**MTH 461/561** Due Tuesday, December 4
Fall 2018

# Exam 2

**Problem 1.** *Let $n \geq 2$ be a positive integer, $\sigma \in S_n$ a cycle of length $n$, and $\lambda$ a Young diagram with $n$ boxes. If $\chi_\lambda$ is the character of the irreducible $S_n$ representation $V_\lambda$, prove that*

$$\chi_\lambda(\sigma) = \begin{cases} (-1)^s, & \text{if } \lambda = (n-s, 1, 1, \cdots, 1) \\ 0 & \text{otherwise.} \end{cases}$$

In particular, $\chi_\lambda(\sigma)$ is non-vanishing exactly when $\lambda$ is a hook.

**Problem 2.** *Let $C \subset S_n$ be a conjugacy class of the symmetric group, and assume $C$ is in fact contained in the alternating group $A_n$ (i.e. consists of even permutations only).*

*Show that the following conditions are equivalent:*

(a) *$C$ decomposes as a union of two equally-sized conjugacy classes of $A_n$.*
(b) *Every permutation commuting with an element of $C$ is even.*
(c) *The cycle decomposition of every element in $C$ consists of cycles of distinct odd lengths.*

Recall:

**Definition 1.** A *projection* $p$ in an algebra $A$ is an idempotent element, i.e. one satisfying $p^2 = p$.

Two projections $p, q \in A$ are *equivalent* if there are elements $a, b \in A$ such that $ab = p$ and $ba = q$. ♦

**Problem 3.** *Prove that two projections of the matrix algebra $M_n(\mathbb{C})$ are equivalent if and only if they have the same rank.*

(Hint: We proved the implication '$\Rightarrow$' in class. For the converse feel free to use the fact that every matrix with vanishing trace is a *commutator*, i.e. of the form $xy - yx$ for matrices $x, y \in M_n(\mathbb{C})$.)