MTH 461/561 Due Thursday, September 27
Fall 2018

Homework 3

Problem 1. *2.5.1 from the textbook.*

To make sense of the next problem you'll need

Definition 1. Let V be a complex vector space. Its *dual* V^* is the vector space of linear functions $V \to \mathbb{C}$.

A linear function $V \to \mathbb{C}$ is sometimes also called a *functional* on V, so that V^* is the space of functionals on V. ♦

Problem 2. *2.5.2 from the book.*

In that problem A is an algebra and you'll see the symbol A^* denoting to the space of functionals on A. What this means is clarified by Definition 1 above.

Problem 3. *Let A be an algebra and U_1 and U_2 be submodules of an A-module U.*

(a) *Prove that*

$$U_1 + U_2 := \{u_1 + u_2 \mid u_1 \in U_1,\ u_2 \in U_2\}$$

is again a submodule of U.

(b) *Show that the map from $U_1 \oplus U_2$ to $U_1 + U_2$ defined by*

$$(u_1, u_2) \mapsto u_1 + u_2$$

is a module morphism.

(c) *Show that the map in part (b) is an isomorphism if and only if $U_1 \cap U_2 = \{0\}$.*

For the next problem to make sense we'll need some preparation.

First, let A_i be a family of algebras for i ranging over some index set I (which can be finite or infinite). We can then construct a new algebra out of this data as follows.

Definition 2. The *direct product* (or just 'product' for short) $\prod_i A_i$ of the algebras A_i is the set of tuples $(a_i)_i$ of elements $a_i \in A_i$ (that is, simply choose one element a_i from each A_i). ♦

The addition and multiplication on the direct product are component-wise:

$$(a_i)_i + (a_i')_i = (a_i + a_i')_i, \quad (a_i)_i \cdot (a_i')_i = (a_i a_i')_i$$

To clarify what's going on you might want to look at the case where the index set I is just $\{1, 2\}$. In that case the product is written as $A_1 \times A_2$, and simply consists of pairs of elements (a_1, a_2) with $a_1 \in A_1$ and $a_2 \in A_2$.

One important family of algebras is that of *matrix algebras*, i.e. those of the form $M_n(\mathbb{C})$ for some positive integer n. We can furthermore grab a bunch of these and form a product, as above:

$$A = \prod_{i \in I} M_{n_i}(\mathbb{C})$$

for positive integers n_i, $i \in I$. We mostly work with finite products, i.e. I is finite.

The next problem is meant to guide you through a proof of Theorem 3.3.1 in the textbook, which describes all irreducible and all finite-dimensional modules over a finite product of matrix algebras. Note that the book denotes

$$\mathrm{Mat}_n(\mathbb{C}) := M_n(\mathbb{C}).$$

Note also that Etingof writes $\oplus$ for the symbol between algebras to indicate their direct product. So his $A_1 \oplus A_2 \oplus \cdots$ is our product $A_1 \times A_2 \times \cdots$. You can use either notation; both appear in the literature.

Problem 4. *3.3.3 from the textbook.*