarly here.)

Problem 3. *Let R be a ring and M a left R-module. Prove that the following conditions on M are equivalent.*

(i) M is flat.
(ii) $\mathrm{Tor}_i^R(N, M) = 0$ for all right R-modules N and all $i \geq 1$;
(iii) $\mathrm{Tor}_1^R(N, M) = 0$ for all right R-modules N.

Problem 4. *Let R be a ring.*

(a) If

$$0 \to K \to F \to N \to 0$$

is a short exact sequence of right R-modules with F flat, then for all left R-modules M and all positive integers n we have

$$\mathrm{Tor}_{n+1}^R(N, M) = \mathrm{Tor}_n^R(K, M).$$

(b) Given a non-negative integer n, prove that the following conditions on a left module M are equivalent.

(i) $\mathrm{Tor}_i^R(N, M) = 0$ *for all right R-modules N and all $i \geq n+1$;*

(ii) $\mathrm{Tor}_{n+1}^R(N, M) = 0$ *for all right R-modules N.*

(iii) For every exact sequence

$$0 \to F_n \to \cdots \to F_0 \to M \to 0$$

of left modules with flat F_i, $0 \leq i \leq n-1$ the last term F_n is also flat.