MTH 461/561 Due Thursday, October 4

Fall 2018

Homework 4

The following problem walks you through the reason why characters of representations are well defined. I'll write $P = (p_{ij})$ to mean that P is the matrix whose (i, j) entry is p_{ij}.

Problem 1. *Let V be an n-dimensional vector space.*

(a) *Let $\mathcal{B} = \{e_i\}$, $1 \le i \le n$ be a basis. For a linear map $T : V \to V$ define the matrix entries $t_{ij} \in \mathbb{C}$ for $1 \le i, j \le n$ by*

$$Te_j = \sum_{i=1}^{n} t_{ij} e_i.$$

Prove that sending T to the matrix $M_{\mathcal{B}}(T)$ whose (i, j) entry is t_{ij} is an algebra isomorphism between the algebra $\mathrm{End}(V)$ of all linear maps $V \to V$ and the algebra $M_n(\mathbb{C})$ of $n \times n$ matrices.

(b) *Now fix a different basis $\mathcal{D} = \{f_j\}$, $1 \le j \le n$ for V and similarly define the matrix $M_{\mathcal{D}}(T)$ associated to $T \in \mathrm{End}(V)$. In other words, the (i, j) entry of $M_{\mathcal{D}}(T)$ is s_{ij}, where*

$$Tf_j = \sum_{i=1}^{n} s_{ij} f_i.$$

Let also P be the change of basis matrix between the two bases, defined by $P = (p_{ij})_{i,j=1}^{n}$ where

$$f_j = \sum_{i=1}^{n} p_{ij} e_i.$$

Prove that we have

$$M_{\mathcal{B}}(T) = P M_{\mathcal{D}} P^{-1}$$

(you can assume P is invertible, i.e. P^{-1} exists).

(c) *Show that the matrices $M_{\mathcal{B}}(T)$ and $M_{\mathcal{D}}(T)$ have the same trace, i.e.*

$$t_{11} + \cdots + t_{nn} = s_{11} + \cdots + s_{nn}.$$

Hint: use the fact that for square matrices X and Y you have $\mathrm{tr}(XY) = \mathrm{tr}(YX)$; you can take this fact for granted.

In other words, the trace of a matrix only depends on the linear map $V \to V$ that it induces, not on the actual realization as a matrix.

Problem 2. *Let* V *be a vector space and* $Q : V \to V$ *a linear map that is idempotent, in the sense that* $Q^2 = Q$*. Show that we have*

$$V = \operatorname{Im} Q \oplus \ker Q,$$

meaning that every element $x \in V$ *has a unique decomposition as* $x = u + w$ *where* $u \in \operatorname{Im} Q$ *and* $w \in \ker Q$*.*

Hint: write $x = Qx + (x - Qx)$*.*

Problem 3 (561 only)**.** *Exercise 3.6.1 from Etingof's book.*

Problem 4. *Let* G *be a group,* $A = \mathbb{C}G$ *the group algebra, and* V *an* A*-module whose structure is given by an algebra map* $\rho : A \to \operatorname{End}(V)$*.*

Prove that the linear map $T : V \to V$ *is an* A*-module morphism if and only if we have*

$$\rho(g)T\rho(g)^{-1} = T, \ \forall g \in G.$$