MTH 461/561 Due Tuesday, September 11
Fall 2018

Homework 1

Recall that we defined the *path algebra* P_Q of a quiver Q as the vector space having the directed paths of Q as a basis. Remember also that we are allowing *empty* paths, i.e. paths p_v attached to a vertex v of the quiver, starting and ending at the vertex v without traversing any edges.

For a path p in Q I will write $s(p)$ for its *source* (i.e. vertex where it starts) and $t(p)$ for its *target* (meaning vertex where it ends).

Definition 1. Let p and q be paths in Q. We said that p is *composable* with q if $t(p) = s(q)$ (meaning: p ends where q begins). ♦

For two elements

$$x = \sum_{\text{paths } p} r_p p \text{ and } x' = \sum_{\text{paths } q} r'_q q$$

of P_Q the product was defined as

$$xx' = \sum_{\text{paths } p,q} r_p r'_q pq \tag{1}$$

where

$$pq = \begin{cases} \text{path } p \text{ followed by path } q & \text{if } p \text{ is composable with } q \\ 0 & \text{otherwise} \end{cases}$$

The following problem is meant to guide you through the proof of the claim made in class that the multiplication on P_Q defined above in (1) has a neutral element if and only if the quiver Q has finitely many vertices.

Problem 1. *Let Q be a quiver and P_Q its algebra.*

(a) *Suppose Q has finitely many vertices, v_1 up to v_n, and denote*

$$x = p_{v_1} + p_{v_2} + \cdots + p_{v_n} \in P_Q \tag{2}$$

(the sum of all of the empty paths, one for each of the vertices). Show that for every path p in Q we have the following relation in the path algebra:

$$xp = p = px \in P_Q.$$

(b) *Deduce from (a) that if Q has finitely many vertices then the element x defined in (2) is a multiplicative unit for the path algebra P_Q.*

(c) *Consider an arbitrary element*

$$x = \sum_{\textit{paths } p} r_p p \in P_Q. \tag{3}$$

Show that for a vertex v in Q we have

$$xp_v = \sum_{\textit{paths } p \textit{ such that } t(p)=v} r_p p,$$

i.e. summation over only those paths that terminate at the vertex v. Similarly,

$$p_v x = \sum_{\textit{paths } p \textit{ such that } s(p)=v} r_p p,$$

meaning summation over only those paths that start at v.

(d) *Now suppose $x \in P_Q$ is a multiplicative unit for the path algebra P_Q, i.e. $xy = y = yx$ for all $y \in P_Q$. In particular we have*

$$xp_v = p_v = p_v x, \ \forall \textit{ vertices } v \textit{ of } Q.$$

From this and part (c) deduce that each summand $r_p p$ of x in the decomposition (3) must be of the form p_v and all p_v (for all vertices v) must appear as summands in (3).

(e) *Conclude from (d) that if P_Q has a multiplicative unit x then Q has finitely many vertices v_1 up to v_n and we have*

$$x = p_{v_1} + \cdots + p_{v_n}.$$