Math 444 A Due Friday, Feb 26
Winter 2016

Bonus Hw 1: rescaling models of NG

When we discussed models of NG in class on Friday, Feb 19, you saw at work a general procedure for producing new models out of old ones that I want to unpack here in the abstract.

Take an arbitrary model $\mathcal{M}$ of NG (like say the Cartesian plane or the Poincaré disk, which are the two discussed in some detail in Chapter 6). Now fix some positive real number t. I will construct a new model $\mathcal{M}_t$ by literally *rescaling* $\mathcal{M}$ by a factor of t. So think model ships or airplanes; you can have two that are visually identical in all respects except one's bigger.

Formally, $\mathcal{M}_t$ looks like this.

Definition 1. Let $\mathcal{M}$ be a model of Neutral Geometry and $t > 0$ a positive real number. The *rescaled* model $\mathcal{M}_t$ is defined as follows:

- The points of $\mathcal{M}_t$ are the same as those of $\mathcal{M}$;
- The lines of $\mathcal{M}_t$ are the same as those of $\mathcal{M}$;
- The angle measures in $\mathcal{M}_t$ coincide with those in $\mathcal{M}$;
- For any two points A and B of $\mathcal{M}_t$ (which are hence also points of $\mathcal{M}$), if the distances between them in $\mathcal{M}$ and $\mathcal{M}_t$ are denoted by $d_{\mathcal{M}}(A,B)$ and $d_{\mathcal{M}_t}(A,B)$ respectively, then the relation between the two distances is

$$d_{\mathcal{M}_t}(A,B) = t \cdot d_{\mathcal{M}}(A,B)$$

(this last expression is just multiplication by the number $t > 0$). ♦

Now, what we did in class was apply this procedure to the hyperbolic plane $\mathcal{H}$ (first introduced Wednesday, Feb 17 and isomorphic to the Poincaré disk in your book) to get its rescaled siblings $\mathcal{H}_t$. With these in hand, the theorem I stated was

Theorem. *Every model of NG is isomorphic either to one of the rescaled hyperbolic planes $\mathcal{H}_t$ or to the Cartesian plane $\mathcal{C}$.*

What should puzzle you a bit at this stage is: why didn't I do this rescaling thing to $\mathcal{C}$ itself? Can't I get *new* models $\mathcal{C}_t$ out of $\mathcal{C}$ the same way I got $\mathcal{H}_t$ out of $\mathcal{H}$? Well, no, because that wouldn't have led to anything new! This is what you're supposed to do for this assignment:

Problem. *Let $\mathcal{C}$ be the Cartesian plane and $t > 0$. Prove that the rescaled version $\mathcal{C}_t$ is isomorphic to $\mathcal{C}$.*

In other words, rescaling $\mathcal{C}$ doesn't affect it: you get a model that's "the same" as $\mathcal{C}$ for all mathematical intents and purposes. You might need to refresh your memory on isomorphisms between models. These are first mentioned on page 29 of the textbook, and discussed further in Chapter 6, on page 132.

As a hint, maybe try to show that turning the point (x, y) in $\mathcal{C}_t$ into the point (tx, ty) of $\mathcal{C}$ is an isomorphism in the sense of that explanation on page 132.

One more thing. As a final thought, I maybe want to look a little at the sort of phenomenon you saw happen in class, where the Cartesian plane $\mathcal{C}$ was somehow recovered as "the limit" of the hyperbolic planes $\mathcal{H}_t$ as $t \to \infty$. One thing that should be a little mysterious is that in the limit you get this drastic change of behavior from the hyperbolic postulate to the Euclidean postulate sort of "suddenly".

This happens all the time in the real world though! Physicists call these types of phenomena *phase transitions*; they consist of drastic changes in the physical behavior of a substance when a certain parameter hits a critical value. The most familiar example will probably be water switching from liquid to solid when it hits 0° (Celsius).

You might want to keep this analogy in mind, because it works on multiple levels, with curvature (the thing we talked about in class) as the parameter; it's sort of like temperature in the liquid water / ice example. So just think of the hyperbolic planes $\mathcal{H}_t$ "crystallizing" into the Euclidean plane as the curvature $-\frac{1}{t^2}$ hits 0 (i.e. t "reaches" infinity).

In a sense, curvature "explains" why you can't rescale your way out of $\mathcal{C}$, in the sense that all $\mathcal{C}_t$ are isomorphic to it. Rescaling also scales curvature, and once you hit zero, you can rescale that all you want, it'll still be zero. This is no proof though; proving that the various $\mathcal{C}_t$ are mutually isomorphic is precisely what the Problem above asks you to do.