MTH 461/561 Due Thursday, October 25
Fall 2018

Homework 6

Problem 1. *Problem 5.1.2 from Etingof.*

Let $n \geq 2$ be a positive integer and G an arbitrary group.

Definition 1. The *wreath product* $G \wr S_n$ is defined as follows.

First, make S_n act on the n-fold product $G^{\times n} = G \times \cdots \times G$ by permuting the factors. Then, set

$$G \wr S_n := G^{\times n} \rtimes S_n,$$

i.e. the semidirect product of the two groups $G^{\times n}$ and S_n with respect to the action of the latter on the former. ♦

Problem 2. *Let $n \geq 2$ be a positive integer, and consider the subgroup G of $GL_n(\mathbb{C})$ consisting of all $n \times n$ matrices with the following properties:*

- *every row contains exactly one non-zero entry;*
- *every column contains exactly one non-zero entry;*
- *the non-zero entries are n^{th} roots of unity.*

Show that G is isomorphic to $\mathbb{Z}_n \wr S_n$.

(Hint: As your n copies of $\mathbb{Z}_n$ take the subgroups $G_i \leq G$ consisting of diagonal matrices whose only diagonal entry $\neq 1$ is (i, i).)

Problem 3. *Let G be a finite group and $H \trianglelefteq G$ a normal subgroup. For two irreducible G-representations V_1 and V_2 write $V_1 \sim V_2$ if the H-representations $\mathrm{Res}_H^G V_1$ and $\mathrm{Res}_H^G V_2$ have at least one common irreducible summand.*

Show that '$\sim$' is an equivalence relation on the set of isomorphism classes of irreducible H-representations.

(Hint: The only mildly difficult part is transitivity. To prove that it holds, use Clifford's theorem to conclude that $V_1 \sim V_2$ if and only if *all* irreducible summands of $\mathrm{Res}_H^G V_1$ are summands of $\mathrm{Res}_H^G V_2$ and vice versa.)